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Topological entropy for set valued maps

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Abstract. Any continuous map T on a compact metric space $\mathbb{X}$ induces in a natural way a continuous map $\overline{T}$ on the space $\mathcal{K}(\mathbb{X})$ of all non-empty compact subsets of $\mathbb{X}$. Let T be a homeomorphism on the interval or on the circle. It is proved that the topological entropy of the induced set valued map $\overline{T}$ is zero or infinity. Moreover, the topological entropy of $\overline{T}|_{\mathcal{C}(\mathbb{X})}$ is zero, where $\mathcal{C}(\mathbb{X})$ denotes the space of all non-empty compact and connected subsets of $\mathbb{X}$. For general continuous maps on compact metric spaces these results are not valid.

1. Preliminary

Many phenomena arising from Population Dynamics, Economy Theory, Social Sciences and Engineering are described by discrete dynamical system

$$x_{n+1} = T(x_n),\ n = 0, 1, 2, \ldots \tag{1}$$

where points x_n belong to a compact metric space $\mathbb{X}$ and $T : \mathbb{X} \to \mathbb{X}$ is a continuous transformation. The main aim of the theory of discrete dynamical systems is centred on the understanding how the trajectories of all points from $\mathbb{X}$ look like.

Nevertheless, to understand many events from the system (1) it needs to know how the subsets (called *collectives*) of $\mathbb{X}$ are moved, not only points (called *individuals*). In this direction we consider the *set valued* discrete dynamical system

$$K_{n+1} = \overline{T}(K_n),\ n = 0, 1, 2, \ldots \tag{2}$$

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where points K_n belong to a set of all non-empty compact subsets of $\mathbb{X}$ and $\overline{T}$ (called *induced map*) is the natural extension of T defined by $\overline{T}(K) = T(K)$ for any compact subset of $\mathbb{X}$.

2. Introduction

Let $(\mathbb{X}, d)$ be a compact metric space and $T : \mathbb{X} \to \mathbb{X}$ be a continuous map. Denote by $\mathcal{K}(\mathbb{X})$ the set of all non-empty compact subsets of $\mathbb{X}$, i.e.

$$\mathcal{K}(\mathbb{X}) = \{K \subseteq \mathbb{X} : K \text{ is compact}\},$$

and let $\mathcal{C}(\mathbb{X})$ be the set of all compact and connected subsets of $\mathbb{X}$.

On $\mathcal{K}(\mathbb{X})$ we will use the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) = \max\big\{ \sup_{x_1 \in K_1} d(x_1, K_2), \sup_{x_2 \in K_2} d(x_2, K_1)\big\}.$$

This space endowed with the Hausdorff metric is compact. The topology induced by the Hausdorff metric is equivalent to the Vietoris topology whose base is defined for any finite collection of non-empty open subsets $U_1, U_2, \ldots, U_n$ of $\mathbb{X}$ by

$$\langle U_1, U_2, \ldots U_n \rangle = \Big\{K \in \mathcal{K}(\mathbb{X}) |\, K \subseteq \bigcup_{i=1}^{n} U_i,\ K \cap U_i \neq \emptyset \text{ for } i = 1, 2, \ldots, n\Big\}.$$

We define the induced map $\overline{T} : \mathcal{K}(\mathbb{X}) \to \mathcal{K}(\mathbb{X})$ by

$$\overline{T}(K) = T(K) = \{T(x) |\, x \in K\}.$$

The main aim is to find all the dynamical properties which are preserved while moving from T to $\overline{T}$ and conversely. Many of them are known. Transitivity and mixing properties were discussed in [6] and [10]. Various notions of chaos were studied in [1], [3], [4], [5], [8], [9], [11], [12], [14], and finally, some properties about topological entropy were proved in [7].

As it was proved in [2] (or [7]) if T has PTE (positive topological entropy) then also $\overline{T}$ does. The converse is not true (see, e.g. [4]). It is also true that if T has PTE than topological entropy of $\overline{T}$ is infinite (see [2] or [7]). The main aim of the present paper is to clarify these observance in more details.

The paper is organized as follows: First, we put some motivations in Section 1 and definitions in Sections 2 and 3. Then in Section 4 we prove results on the interval, on the circle and on continua. We close in Section 5 with counter examples for general compact metric spaces.

3. Definitions

Let $T : \mathbb{X} \to \mathbb{X}$ be a continuous map of a compact metric space $(\mathbb{X}, d)$. Define $d_n(x, y) = \max_{0\le i\le n-1} d(T^i(x), T^i(y))$ for any natural number n and $x, y \in \mathbb{X}$. For a natural number n, $\varepsilon > 0$ and a compact subset K of $\mathbb{X}$, a subset F of $\mathbb{X}$ is said to (n, ε) *span* K with respect to T if for any $x \in K$ there is $y \in F$ such that $d_n(x, y) \le \varepsilon$. Denote by $r_n(\varepsilon, K)$ the smallest cardinality of any (n, ε) spanning set for K with respect to T.

Definition 1. The *topological entropy* $\mathrm{h}(T)$ of T is defined by

$$\mathrm{h}(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon, \mathbb{X})\,. \tag{3}$$

Now let $\mathcal{A}$ be any finite set (an alphabet) containing n elements (called symbols). An *infinite word* is a map $w\colon \mathbb{N}_0 \to \mathcal{A}$, hence it is an infinite sequence $(w_0, w_1, w_2, \ldots)$ where $w_i \in \mathcal{A}$ for any $i \in \mathbb{N}_0$. The set of all infinite words over the alphabet $\mathcal{A}$ is denoted by Σ_n.

The set $\mathcal{A}$ is endowed with the discrete topology. Then Σ_n is metrizable by the following metric. For any $x, y \in \Sigma_n$ put

$$d(x, y) = \begin{cases} 2^{-k}, & \text{if } \ x \neq y, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where k is the length of the maximal common prefix of x and y. Now, define a shift map $\sigma_n\colon \Sigma_n \to \Sigma_n$ by

$$(\sigma_n(x))_i = x_{i+1}$$

The pair (Σ_n, σ_n) is said to be *the one-sided shift on n symbols.*

If we consider the space $\mathcal{A}^{\mathbb{Z}}$ of bi-infinite words (i.e. sequences $\mathbb{Z} \to \mathcal{A}$), then we can build an analogous theory of shift spaces. Then the pair $(\{0, \ldots, n-1\}^{\mathbb{Z}}, \sigma_n)$ is called *two-sided shift on n symbols.*

We denote by $O(x)$ full orbit of x under T, by $\omega_T(x)$ an omega limit set of x under T and by $\omega(T)$ union of all omega limit sets. By $\mathrm{Fix}(T)$ we denote the set of all fixed points and by $\mathrm{Per}(T)$ the set of all periodic points.

4. Results on continua

In this section we characterize topological entropy of homeomorphisms and their induced maps on the interval $\mathbb{I} = [0, 1]$, the unit circle $\mathbb{S}^1$ and on continua $\mathbb{X}$. The following proposition which follows directly from Theorem 7.5 and Corollary 8.6.1 in [13] will be used in our proofs.

Proposition 1. *Let $\mathbb{X}$ be a compact metric space and T a continuous map. If $\omega(T) \subseteq M_1 \cup M_2 \cup \cdots \cup M_n$, where the sets M_i are closed and invariant under T for each i, then $\mathrm{h}(T) = \max_{1\leq i\leq n} \mathrm{h}(T|_{M_i})$.*

From the proof of Corollary 8.6.1 in [13] we get the following result.

Proposition 2. *Let $T : \mathbb{X} \to \mathbb{X}$ be a continuous map where $\mathbb{X}$ is a compact metric space. Assume that for any $x \in \mathbb{X}$ there is a $y \in \mathrm{Fix}(T)$ with $\lim_{n\to\infty} T^n(x) = y$. Then $\mathrm{h}(T) = 0$.*

Firstly, we would like to concentrate our attention on homeomorphisms defined on $\mathbb{I}$ and we distinguish between induced homeomorphisms on $\mathcal{K}(\mathbb{I})$ and $\mathcal{C}(\mathbb{I})$.

Theorem 1. *Let T be a homeomorphism on $\mathbb{I}$. Then $\mathrm{h}(T) = \mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{I})}) = 0$.*

Proof. It follows from Corollary 7.14.1 in [13] that $\mathrm{h}(T) = 0$. The map T^2 is increasing. If T^2 is the identity then $\overline{T}^2$ is also the identity and hence $2\mathrm{h}(\overline{T}) = \mathrm{h}(\overline{T}^2) = 0$. Therefore it remains to consider the case that T^2 is not the identity.

Let $K \in \mathcal{C}(\mathbb{I})$. Then $K = [k_1, k_2]$. Set $x_1 = \inf\{x \in [k_1, 1] : x \in \mathrm{Fix}(T^2)\}$ if $T^2(k_1) \geq k_1$, and $x_1 = \sup\{x \in [0, k_1] : x \in \mathrm{Fix}(T^2)\}$ otherwise. Analogously define $x_2 = \inf\{x \in [k_2, 1] : x \in \mathrm{Fix}(T^2)\}$ if $T^2(k_2) \geq k_2$, and $x_2 = \sup\{x \in [0, k_2] : x \in \mathrm{Fix}(T^2)\}$ otherwise. Then $\lim_{n\to\infty}(\overline{T}^2)^n(K) = [x_1, x_2]$. Hence Proposition 2 gives $2\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{I})}) = \mathrm{h}\big((\overline{T}|_{\mathcal{C}(\mathbb{I})})^2\big) = 0$, completing the proof. □

Now we will prove a result on continua $\mathbb{X}$. It will state that for a homeomorphism T on $\mathbb{X}$ the existence of a point $x \notin \mathrm{Fix}(T)$ with $\lim_{n\to\infty} T^n(x)$ and $\lim_{n\to-\infty} T^n(x)$ exists implies that $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{I})}) = \infty$. Note that these two limits may be equal or may be different. In any case both have to be fixed points of T. If both limits coincide the two-sided orbit of x would be homoclinic.

Lemma 1. *Let $\mathbb{X}$ be a continuum and let $T : \mathbb{X} \to \mathbb{X}$ be a homeomorphism. Assume that there exist $(x_n)_{n=-\infty}^{\infty}$ with $x_{n_1} \neq x_{n_2}$ for $n_1 \neq n_2$ and $T(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Moreover assume that $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} x_{-n}$ exist. Then there exist $(y_{k,n})_{k=0}^{\infty}{}_{n=-\infty}^{\infty}$ with $y_{k_1,n_1} \neq y_{k_2,n_2}$ for $(n_1, k_1) \neq (n_2, k_2)$, $T(y_{k,n}) = y_{k,n+1}$ for all $k \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ and $\lim_{n\to\infty} y_{k,n} = a$ and $\lim_{n\to-\infty} y_{k,n} = b$ for all $k \in \mathbb{N}_0$.*

Proof. Consider an arc γ_0 between x_0 and x_1 such that $\mathrm{Per}(T)\cap\gamma_0 = \emptyset$. As T is a homeomorphism there are $(\gamma_n)_{n=-\infty}^{\infty}$ such that γ_n is an arc between x_{n-1} and x_n and $T(\gamma_n) = \gamma_{n+1}$ for all n. Choose a sequence $(y_k)_{k=0}^{\infty}$ in $\gamma_0 \setminus \{x_1\}$ with $y_{k_1} \neq y_{k_2}$ for $k_1 \neq k_2$. For $k \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ define $y_{k,n} = T^n(y_k)$. Then $(y_{k,n})_{k=0}^{\infty}{}_{n=-\infty}^{\infty}$ obviously satisfy the desired properties. □

Now we are able to present the above-mentioned result. The proof is similar to the proof of Theorem 10 in [4].

Theorem 2. *Let $\mathbb{X}$ be a continuum and let $T : \mathbb{X} \to \mathbb{X}$ be a homeomorphism. Suppose that there exist $(x_n)_{n=-\infty}^{\infty}$ with $x_{n_1} \neq x_{n_2}$ for $n_1 \neq n_2$ and $T(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Moreover assume that $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} x_{-n}$ exist. Then $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = \infty$.*

Proof. By Lemma 1 there are $(y_{k,n})_{k=0}^{\infty}{}_{n=-\infty}^{\infty}$ with $y_{k_1,n_1} \neq y_{k_2,n_2}$ for $(n_1, k_1) \neq (n_2, k_2)$, $T(y_{k,n}) = y_{k,n+1}$ for all $k \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ and $\lim_{n\to\infty} y_{k,n} = a$ and $\lim_{n\to-\infty} y_{k,n} = b$ for all $k \in \mathbb{N}_0$, where $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to-\infty} x_n = b$. Fix an $r \in \mathbb{N}$. Then $L_r = \bigcup_{k=0}^{r-1} \overline{O(y_{k,0})} = \{y_{k,n} : 0 \leq k \leq r-1, n \in \mathbb{Z}\} \cup \{a, b\}$.

We construct a conjugation φ between $\mathcal{K}(L_r)$ and $(\{0,1\}^r)^{\mathbb{Z}}$. If $K \in \mathcal{K}(L_r)$ define $\varphi(K) = \big((w_{0,n}, w_{1,n}, \ldots, w_{r-1,n})\big)_{n=-\infty}^{\infty}$ where $w_{k,n} = 1$ if $y_{k,n} \in K$ and $w_{k,n} = 0$ otherwise. This map is bijective and satisfies $\varphi \circ \overline{T} = \sigma \circ \varphi$ where σ is the shift map. Obviously σ is conjugated to the two-sided shift on 2^r symbols. Therefore $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) \geq \mathrm{h}(\overline{T}|_{L_r}) = \log(2^r)$. Since this holds for any r we obtain $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = \infty$. □

Next we consider again homeomorphisms on the interval. In Theorem 1 it was proved that $\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{X})}) = 0$. The situation is different if we consider $\overline{T}|_{\mathcal{K}(\mathbb{X})}$.

Theorem 3. *Let $T : \mathbb{I} \to \mathbb{I}$ be a homeomorphism such that T^2 is not the identity. Then $\mathrm{h}(T) = 0$ and $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{I})}) = \infty$.*

Proof. As in Theorem 1 $\mathrm{h}(T) = 0$ (Corollary 7.14.1 from [13]). Since T^2 is not the identity there is an x with $T^2(x) < x$ or $T^2(x) > x$. Without loss of generality we may assume that $T^2(x) < x$. Then $(T^{2n}(x))_{n=-\infty}^{\infty}$ satisfies that $T^{2n_2}(x) < T^{2n_1}(x)$ for $n_1 < n_2$, $\lim_{n\to\infty} T^{2n}(x)$ is a fixed point of T^2 and $\lim_{n\to-\infty} T^{2n}(x)$ is a fixed point of T^2. Therefore Theorem 2 implies that $2\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{I})}) = \mathrm{h}(\overline{T^2}|_{\mathcal{K}(\mathbb{I})}) = \infty$ completing the proof. □

Remark 1. Let us note as a consequence of Theorems 1 and 3 that the topological entropy of homeomorphisms on $\mathbb{I}$ is supported on the subspace of non-connected subsets of $\mathbb{X}$.

Next we consider homeomorphisms on the unit circle $\mathbb{S}^1$. We like to find the topological entropies of its induced maps with respect to the spaces $\mathcal{K}(\mathbb{S}^1)$ and $\mathcal{C}(\mathbb{S}^1)$. At first we investigate $\overline{T}|_{\mathcal{C}(\mathbb{S}^1)}$.

Theorem 4. *Let T be a homeomorphism on $\mathbb{S}^1$. Then $\mathrm{h}(T) = \mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{S}^1)}) = 0$.*

Proof. By Theorem 7.14 from [13] we have that $\mathrm{h}(T) = 0$. It remains to show that $\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{S}^1)}) = 0$.

For $x, y \in \mathbb{S}^1$ let $[x, y]$ be all points of $\mathbb{S}^1$ lying in counterclockwise sense between x and y ($[x, y] = \{x\}$ if $x = y$). Then each non-empty $A \in \mathcal{C}(\mathbb{S}^1)$ satisfies $A = [x, y]$ for some x, y or $A = \mathbb{S}^1$. Define $\widehat{\mathcal{C}}(\mathbb{S}^1) = \{(x, [x, y]) : x, y \in \mathbb{S}^1\} \cup \{(x, \mathbb{S}^1) : x \in \mathbb{S}^1\}$ and define $\widehat{T} : \widehat{\mathcal{C}}(\mathbb{S}^1) \to \widehat{\mathcal{C}}(\mathbb{S}^1)$ by $\widehat{T}(x, A) = (T(x), \overline{T}(A))$. Then $\widehat{T}$ is continuous and $\overline{T}|_{\mathcal{C}(\mathbb{S}^1)}$ is a factor of $\widehat{T}$. Now define $\varphi(x, [x, y]) = (x, y)$ for $x, y \in \mathbb{S}^1$ and $\varphi(x, \mathbb{S}^1) = (x, x)$ for $x \in \mathbb{S}^1$. The map $\varphi : \widehat{\mathcal{C}}(\mathbb{S}^1) \to \mathbb{S}^1 \times \mathbb{S}^1$ is continuous and satisfies $\varphi \circ \widehat{T} = (T \times T) \circ \varphi$. Moreover, every element of $\mathbb{S}^1 \times \mathbb{S}^1$ has at most two pre-images under φ and φ^{-1} is continuous on $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{(x, x) : x \in \mathbb{S}^1\}$. Therefore $\mathrm{h}(\widehat{T}) = \mathrm{h}(T \times T)$. From Theorem 7.10 in [13] we obtain $\mathrm{h}(T \times T) = \mathrm{h}(T) + \mathrm{h}(T) = 0$. As $\overline{T}|_{\mathcal{C}(\mathbb{S}^1)}$ is a factor of $\widehat{T}$ its entropy is less than $\mathrm{h}(\widehat{T}) = 0$, and hence $\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{S}^1)}) = 0$. □

On the interval there were two possibilities for $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{S}^1)})$ by Theorem 3. If T^2 is the identity this entropy equals zero, otherwise it equals infinity. Our next result states that also for homeomorphisms on the circle the entropy is either zero or infinity.

Theorem 5. *Let T be a homeomorphism on $\mathbb{S}^1$. Then either $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{S}^1)}) = 0$ or $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{S}^1)}) = \infty$.*

Proof. Assume at first that T has a periodic point. Then there is an $x \in \mathbb{S}^1$ and a $k \in \mathbb{N}$ with $T^k(x) = x$. Every $z \in \mathbb{S}^1$ can be written as $z = xe^{2\pi it}$ for some $t \in [0, 1]$. The map $\varphi(z) = t$ conjugates T^k to a homeomorphism on $\mathbb{I}$. Hence Theorem 3 implies that $k\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{S}^1)}) = \mathrm{h}(\overline{T}^k|_{\mathcal{K}(\mathbb{S}^1)}) \in \{0, \infty\}$.

It remains to consider the case that T has no periodic points. If T is conjugated to an irrational rotation then $\overline{T}|_{\mathcal{K}(\mathbb{S}^1)}$ is conjugated to an isometry (as any rotation is an isometry). Therefore $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{S}^1)}) = 0$ in this case. Otherwise there exists an $x \in \mathbb{S}^1$ which is not contained in the ω-limit set of T. Denote the α-limit set of x by A and the ω-limit set of x by B. Then the two-sided orbit of x is disjoint from $A \cup B$. Now a proof analogous to the proof of Theorem 2 (and the proof of Lemma 1) shows that $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{S}^1)}) = \infty$ in this case. □

Remark 2. From Theorems 1, 3, 4 and 5 one can derive the following results for a homeomorphism T on a graph $\mathbb{X}$. We have again that $\mathrm{h}(T) = \mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{X})}) = 0$. Furthermore $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) \in \{0, \infty\}$.

To the end of this section we present examples showing differences in the behaviour for continuous interval maps. Our first example shows that Theorem 1 is not true for continuous maps on the interval. According to the second example a one-sided analogon of Theorem 2 does not hold.

Example 1. Let T be the full tent map defined by $T(x) = 1 - |2x - 1|$ on $\mathbb{I}$. It is well known that $\mathrm{h}(T) = \log 2$. Let $K_1 \subseteq \mathcal{C}(\mathbb{I})$ be a non-degenerated interval. Then $T^n(K_1) \to \mathbb{I}$, for $n \to \infty$. If $K_2 \subseteq \mathcal{C}(\mathbb{I})$ is a singleton then $T(K_2) = \overline{T}(K_2)$ and $\mathrm{h}(T) = \mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{I})}) = \log 2$. Obviously, by Proposition 3 below $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{I})}) = \infty$.

Remark 3. For the proof of the above constructions we need the map T to be homeomorphism. The constructions are not valid for continuous maps. If T is a continuous map, e.g. $T(x) = \frac{1}{2}x$, then the essential subspace of $\mathcal{K}(\mathbb{X})$ is conjugated to the right one-sided shift on two symbols. Put $x \in \mathrm{int}(\mathbb{I})$ and $K = \{T^j(x) : j \in N \subseteq \mathbb{N}\}$. Then $\overline{T}(K) = \{T^{j+1}(x) : j \in N \subseteq \mathbb{N}\}$ and T is conjugated to the right one-sided shift on two symbols. Hence, $\sigma_r(w_0, w_1, w_2, \dots) = (0, w_0, w_1, w_2, \dots)$ since $x \notin \overline{T}(K)$. So, $\overline{T}^n(K)$ tends to the fixed point $\{0\}$ and $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = 0$.

5. Results on compact metric spaces

In this section we formulate theorems on general compact metric spaces and we find some counter examples. At first we present a result proved in [2].

Proposition 3. *Let T be a continuous map on $\mathbb{X}$ such that $\mathrm{h}(T) > 0$, then $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = \infty$.*

Remark 4. Let us note, that the above proposition is not valid on $\mathcal{C}(\mathbb{X})$, see Example 1.

Finally, we construct examples of homeomorphisms such that they have zero topological entropy and their induced maps have positive topological entropy.

Theorem 6. *For any positive integer n there is a disconnected compact metric space $\mathbb{X}$ and a homeomorphism T on $\mathbb{X}$ such that $\mathrm{h}(T) = 0$, $\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{X})}) = 0$ and $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = n \log 2$.*

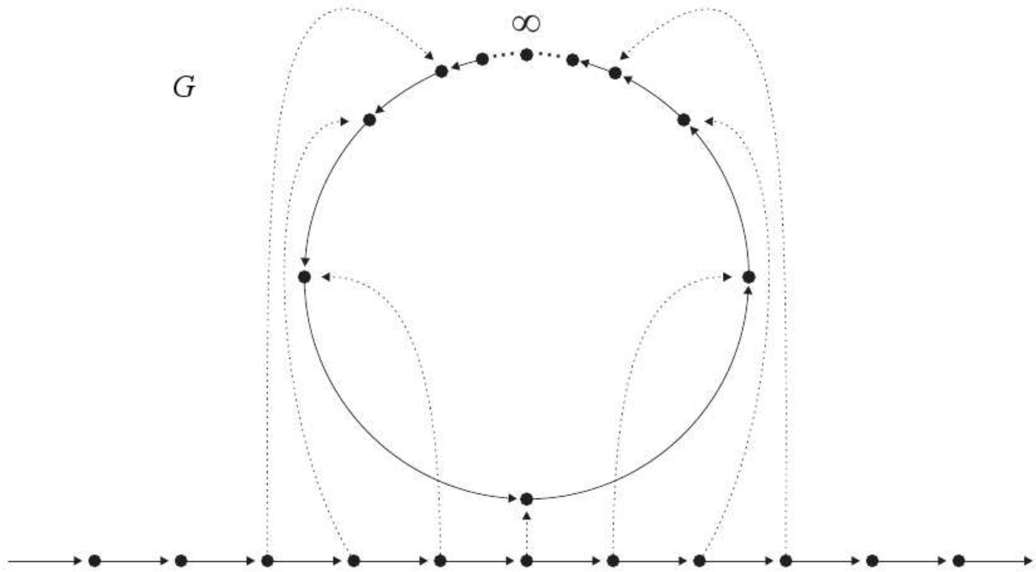
Proof. Let $\mathbb{Y}$ be the one point compactification of $\mathbb{Z}$ and define $G : \mathbb{Y} \to \mathbb{Y}$ by

$$G(k) = \begin{cases} k+1, & \text{if} \quad k \in \mathbb{Z}, \\ \infty, & \text{if} \quad k = \infty, \end{cases} \tag{5}$$

(see the picture below). By [7] $\mathrm{h}(\overline{G}) = \log 2$ (in [7] a subsystem conjugate to the two-shift has been constructed).

Now, by collapsing $\{1, 2, \dots n\} \times \{\infty\}$ in $\{1, 2, \dots n\} \times \mathbb{Y}$ into a single point, we get a compact metric space $\mathbb{X}$. One can think of it as a subspace of $\mathbb{R}^3$. The topology on $\mathbb{X}$ is given by the metric inherited from $\mathbb{R}^3$. We can imagine the space $\mathbb{X}$ as a union of slices $\mathbb{Y}_i = \mathbb{Y} \times \{\infty\}$ with one common point ∞. Let $T : \mathbb{X} \to \mathbb{X}$ be the map such that T restricted to $\mathbb{Y}_i$ is equal to G, for each i.

Consequently, by the same arguments as in Theorem 2 we have $\mathrm{h}(\overline{T}) = \log(2^n) = n \log 2$. ☐



Proposition 4. *There is a disconnected compact metric space* $\mathbb{X}$ *and a homeomorphism* T *on* $\mathbb{X}$ *such that* $\mathrm{h}(T) = 0$, $\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{X})}) = 0$ *and* $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = \infty$.

Proof. If we replace the set $\{1, 2, \dots, n\}$ by the set $D = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ in the proof of Theorem 6, then we get the assertion. ☐

6. Concluding Remarks

As it was proved, it is possible that $\mathrm{h}(\overline{T}|_{\mathcal{C}(\mathbb{X})})$ is 0, finite or infinite for a continuum $\mathbb{X}$ (see Example 1, Theorem 1). The natural question is whether the same is true for $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})})$ on a continuum $\mathbb{X}$. We analyzed this situation for the interval $\mathbb{I}$, circle $\mathbb{S}^1$ and a finite graph. So the following problem remains still open.

Open problem. Is it possible that $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{D})})$ is not zero and finite for a dendrite $\mathbb{D}$?

To the end we can pose a hypothesis.

Hypothesis. Let T be a homeomorphism on a continuum $\mathbb{X}$. Then $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = \infty$ or $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) = 0$.

As it seems (see Section 5) the assumption of continua makes sense. In this direction one can ask the following question.

Open problem. Which topological spaces $\mathbb{X}$ satisfy that $\mathrm{h}(\overline{T}|_{\mathcal{K}(\mathbb{X})}) \in \{0, \infty\}$ for all continuous maps T (or for all homeomorphisms T).

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