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Topological transitivity for a class of monotonic mod one transformations

Peter Raith and Angela Stachelberger

Abstract. Suppose that $f : [0,1] \to [0,2]$ is a continuous strictly increasing piecewise differentiable function, and define $T_f x := f(x) \pmod 1$. Let $\beta \geq \sqrt[3]{2}$. It is proved that T_f is topologically transitive if $\inf f' \geq \beta$ and $f(0) \geq \frac{1}{\beta+1}$. Counterexamples are provided if the assumptions are not satisfied. For $\sqrt[3]{2} \leq \beta < \sqrt{2}$ and $0 \leq \alpha \leq 2 - \beta$ it is shown that $\beta x + \alpha \pmod 1$ is topologically transitive if and only if $\alpha < \frac{1}{\beta^2+\beta}$ or $\alpha > 2 - \beta - \frac{1}{\beta^2+\beta}$.

Introduction

Let $f : [0,1] \to [0,2]$ be a continuous strictly increasing function. Define

$$T_f x := f(x) \pmod 1 := f(x) - [f(x)] \,, \tag{1}$$

where $[y]$ denotes the largest integer smaller than or equal to y. Such a map is called a *monotonic mod one transformation* (with two monotonic pieces). A general monotonic mod one transformation is also defined as in (1), but $f : [0,1] \to \mathbb{R}$. Assume that f is a piecewise differentiable function, that means f is differentiable on $(0,1) \setminus F$ where F is a finite set. The map T_f is called *topologically transitive* if there is an $x \in [0,1]$ such that $\{{T_f}^n x : n \in \mathbb{N}\}$ is dense in $[0,1]$. This is equivalent to the property that there is an x whose ω-limit set equals $[0,1]$, where the ω-limit set is the set of all limit points of the sequence $({T_f}^n x)_{n\in\mathbb{N}}$. For further properties of topological transitivity see e.g. [1], [5], and [12].

The aim of this paper is to present conditions for f implying topological transitivity (obviously there will not be equivalent conditions). These conditions are related to the derivative of f. Set $\inf f' := \inf\{f'(x) : x \in (0,1) \setminus F\}$.

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In fact, the condition $\inf f' \geq \beta$ used throughout this paper could be replaced by the weaker condition $|f(x) - f(y)| \geq \beta|x - y|$ for all $x, y \in [0, 1]$. As in the proofs only the fact that an interval of length d is mapped to an interval of length at least βd is used they work also in the more general case. However, in the statements of the results we use $\inf f' \geq \beta$, since this is the easier formulation.

Similar problems have been treated in [6], [9], and [10]. In [6] and [9] conditions implying the topological transitivity of piecewise monotonic maps are investigated. These results imply that a general monotonic mod one transformation is topologically transitive if $\inf f' > 2$ (Corollary 1.1 in [9]). It has been proved in [10] that a monotonic mod one transformation with two monotonic pieces satisfying $\inf f' \geq \sqrt{2}$ is topologically transitive (Theorem 1 in [10]).

Throughout this paper we will only consider monotonic mod one transformations with two monotonic pieces. The main result (Theorem 1) states that a monotonic mod one transformation satisfying $\inf f' \geq \beta$ for some $\beta \geq \sqrt[3]{2}$ and $f(0) \geq \frac{1}{\beta+1}$ is topologically transitive. In particular (Corollary 1.1) any monotonic mod one transformation with $\inf f' \geq \sqrt[3]{2}$ and $f(0) \geq \frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3} = \frac{1}{\sqrt[3]{2}+1}$ is topologically transitive. An example is presented where $\beta < \sqrt[3]{2}$, $\inf f' \geq \beta$, $f(0) \geq \frac{1}{\beta+1}$ and T_f is not topologically transitive. Finally we give an example with $\inf f' \geq \sqrt[3]{2}$ and $f(0) < \frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3}$ where T_f is not topologically transitive.

Then the special case $\beta x + \alpha \pmod 1$ is investigated. In this case the results are slightly different to the general case. Suppose that $\sqrt[3]{2} \leq \beta < \sqrt{2}$ and that $0 \leq \alpha \leq 2 - \beta$. Then Theorem 4 states that $\beta x + \alpha \pmod 1$ is topologically transitive if and only if $\alpha < \frac{1}{\beta^2+\beta}$ or $\alpha > 2 - \beta - \frac{1}{\beta^2+\beta}$.

1. The Markov diagram of monotonic mod one transformations

Consider again a continuous strictly increasing piecewise differentiable function $f : [0, 1] \to [0, 2]$ and let T_f be as in (1). If $\inf f' > 1$ then there exists a unique $c \in (0, 1)$ with $f(c) = 1$. Define $\mathcal{Z} := \{(0, c), (c, 1)\}$. For each $Z \in \mathcal{Z}$ the map $T_f|_Z$ is continuous and strictly increasing. Note that T_f is discontinuous at c.

A *topological dynamical system* (X, S) is a continuous map $S : X \to X$ on a compact metric space (see e.g. [12]). As T_f is not continuous $([0, 1], T_f)$ is not a topological dynamical system. In order to get a topological dynamical system we use a standard doubling points construction as in [7] or [11]. For details we refer to the papers mentioned above.

To investigate the orbit structure of a piecewise monotonic map Franz Hofbauer introduced the Markov diagram in [2] (see e.g. [2] and [4]). It is an at most countable oriented graph. For the convenience of the reader we describe it for monotonic mod one transformations. Let $Z_0 \in \mathcal{Z}$ and let D be an open subinterval of Z_0. We call a nonempty C a *successor* of D, if there exists a $Z \in \mathcal{Z}$ with $C = T_f D \cap Z$. In this case we write $D \to C$. Now let $\mathcal{D}$ be the smallest set with $\mathcal{Z} \subseteq \mathcal{D}$ satisfying $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then the oriented graph $(\mathcal{D}, \to)$ is called the *Markov diagram* of T_f. The set $\mathcal{D}$ is at most countable and its elements are open subintervals of elements of $\mathcal{Z}$. A subset $\mathcal{C} \subseteq \mathcal{D}$ is called *irreducible*, if for every $C, D \in \mathcal{C}$ there exists a finite path $C_0 \to C_1 \to \cdots \to C_n$ in $\mathcal{C}$ with $C_0 = C$ and $C_n = D$. We call $\mathcal{C} \subseteq \mathcal{D}$ *closed* if $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $C \to D$ imply that $D \in \mathcal{C}$. In the proofs we need the following result of Franz Hofbauer which is also true for general piecewise monotonic maps.

Lemma 1. *Assume that* $f : [0,1] \to [0,2]$ *is continuous and strictly increasing, and let* $(\mathcal{D}, \to)$ *be the Markov diagram of* T_f*. Moreover, suppose that there is an irreducible and closed* $\mathcal{C} \subseteq \mathcal{D}$ *with* $\bigcup_{C \in \mathcal{C}} \overline{C} = [0,1]$*. Then* T_f *is topologically transitive.*

Proof. This result follows from (i) of Theorem 11 and Theorem 1 in [4]. ☐

For monotonic mod one transformations the Markov diagram has a special structure. More details of the Markov diagram of a monotonic mod one transformation can be found in [3] and [8]. However we will not need details of this special structure.

2. Topological transitivity

It is useful to modify the orbits of 0 and 1 in the following way for the map T_f defined in (1). For $n \in \mathbb{N}$ set ${T_f}^n 0 := \lim_{x \to 0^+} {T_f}^n x$ and ${T_f}^n 1 := \lim_{x \to 1^-} {T_f}^n x$.

If C is an interval denote by $|C|$ the length of C.

Lemma 2. *Assume that* $f : [0,1] \to [0,2]$ *is continuous, strictly increasing, piecewise differentiable and satisfies* $\inf f' > 1$*. Moreover, let* $n \in \mathbb{N}$*,* $n \geq 2$*, and suppose that* $T_f 0 > {T_f}^2 0 > \cdots > {T_f}^{n-2} 0 \geq c$*. Let* $C \in \mathcal{D}$ *be so that* c *is an endpoint of* C*. If* $C \subseteq (c,1)$ *set* $C_0 := C$*,* $C_1 = T_f C \cap (0,c)$ *and* $C_j := T_f C_{j-1} \cap (c,1)$ *for* $j = 2,3,\ldots,n$*, and if* $C \subseteq (0,c)$ *set* $C_0 := C$*,* $C_j := T_f C_{j-1} \cap (c,1)$ *for* $j = 1,2,\ldots,n-1$ *and* $C_n = T_f C_{n-1} \cap (0,c)$*. Suppose that* $C_0 \subseteq C_n$ *and* $|C_n| > |C_0|$*. Then there exists a path* $C \to C_1 \to \cdots \to C_k$ *with* $C_k \in \{(0,c), (c, {T_f}^{n-2} 1)\}$*.*

Proof. Define $\beta := \inf f'$ and $\delta := |C_n| - |C_0|$. Assume that $C \subseteq (c,1)$. Then the left endpoint of C_j is ${T_f}^{j-1}0$ for $j = 1, 2, \dots, n-1$. For $j = 1, 2, \dots, n$ let $Z_j \in \mathcal{Z}$ be so that $C_j \subseteq Z_j$. Now define $C_{tn+j} = T_f C_{tn+j-1} \cap Z_j$ for $t \in \mathbb{N}$ and $j = 1, 2, \dots, n$. Next we prove by induction that $C_{(t-1)n+j} \subseteq C_{tn+j}$. By our assumption $C_0 \subseteq C_n$ and hence $C_1 = T_f C_0 \cap Z_1 \subseteq T_f C_n \cap Z_1 = C_{n+1}$. Now let $t > 1$ or $j > 1$. Then $C_{(t-1)n+j-1} \subseteq C_{tn+j-1}$ and therefore $C_{(t-1)n+j} = T_f C_{(t-1)n+j-1} \cap Z_j \subseteq T_f C_{tn+j-1} \cap Z_j = C_{tn+j}$.

Now we claim that for every $t \in \mathbb{N}$ there is an $p \leq tn$ such that C_p has a successor in $\{(0,c), (c, {T_f}^{n-2}1)\}$ or $|C_{tn}| \geq |C_0| + \beta^{n(t-1)}\delta$ and $\sup C_{tn} \geq \sup C_{(t-1)n} + \beta^{(t-1)n}\delta$. This is obvious in the case $t = 1$ by our assumptions. Let $t > 1$. If C_p has a successor in $\{(0,c), (c, {T_f}^{n-2}1)\}$ for some $p \leq (t-1)n$ we are done. Otherwise $C_{(t-1)n} \subseteq (c,1)$. If $C_{(t-1)n} \to (0,c)$ we are done. Assume that $C_{(t-1)n+1}$ is the only successor of $C_{(t-1)n}$. Then $C_{(t-1)n+j}$ is the only successor of $C_{(t-1)n+j-1}$ for $j = 2, 3, \dots, n-1$ and $\sup C_{(t-1)n+j} - \sup C_{(t-2)n+j} \geq \beta^j(\sup C_{(t-1)n} - \sup C_{(t-2)n}) \geq \beta^{(t-2)n+j}\delta$. Hence $|C_{tn}| = \sup C_{tn} - c \geq \beta^{(t-1)n}\delta + |C_0|$.

Since $\beta > 1$, $|C_0| + \beta^{(t-1)n}\delta$ tends to infinity, if $t \to \infty$. As $|C_{tn}| \leq 1$ for all n this implies that there is a k with $C_k \in \{(0,c), (c, {T_f}^{n-2}1)\}$. A similar reasoning works in the case $C \subseteq (0,c)$. □

Lemma 3. *Assume that* $f : [0,1] \to [0,2]$ *is continuous, strictly increasing, piecewise differentiable and satisfies* $\inf f' > 1$. *Then there exists an* $r \in \mathbb{N}$ *with* ${T_f}^r 0 < c$. *Let* $r(f)$ *be the smallest* $r \in \mathbb{N}$ *with* ${T_f}^r 0 < c$. *Then* $T_f 0 > {T_f}^2 0 > \dots > {T_f}^{r(f)-1} 0 \geq c$ *and* ${T_f}^j 0 < {T_f}^j 1$ *for* $j = 1, 2, \dots, r(f)$.

Proof. Set $\beta := \inf f'$. As $f(1) - f(0) \geq \beta$ we get $T_f 1 = f(1) - 1 \geq f(0) + \beta - 1 = T_f 0 + (\beta - 1) > T_f 0$. If $T_f 0 < c$ we have $r(f) = 1$ and we are done. Assume that ${T_f}^j 0 \geq c$ for $j = 1, 2, \dots, s$. Note that for $c \leq x < 1$ we get $1 - T_f x \geq T_f 1 - T_f x = f(1) - f(x) \geq \beta(1 - x) > 1 - x$ and therefore $T_f x < x$. Hence $T_f 0 > {T_f}^2 0 > \dots > {T_f}^{s+1} 0$. Moreover, using $T_f 0 < T_f 1$ and induction we get ${T_f}^j 0 < {T_f}^j 1$ for $j = 1, 2, \dots, s+1$. Also using induction we get $1 - {T_f}^j 0 \geq \beta^{j-1}(1 - T_f 0)$, and therefore $0 \leq {T_f}^j 0 \leq 1 - \beta^{j-1}(1 - T_f 0)$ for $j = 1, 2, \dots, s+1$. Since $1 - T_f 0 > 0$ and β^{j-1} tends to infinity for $j \to \infty$ there must be an r with ${T_f}^r 0 < c$. This completes the proof. □

Remark. If $\inf f' \geq \sqrt[3]{2}$ then $r(f) \leq 6$. To see this set $\beta := \inf f'$ and $r := r(f)$. As shown above $|(T_f 0, 1)| \geq T_f 1 - T_f 0 \geq \beta - 1$. Moreover $1 \geq |({T_f}^r 0, {T_f}^{r-1} 1)| \geq \beta^{r-1}(\beta - 1) = \beta^r - \beta^{r-1}$. Observe that $x \mapsto x^r - x^{r-1} - 1$ is strictly increasing for $x > 1$. Since $\sqrt[3]{2} > \frac{5}{4}$ this implies that $\beta^7 - \beta^6 - 1 \geq (\sqrt[3]{2})^7 - (\sqrt[3]{2})^6 - 1 = 4\sqrt[3]{2} - 5 > 0$, and therefore $1 \geq \beta^7 - \beta^6$ can not be satisfied.

Now we show the following result.

Lemma 4. *Suppose that* $f(x) := \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$. *Then* T_f *is topologically transitive.*

Proof. Set $\alpha := \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$. Then $T_f 0 = \alpha$, $T_f 1 = \sqrt[3]{2} + \alpha - 1 = \frac{1}{\sqrt[3]{2}}$ and $f(c) = 1$ for $c := \frac{1-\alpha}{\sqrt[3]{2}} = 1 - \frac{1}{\sqrt[3]{4}}$. Obviously $T_f 1 > \alpha > c$. Therefore follows ${T_f}^2 1 = \alpha = T_f 0$, ${T_f}^2 0 = 1 - \frac{1}{\sqrt[3]{2}} < c$ and ${T_f}^3 0 = \frac{1}{\sqrt[3]{2}} = T_f 1$. This implies that $T_f({T_f}^2 0, c) = (T_f 1, 1)$, $T_f(c, T_f 0) = (0, {T_f}^2 0)$, $T_f(T_f 1, 1) = T_f(0, {T_f}^2 0) = (T_f 0, T_f 1)$ and $T_f(T_f 0, T_f 1) = ({T_f}^2 0, T_f 0)$. Hence

$$\mathcal{C} := \{(0, {T_f}^2 0), ({T_f}^2 0, c), (c, T_f 0), (T_f 0, T_f 1), (T_f 1, 1)\}$$

is an irreducible and closed subset of the Markov diagram of T_f and $\bigcup_{C\in\mathcal{C}} \overline{C} = [0, 1]$. Now Lemma 1 implies that T_f is topologically transitive. □

Lemma 5. *Assume that* $f : [0, 1] \to [0, 2]$ *is continuous, strictly increasing and piecewise differentiable. Further assume that* $\beta > 0$, $\inf f' \geq \beta$ *and* $f(0) \geq \frac{1}{\beta+1}$. *Set* $\alpha := f(0)$. *Then* $\alpha \geq \frac{1}{\beta+1} \geq c$.

Proof. Because of $\alpha \geq \frac{1}{\beta+1}$ we get that $\frac{1-\alpha}{\beta} \leq \frac{1}{\beta+1}$. Since $1 = f(c) \geq \alpha + \beta c$ we obtain $c \leq \frac{1-\alpha}{\beta} \leq \frac{1}{\beta+1} \leq \alpha$. □

Remark. In particular Lemma 5 states that $r(f) \geq 2$ under the assumptions of Lemma 5.

Lemma 6. *Suppose that* $f : [0, 1] \to [0, 2]$ *is continuous, strictly increasing and piecewise differentiable. Further assume that* $\beta \geq \sqrt[3]{2}$, $\inf f' \geq \beta$ *and* $f(0) \geq \frac{1}{\beta+1}$. *Moreover assume that* $d_1 < c < d_2$, $C_0 := (d_1, c)$, $D_0 := (c, d_2)$, $|C_0| = |D_0|$, $T_f d_2 \leq c$, ${T_f}^2 d_1 \geq c$ *and* ${T_f}^3(d_1, c) = {T_f}^3(c, d_2) = (d_1, d_2)$. *Then* $f(x) = \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ *for all* $x \in [0, 1]$.

Proof. From Lemma 5 it follows that $T_f d_1 \geq T_f 0 \geq c$, hence $C_1 := T_f C_0 = (T_f d_1, 1) \subseteq (c, 1)$. As ${T_f}^2 d_1 \geq c$ we get that $C_2 := T_f C_1 = ({T_f}^2 d_1, T_f 1) \subseteq (c, 1)$. Furthermore $T_f C_2 = (d_1, d_2)$ by our assumptions. Since $T_f d_2 \leq c$ we obtain that $D_1 := T_f D_0 = (0, T_f d_2) \subseteq (0, c)$ and therefore $D_2 := T_f D_1 = (T_f 0, {T_f}^2 d_2) \subseteq (c, 1)$. Again our assumptions give $T_f D_2 = (d_1, d_2)$. In particular we have $T_f d_2 \in [0, c]$ and $T_f 0, T_f 1, T_f d_1, {T_f}^2 d_1, {T_f}^2 d_2 \in [c, 1]$. As $\inf f' \geq \beta \geq \sqrt[3]{2}$ we get $|T_f C_2| \geq 2|C_0|$ and $|T_f D_2| \geq 2|D_0|$. Now $|C_0| = |D_0|$ and ${T_f}^3 C_0 = {T_f}^3 D_0 = (d_1, d_2)$ imply that there are $\alpha_0, \alpha_1, \alpha_2, \widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\alpha}_2$ with $f(x) = \sqrt[3]{2}x + \alpha_j$ for $x \in C_j$ and $f(x) = \sqrt[3]{2}x + \widehat{\alpha}_j$ for $x \in D_j$ if $j \in \{0, 1, 2\}$.

Assume that $T_f d_2 < d_1$. Because of ${T_f}^3 d_1 = d_1$ and ${T_f}^4 d_2 = T_f {T_f}^3 d_2 = T_f d_2$ we get

$$|d_1 - T_f d_2| = |{T_f}^3 d_1 - {T_f}^4 d_2| \geq 2|d_1 - T_f d_2|$$

which is a contradiction. Hence $T_f d_2 \geq d_1$ and therefore $T_f{}^2 d_2 \geq T_f d_1$ and $d_2 = T_f{}^3 d_2 \geq T_f{}^2 d_1$. This implies that $\overline{C_0 \cup C_1 \cup C_2 \cup D_0 \cup D_1 \cup D_2} = [0,1]$. Therefore $f(x) = \sqrt[3]{2}x + \alpha$ for some α. Now the conditions on d_1 and d_2 imply that $\alpha = \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ completing the proof. □

Lemma 7. *Suppose that $f : [0,1] \to [0,2]$ is continuous, strictly increasing and piecewise differentiable. Further assume that $\beta \geq \sqrt[3]{2}$, $\inf f' \geq \beta$ and $f(0) \geq \frac{1}{\beta+1}$. Moreover assume that $f(x) \neq \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ for some x. Let $C \in \mathcal{D}$ with c being an endpoint of C. Then there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_k$ such that one endpoint of C_k is c and either $C_k \in \{(0,c),(c,T_f{}^{r(f)-1}1)\}$ or $|C_0| \leq \beta|C_k|$.*

Proof. Set $r := r(f)$. We have either $C_0 = (d,c)$ or $C_0 = (c,d)$.

In the first case $T_f d \geq T_f 0 \geq c$ by Lemma 5. Setting $C_j := (T_f{}^j d, T_f{}^{j-1}1)$ for $j = 1,2,\dots,r-1$ we get that C_j is the only successor of C_{j-1} for $j = 1,2,\dots,r-1$. Then $T_f C_{r-1} = (T_f{}^r d, T_f{}^{r-1}1)$ and $|T_f C_{r-1}| \geq \beta^r |C_0|$. If $T_f{}^r d \leq c$ then $C_{r-1} \to (c, T_f{}^{r-1}1)$ and we are done. Otherwise $C_r := T_f C_{r-1}$ is the only successor of C_{r-1}. By Lemma 1 of [10] there is a minimal $s \geq r$ such that $C_j = T_f C_{j-1}$ is the only successor of C_{j-1} for $j = 1,2,\dots,s$ and C_s has two different successors. As $|T_f C_s| \geq \beta^{s+1}|C_0|$ we get that C_s has a successor C_{s+1} with $|C_{s+1}| \geq \frac{\beta^{s+1}}{2}|C_0|$ and c is an endpoint of C_{s+1}. If $s \geq 3$ then $\frac{\beta^{s+1}}{2} \geq \beta$ since $\beta \geq \sqrt[3]{2}$ and we are done.

Otherwise $s = r = 2$ and C_2 has the successors $(T_f{}^3 d, c)$ and $(c, T_f{}^2 1)$ and $|T_f C_2| \geq \beta^3 |C_0| \geq 2|C_0|$. In the case $|(T_f{}^3 d, c)| > |C_0|$ Lemma 2 with $n = 3$ implies the desired result. Otherwise set $C_3 := (c, T_f{}^2 1)$. Note that $|C_3| \geq |C_0|$. If $T_f{}^3 1 \geq c$ then $C_3 \to (0,c)$ and we are done. Now consider the case $T_f{}^3 1 < c$. Then $C_4 := (0, T_f{}^3 1)$ is the only successor of C_3, and $C_5 := (T_f 0, T_f{}^4 1)$ is the only successor of C_4. Moreover $|T_f C_5| \geq \beta^3 |C_3| \geq 2|C_3|$. If C_5 has only one successor then by Lemma 1 of [10] there is a minimal $s \geq 3$ such that $C_{3+j} = T_f C_{3+j-1}$ is the only successor of C_{3+j-1} and C_{3+s} has two different successors. Because of $|T_f C_{3+s}| \geq \beta^{s+1}|C_3|$ we get that C_{3+s} has a successor C_{3+s+1} with $|C_{3+s+1}| \geq \frac{\beta^{s+1}}{2}|C_3| \geq \beta|C_3| \geq \beta|C_0|$ and c is an endpoint of C_{3+s+1}. We are done in this case. Otherwise C_5 has the two successors $(T_f{}^2 0, c)$ and $(c, T_f{}^5 1)$. If $|(c, T_f{}^5 1)| > |C_3|$ the desired result is implied by Lemma 2 with $n = 3$. Otherwise set $C_6 := (T_f{}^2 0, c)$. Observe that $|C_6| \geq |C_3| \geq |C_0|$, which implies $C_0 \subseteq C_6$. Hence $C_7 := (T_f{}^3 0, 1) \supseteq C_1$ is the only successor of C_6 and $T_f C_7 = (T_f{}^4 0, T_f 1) \supseteq C_2$. In the case $T_f{}^4 0 \leq c$ we get $C_7 \to (c, T_f 1)$ and we are done as $r = 2$. Otherwise $C_8 := T_f C_7$ is the only successor of C_7 and $C_3 \subseteq T_f C_8 = (T_f{}^5 0, T_f{}^2 1)$. Hence C_8 has the successors $C_9 := (T_f{}^5 0, c)$ and $(c, T_f{}^2 1) = C_3$. As $|T_f C_8| \geq \beta^3 |C_6| \geq 2|C_6|$ and $|C_6| \geq |C_3|$ we get that $|C_9| = |T_f C_8| - |C_3| \geq 2|C_6| - |C_3| \geq |C_6|$. If $|C_9| = |C_6|$

we get $C_9 = C_6$ and we have the situation described in Lemma 6. Then Lemma 6 implies that $f(x) = \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ for all $x \in [0,1]$ contradicting our assumption. Hence $|C_9| > |C_6|$. Applying Lemma 2 with $n = 3$ we obtain the desired result.

Now we consider the case $C_0 = (c, d)$. If $T_f d \geq c$ we have $C_0 \to (0, c)$ and we are done. It remains to consider the case $T_f d < c$. Then $C_1 := (0, T_f d)$ is the only successor of C_0. Setting $C_j := ({T_f}^{j-1}0, {T_f}^{j}d)$ for $j = 2, 3, \ldots, r$ we get that C_j is the only successor of C_{j-1} for $j = 1, 2, \ldots, r$. Moreover $T_f C_r = ({T_f}^{r}0, {T_f}^{r+1}d)$ and $|T_f C_r| \geq \beta^{r+1}|C_0|$. By Lemma 1 of [10] there is a minimal $s \geq r$ such that $C_j = T_f C_{j-1}$ is the only successor of C_{j-1} for $j = 1, 2, \ldots, s$ and C_s has two different successors. As $|T_f C_s| \geq \beta^{s+1}|C_0|$ we get that C_s has a successor C_{s+1} with $|C_{s+1}| \geq \frac{\beta^{s+1}}{2}|C_0|$ and c is an endpoint of C_{s+1}. If $s \geq 3$ then $\frac{\beta^{s+1}}{2} \geq \beta$ since $\beta \geq \sqrt[3]{2}$ and we are done.

It remains to assume that $s = r = 2$. Then C_2 has the successors $({T_f}^{2}0, c)$ and $(c, {T_f}^{3}d)$, and $|T_f C_2| \geq \beta^3|C_0| \geq 2|C_0|$. If $|(c, {T_f}^{3}d)| > |C_0|$ then Lemma 2 with $n = 3$ implies the desired result. Otherwise set $C_3 := ({T_f}^{2}0, c)$. Hence $C_3 = (d, c)$ for $d = {T_f}^{2}0$ and $|C_3| \geq |C_0|$. In this case we have shown above that there exists a finite path $C_3 \to C_4 \to \cdots \to C_k$ such that one endpoint of C_k is c and either $C_k \in \{(0, c), (c, {T_f}^{r(f)-1}1)\}$ or $|C_0| \leq |C_3| \leq \beta|C_k|$. This completes the proof. ☐

Lemma 8. *Suppose that $f : [0,1] \to [0,2]$ is continuous, strictly increasing and piecewise differentiable. Further assume that $\beta \geq \sqrt[3]{2}$, $\inf f' \geq \beta$ and $f(0) \geq \frac{1}{\beta+1}$. Moreover assume that $f(x) \neq \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ for some x. Let $C \in \mathcal{D}$.*

(1) *If $r(f) \geq 3$, then there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$ with $C_n = (0, c)$. Moreover ${T_f}^{r(f)}1 > c$.*

(2) *If $r(f) = 2$, then there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$ with $C_n \in \{(0, c), (c, T_f 1)\}$.*

Proof. Set $r := r(f)$. By Lemma 1 of [10] there exists a path $C_0 \to C_1 \to \cdots \to C_l$ in $\mathcal{D}$ with c is an endpoint of C_l. Now we prove by induction that for every $t \in \mathbb{N}$ there is a path $C_l \to C_{l+1} \to \cdots \to C_{n_t}$ such that c is an endpoint of C_{n_t} and either $C_{n_t} \in \{(0, c), (c, {T_f}^{r-1}1)\}$ or $|C_{n_t}| \geq \beta^t|C_l|$. For $t = 1$ this follows immediately from Lemma 7. Now let $t > 1$. If $C_{n_{t-1}} \in \{(0, c), (c, {T_f}^{r-1}1)\}$ we are done. Otherwise by Lemma 7 there exists a path $C_{n_{t-1}} \to C_{n_{t-1}+1} \to \cdots \to C_{n_t}$ such that c is an endpoint of C_{n_t} and either $C_{n_t} \in \{(0, c), (c, {T_f}^{r-1}1)\}$ or $|C_{n_t}| \geq \beta|C_{n_{t-1}}| \geq \beta^t|C_l|$.

As β^t tends to infinity for $t \to \infty$ there exists a path $C_0 := C \to C_1 \to \cdots \to C_n$ with $C_n \in \{(0, c), (c, {T_f}^{r-1}1)\}$. In the case $r = 2$ we obtain (2).

Finally, assume that $r > 2$. In order to show (1) it suffices to show that $(c, {T_f}^{r-1}1) \to (0, c)$. Setting $D_0 := (0, c)$, $D_j := ({T_f}^j 0, {T_f}^{j-1}1)$ for $j = 1, 2, \dots, r-1$ we have that D_j is the only successor of D_{j-1} for $j = 1, 2, \dots, r-1$. Moreover $T_f D_{r-1} = ({T_f}^r 0, {T_f}^{r-1}1)$ satisfies $|T_f D_{r-1}| \geq \beta^r |D_0| \geq 2c$. Since ${T_f}^r 0 \geq 0$ this implies that $|(c, {T_f}^{r-1}1)| \geq |T_f D_{r-1}| - |(0, c)| \geq c$. Hence $|(0, {T_f}^r 1)| = |T_f(c, {T_f}^{r-1}1)| \geq \beta c$. Therefore ${T_f}^r 1 > c$ and this implies that $(c, {T_f}^{r-1}1) \to (0, c)$. □

Theorem 1. *Let $f : [0, 1] \to [0, 2]$ be a continuous, strictly increasing and piecewise differentiable function. Moreover assume that $\beta \geq \sqrt[3]{2}$, $\inf f' \geq \beta$ and $f(0) \geq \frac{1}{\beta+1}$. Then T_f is topologically transitive.*

Proof. If $f(x) := \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ then topological transitivity follows from Lemma 4. Otherwise denote by $(\mathcal{D}, \to)$ the Markov diagram of T_f. Let $C \in \mathcal{D}$ and set $r := r(f)$.

Assume at first that ${T_f}^2 1 < c$. Using Lemma 3 and Lemma 5 this implies ${T_f}^2 0 < {T_f}^2 1 < c$, $c \leq {T_f}^3 0 < {T_f}^3 1$ and $r = 2$. Define $\mathcal{C}$ as the set of all $D \in \mathcal{D}$ such that there is a path $D_0 := (c, T_f 1) \to D_1 \to \cdots \to D_n = D$ which is obviously closed. Set $C_0 := (c, T_f 1)$, $C_1 := (0, {T_f}^2 1)$, $C_2 := (T_f 0, {T_f}^3 1)$, $C_3 := ({T_f}^2 0, c)$ and $C_4 := ({T_f}^3 0, 1)$. Because of ${T_f}^2 0 < c$ we have $(0, c) \to (T_f 0, 1) \to C_0$. By (2) of Lemma 8 for any $C \in \mathcal{D}$ there exists a path $D_0 := C \to D_1 \to \cdots \to D_p$ with $D_p \in \{(0, c), (c, T_f 1)\}$. Therefore the set $\mathcal{C}$ is closed and irreducible. Since $1 \geq |T_f(c, 1)| \geq \beta|(c, 1)| = \beta(1 - c)$ we get $1 - c \leq \frac{1}{\beta}$. Moreover $|C_0| \geq \beta - 1$ as $T_f 1 \geq T_f 0 + \beta - 1 \geq c + \beta - 1$ by Lemma 5. By our assumptions C_1 is the only successor of C_0 and C_2 is the only successor of C_1. Therefore $|C_2| \geq \beta^2(\beta - 1)$ and $|T_f C_2| = |({T_f}^2 0, {T_f}^4 1)| \geq \beta^3(\beta - 1) > \frac{1}{\beta+1}$ as $x^5 - x^3 - 1 > 0$ for $x \geq \sqrt[3]{2}$. Hence ${T_f}^4 1 > \frac{1}{\beta+1} \geq c$ by Lemma 5 and C_2 has the two different successors C_3 and $(c, {T_f}^4 1)$. Moreover $C_3 \to C_4$. Since $C_0, C_1, C_2, C_3, C_4 \in \mathcal{C}$ and $\bigcup_{k=0}^4 \overline{C_k} = [0, 1]$ the topological transitivity is implied by Lemma 1.

Now assume that ${T_f}^2 0 > c$. Then $r \geq 3$ and $c \leq {T_f}^k 0 < {T_f}^k 1$ for $k = 1, 2, \dots, r-1$ by Lemma 3. Set $C_0 := (0, c)$, $C_k := ({T_f}^k 0, {T_f}^{k-1} 1)$ for $k = 1, 2, \dots, r-1$, and $C_r := (c, {T_f}^{r-1} 1)$. Define $\mathcal{C}$ as the set of all $D \in \mathcal{D}$ such that there is a path $D_0 := (0, c) \to D_1 \to \cdots \to D_n = D$. Obviously $\mathcal{C}$ is closed. By (1) of Lemma 8 for any $C \in \mathcal{D}$ there exists a path $D_0 := C \to D_1 \to \cdots \to D_p$ with $D_p = (0, c)$. Hence the set $\mathcal{C}$ is closed and irreducible. Observe that $C_0 \to C_1 \to \cdots \to C_{r-1} \to C_r$. Since $C_0, C_1, \dots, C_r \in \mathcal{C}$ and $\bigcup_{k=0}^r \overline{C_k} = [0, 1]$ the map T_f is topologically transitive by Lemma 1.

Finally it remains to consider the case ${T_f}^2 0 \leq c$ and ${T_f}^2 1 \geq c$. Set $C_0 := (0, c)$, $C_1 := (T_f 0, 1)$ and $C_2 := (c, T_f 1)$. Define $\mathcal{C}$ as the set of all

$D \in \mathcal{D}$ such that there is a path $D_0 := (0,c) \to D_1 \to \cdots \to D_n = D$. Observe that $C_0 \to C_1 \to C_2 \to C_0$. By Lemma 8 for any $C \in \mathcal{D}$ there exists a path $D_0 := C \to D_1 \to \cdots \to D_p$ with $D_p \in \{(0,c),(c,T_f 1)\}$. This implies that $\mathcal{C}$ is closed and irreducible. Moreover $C_0, C_1, C_2 \in \mathcal{C}$ and $\bigcup_{k=0}^{2} \overline{C_k} = [0,1]$. Therefore T_f is topologically transitive by Lemma 1. □

Corollary 1.1. *Let $f : [0,1] \to [0,2]$ be a continuous, strictly increasing and piecewise differentiable function. Moreover assume that* $\inf f' \geq \sqrt[3]{2}$ *and* $f(0) \geq \frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3}$. *Then T_f is topologically transitive.*

Proof. Observe that $\frac{1}{\sqrt[3]{2}+1} = \frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3}$. Setting $\beta = \sqrt[3]{2}$ in Theorem 1 gives the desired result. □

Remark. Using the conjugation $h(x) := 1-x$ we see that T_f is conjugated to $T_{\widehat{f}}$, where $\widehat{f}(x) := 2 - f(1-x)$. Obviously $f(0) \geq \alpha$ is equivalent to $\widehat{f}(1) \leq 2-\alpha$. Hence Theorem 1 implies that for every continuous, strictly increasing and piecewise differentiable function $f : [0,1] \to [0,2]$ with $\inf f' \geq \beta$ and $f(1) \leq 2 - \frac{1}{\beta+1}$ the map T_f is topologically transitive.

3. Counterexamples

Let $1 < \beta < \sqrt[3]{2}$. Define

$$f(x) := \beta x + \frac{\beta^2+1}{\beta^3+\beta^2+\beta}. \tag{2}$$

Setting $c := \frac{\beta^3+\beta-1}{\beta^4+\beta^3+\beta^2}$ we obtain $f(c) = 1$. Such a map T_f is shown in Figure 1. Now define $A := [0, {T_f}^3 1] \cup [{T_f}^2 0, {T_f}^2 1] \cup [T_f 0, T_f 1] \cup [{T_f}^3 0, 1]$. Note that $T_f 0 = \frac{\beta^2+1}{\beta^3+\beta^2+\beta} > c$, ${T_f}^2 0 = \frac{1}{\beta^3+\beta^2+\beta} < c$, ${T_f}^3 0 = \frac{1}{\beta} > c$, ${T_f}^4 0 = \frac{\beta^2+1}{\beta^3+\beta^2+\beta} = T_f 0$, $T_f 1 = \frac{\beta^4+\beta^2-\beta+1}{\beta^3+\beta^2+\beta} > c$, ${T_f}^2 1 = \frac{\beta^5-\beta^2+1}{\beta^3+\beta^2+\beta} > c$, ${T_f}^3 1 = \frac{\beta^6-2\beta^3+1}{\beta^3+\beta^2+\beta} < c$ and ${T_f}^4 1 = \frac{\beta^7-2\beta^4+\beta^2+\beta+1}{\beta^3+\beta^2+\beta} \leq T_f 1$. Moreover, ${T_f}^3 1 < {T_f}^2 0$ which implies that $[0,1] \setminus A \neq \emptyset$. Since ${T_f}^4 1 \leq T_f 1$ and ${T_f}^4 0 = T_f 0$ we get that $T_f A \subseteq A$. Therefore T_f is not topologically transitive.

Observe that $f(0) = T_f 0 = \frac{\beta^2+1}{\beta^3+\beta^2+\beta}$. The function $g : \{x \in \mathbb{R} : x > 0\} \to \mathbb{R}$, $g(x) := \frac{x^2+1}{x^3+x^2+x}$ is strictly decreasing as $g'(x) = \frac{-x^4-2x^2-2x-1}{(x^3+x^2+x)^2} < 0$. Hence $f(0) = \frac{\beta^2+1}{\beta^3+\beta^2+\beta} = g(\beta) > g(\sqrt[3]{2}) = \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ because $\beta < \sqrt[3]{2}$. Note that $\frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2} > \frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3}$ ($\frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2} \approx 0.533779$, $\frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3} \approx 0.442493$). Moreover $f(0) = \frac{\beta^2+1}{\beta^3+\beta^2+\beta} > \frac{1}{\beta+1}$. This shows the following result.

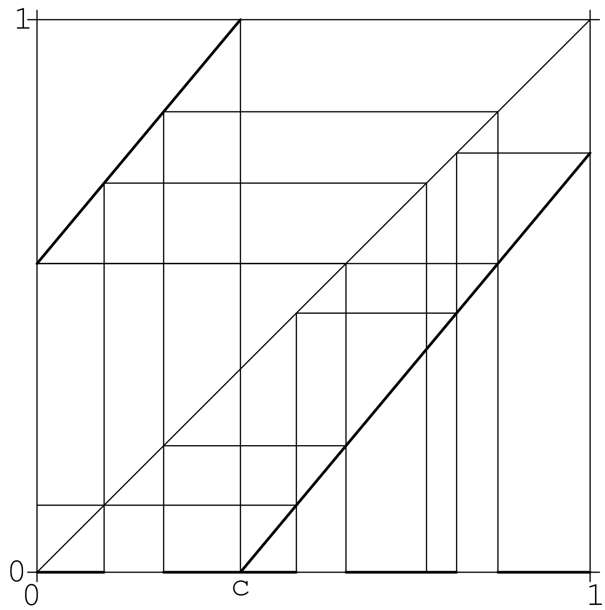

Figure 1: This is T_f for f from (2).

Theorem 2. *For $1 < \beta < \sqrt[3]{2}$ there exists a continuous strictly increasing piecewise differentiable function $f : [0,1] \to [0,2]$ with $f(0) > \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$, $f(0) > \frac{1}{\beta+1}$ and $\inf f' \geq \beta$ such that T_f is not topologically transitive.*

Remark. The proof above shows that the function f in Theorem 2 can be chosen as $f(x) = \beta x + \alpha$ for suitable α and β.

Now let β_0 be the largest zero of the polynomial $x^3 - x - 1$. Assume that $\sqrt[3]{2} \leq \beta \leq \beta_0$ and $\alpha < \frac{1}{\beta+1}$. We will define a continuous strictly increasing function $f : [0,1] \to [0,2]$ in the following way. Choose $\delta > 0$ such that

$$\delta < \min\left\{\frac{1}{\beta+1} - \alpha, \frac{1}{(\beta+1)(\beta^2+1)}, \frac{1}{3+\beta^3+\beta^2}, \frac{\beta^2-1}{\beta^4+\beta^3+\beta^2-1}\right\}.$$

We define f as the join the dots map with the dots

$$(0, \tfrac{1}{1+\beta} - \delta), (\beta^2\delta, \tfrac{1}{1+\beta} - \delta + \beta^3\delta), (c-\delta, 1-\beta\delta), (c+\beta\delta, 1+\beta^2\delta),$$
$$(\tfrac{1}{\beta}(1 - \tfrac{3\delta}{1+\beta}), 1 + \tfrac{1-2\delta}{1+\beta}), (\tfrac{1}{\beta}(1 + \tfrac{\beta^4-1}{1+\beta}\delta), 1 + \tfrac{1+\beta^4\delta}{1+\beta}),$$
$$(1-\beta\delta, \tfrac{1}{\beta}(\beta + 1 - \tfrac{3\delta}{1+\beta})) \text{ and } (1, \tfrac{1}{\beta}(\beta+1+\tfrac{(\beta^4-1)\delta}{1+\beta})).$$

Such a map T_f is shown in Figure 2. Note that $\inf f' \geq \beta$. Furthermore $A := [0, \beta^2\delta] \cup [c-\delta, c+\beta\delta] \cup [\frac{1}{\beta}(1 - \frac{3\delta}{1+\beta}), \frac{1}{\beta}(1 + \frac{\beta^4-1}{1+\beta}\delta)] \cup [1-\beta\delta, 1]$ is T_f-invariant, and $\beta^2\delta < c - \delta$. Moreover note that $f(0) = \frac{1}{1+\beta} - \delta \geq \alpha$ by

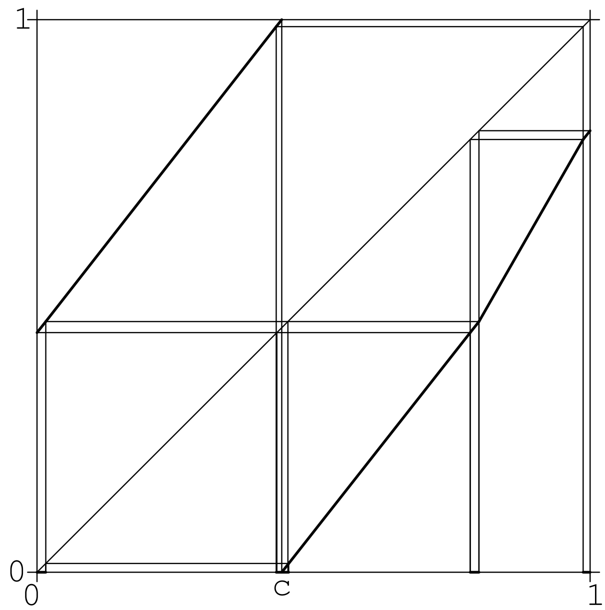

Figure 2: This is T_f for f from Theorem 3.

the choice of δ. Therefore T_f is not topologically transitive. Hence we have shown the following result.

Theorem 3. *For $\sqrt[3]{2} \le \beta \le \beta_0$ and $\alpha < \frac{1}{\beta+1}$ there exists a continuous strictly increasing piecewise differentiable function $f : [0,1] \to [0,2]$ with $f(0) \ge \alpha$ and $\inf f' \ge \beta$ such that T_f is not topologically transitive.*

Corollary 3.1. *For $\alpha < \frac{1+\sqrt[3]{4}-\sqrt[3]{2}}{3}$ there exists a continuous strictly increasing piecewise differentiable function $f : [0,1] \to [0,2]$ with $f(0) \ge \alpha$ and $\inf f' \ge \sqrt[3]{2}$ such that T_f is not topologically transitive.*

4. The special case $\beta x + \alpha \pmod 1$

Finally we investigate the special case $\beta x + \alpha \pmod 1$. In this case the situation is slightly different. For any $\beta \ge \sqrt[3]{2}$ we will determine the set of all α such that $\beta x + \alpha \pmod 1$ is topologically transitive. Note that for every $\beta > 0$ the map $f(x) := \beta x + \alpha$ satisfies $f([0,1]) \subseteq [0,2]$ if and only if $0 \le \alpha \le 2 - \beta$. Furthermore observe that $c = \frac{1-\alpha}{\beta}$ in this case.

Lemma 9. *Assume that $1 < \beta < \sqrt{2}$ and $\frac{1}{\beta^2+\beta} \le \alpha < \frac{1}{\beta+1}$, and set $f(x) := \beta x + \alpha$. Then $T_f 0 < c < T_f 1 < T_f{}^2 0$ and $T_f{}^3 0 \ge T_f 0$. Moreover, $T_f{}^2 1 < c$*

if and only if $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$. *If* $\alpha \le 2 - \beta - \frac{1}{\beta^2+\beta}$ *then* ${T_f}^2 1 < c$ *and* ${T_f}^3 1 \le T_f 1$.

Proof. Because of $\alpha < \frac{1}{\beta+1}$ we get $T_f 0 = \alpha < \frac{1-\alpha}{\beta} = c$, and therefore ${T_f}^2 0 = (\beta+1)\alpha$. Using $(\beta-1)^2(\beta+1) > 0$ one obtains $\frac{-\beta^2+\beta+1}{\beta+1} < \frac{1}{\beta^2+\beta} \le \alpha$ and hence $T_f 1 = \alpha + \beta - 1 > \frac{1-\alpha}{\beta} = c$ and ${T_f}^2 1 = (\beta+1)\alpha + \beta^2 - \beta - 1$. Since $\beta < \sqrt{2}$ we obtain $(\beta-1)(\beta+1) < 1$ which implies $\frac{\beta-1}{\beta} < \frac{1}{\beta^2+\beta} \le \alpha$ and therefore $T_f 1 = \alpha + \beta - 1 < (\beta+1)\alpha = {T_f}^2 0$. In particular ${T_f}^2 0 > c$ which implies ${T_f}^3 0 = (\beta^2+\beta+1)\alpha - 1$. As $\alpha \ge \frac{1}{\beta^2+\beta}$ we get ${T_f}^3 0 = (\beta^2+\beta+1)\alpha - 1 \ge \alpha = T_f 0$.

The property $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ is equivalent to ${T_f}^2 1 = (\beta+1)\alpha + \beta^2 - \beta - 1 < \frac{1-\alpha}{\beta} = c$. In this case one obtains that ${T_f}^3 1 = (\beta^2+\beta+1)\alpha + \beta^3 - \beta^2 - \beta$. Observe that $2 - \beta - \frac{1}{\beta^2+\beta} = \frac{-\beta^3+\beta^2+2\beta-1}{\beta^2+\beta} < \frac{-\beta^3+\beta^2+\beta+1}{\beta^2+\beta+1} = 1 - \frac{\beta^3}{\beta^2+\beta+1}$. Hence ${T_f}^2 1 < c$ if $\alpha \le 2 - \beta - \frac{1}{\beta^2+\beta}$. Moreover in this case we get ${T_f}^3 1 = (\beta^2+\beta+1)\alpha + \beta^3 - \beta^2 - \beta \le \alpha + \beta - 1 = T_f 1$. □

Remark. Observe that $\frac{1}{\beta+1} \le 1 - \frac{\beta^3}{\beta^2+\beta+1}$ for $1 < \beta \le \beta_0$ where β_0 be the largest zero of the polynomial $x^3 - x - 1$ as $\beta(\beta^3 - \beta - 1) \le 0$. Hence $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ is satisfied automatically in this case if $\alpha < \frac{1}{\beta+1}$.

Suppose that $\sqrt[3]{2} \le \beta < \sqrt{2}$ and $\frac{1}{\beta^2+\beta} \le \alpha \le 2 - \beta - \frac{1}{\beta^2+\beta}$, and define $f(x) := \beta x + \alpha$. Such a map is shown in Figure 3. By Lemma 9 the set $A := [0, {T_f}^2 1] \cup [T_f 0, T_f 1] \cup [{T_f}^2 0, 1]$ satisfies $T_f A \subseteq A$ and $[0,1] \setminus A \ne \emptyset$. Hence T_f is not topologically transitive.

Note that $(\beta-1)^2(\beta+1) > 0$ implies $2 - \beta - \frac{1}{\beta^2+\beta} = \frac{-\beta^3+\beta^2+2\beta-1}{\beta^2+\beta} < \frac{1}{\beta+1}$. Moreover, for $\beta < \sqrt{2}$ we obtain $(\beta-1)(\beta^2-2) < 0$ which implies that $\frac{1}{\beta^2+\beta} < \frac{-\beta^3+\beta^2+2\beta-1}{\beta^2+\beta} = 2 - \beta - \frac{1}{\beta^2+\beta} < 1 - \frac{\beta^3}{\beta^2+\beta+1}$.

Lemma 10. *Suppose that* $\sqrt[3]{2} \le \beta < \sqrt{2}$, $2 - \beta - \frac{1}{\beta^2+\beta} < \alpha < \frac{1}{\beta+1}$ *and* $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$, *and set* $f(x) := \beta x + \alpha$. *Then* $(0,c) \to (c,1)$, $(c,1) \to (0,c)$, $(0,c) \to (T_f 0, c)$, $({T_f}^2 0, 1)$ *is the unique successor of* $(T_f 0, c)$, $(c,1) \to (c, T_f 1)$, $(0, {T_f}^2 1)$ *is the unique successor of* $(c, T_f 1)$ *and* ${T_f}^3 1 > T_f 1$. *Furthermore for every* $C \in \mathcal{D}$ *there exists a finite path* $C_0 \to C_1 \to \cdots \to C_k$ *with* $C_0 = C$ *and* $C_k \in \{(0,c), (T_f 0, c), (c, T_f 1)\}$.

Proof. By Lemma 9 we get $(0,c) \to (c,1)$, $(c,1) \to (0,c)$, $(0,c) \to (T_f 0, c)$, $({T_f}^2 0, 1)$ is the unique successor of $(T_f 0, c)$, $(c,1) \to (c, T_f 1)$ and $(0, {T_f}^2 1)$ is the unique successor of $(c, T_f 1)$. Moreover, using that $2 - \beta - \frac{1}{\beta^2+\beta} < \alpha$ we obtain that ${T_f}^3 1 = (\beta^2+\beta+1)\alpha + \beta^3 - \beta^2 - \beta > \alpha + \beta - 1 = T_f 1$.

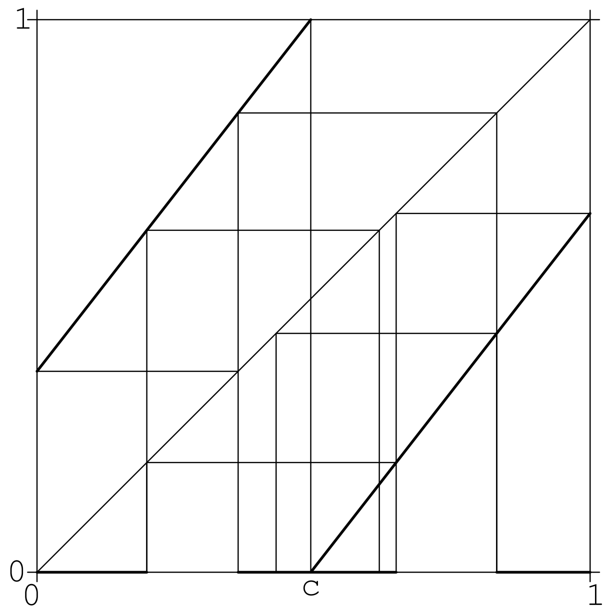

Figure 3: This is $\beta x + \alpha \pmod 1$ with $\sqrt[3]{2} \leq \beta < \sqrt{2}$ and $\frac{1}{\beta^2+\beta} \leq \alpha \leq 2 - \beta - \frac{1}{\beta^2+\beta}$.

Now we claim that for every $C \in \mathcal{D}$ having c as an endpoint there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$ with c is an endpoint of C_n and $C_n \in \{(0,c), (T_f 0, c), (c, T_f 1)\}$ or $|C_n| \geq \beta |C|$. Assume at first that $C = (d, c)$ for some d. If $T_f d \leq c$ we are done as $C \to (c, 1) \to (0, c)$. Otherwise $C_1 := (T_f d, 1) = T_f C$ is the unique successor of $C_0 := C$. In the case ${T_f}^2 d \leq c$ we are done because of $C_1 \to (c, T_f 1)$. If ${T_f}^2 d > c$ then $C_2 := {T_f}^2 C = ({T_f}^2 0, T_f 1) \subseteq (c, T_f 1)$ is the unique successor of C_1 and therefore $C_3 := {T_f}^3 C$ is the unique successor of C_2. By Lemma 1 of [10] there is a minimal $s \geq 3$ such that $C_j := {T_f}^j C$ is the unique successor of C_{j-1} for $j = 1, 2, \ldots, s$ and C_s has two different successors. Each of these successors has c as an endpoint. Because of $T_f C_s = {T_f}^{s+1} C$ we get $|T_f C_s| = \beta^{s+1} |C|$ and therefore C_s has a successor C_{s+1} with $|C_{s+1}| \geq \frac{\beta^{s+1}}{2} |C|$. As $s \geq 3$ and $\beta \geq \sqrt[3]{2}$ we get $\frac{\beta^{s+1}}{2} \geq \beta$ and hence $|C_{s+1}| \geq \beta |C|$.

Similarly for $C = (c, d)$ for some d we have either $C \to (0, c)$ and are done or $C_1 := T_f C = (0, T_f d)$ is the unique successor of $C_0 := C$. If ${T_f}^2 d \geq c$ in the second case then $C_1 \to (T_f 0, c)$ and we are done. It remains to consider the case $T_f d < c$ and ${T_f}^2 d < c$. Then $C_2 := {T_f}^2 C = (T_f 0, {T_f}^2 d) \subseteq (T_f 0, c)$ is the unique successor of C_1 and $C_3 := {T_f}^3 C$ is the unique successor of C_2. By the same argument as above there exists an $s \geq 3$ and a finite path $C_0 := C \to C_1 \to \cdots \to C_{s+1}$ such that c is an endpoint of C_{s+1} and $|C_{s+1}| \geq \beta |C|$.

Next we prove by induction that for every $C \in \mathcal{D}$ having c as an endpoint and for any $t \in \mathbb{N}$ there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_{n_t}$ with c is an endpoint of C_{n_t} and $C_{n_t} \in \{(0,c),(T_f0,c),(c,T_f1)\}$ or $|C_{n_t}| \geq \beta^t|C|$. For $t = 1$ this is exactly the property proved above. Let $t > 1$. If $C_{n_{t-1}} \in \{(0,c),(T_f0,c),(c,T_f1)\}$ set $n_t := n_{t-1}$ and we are done. Otherwise by the property proved above there exists a finite path $C_{n_{t-1}} \to C_{n_{t-1}+1} \to \cdots \to C_{n_t}$ with c is an endpoint of C_{n_t} and $C_{n_t} \in \{(0,c),(T_f0,c),(c,T_f1)\}$ or $|C_{n_t}| \geq \beta|C_{n_{t-1}}| \geq \beta^t|C|$.

Finally let $C \in \mathcal{D}$. By Lemma 1 of [10] there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_l$ such that C_{l-1} has two different successors. Therefore c is an endpoint of C_l. Choose $t \in \mathbb{N}$ with $\beta^t|C_l| > 1$. Then there exists a finite path $C_l \to C_{l+1} \to \cdots \to C_k$ with $C_k \in \{(0,c),(T_f0,c),(c,T_f1)\}$ or $|C_k| \geq \beta^t|C_l|$. As $\beta^t|C_l| > 1$ the second case cannot occur. This completes the proof. □

Lemma 11. *Assume that* $\sqrt[3]{2} \leq \beta < \sqrt{2}$ *and* $1 - \frac{\beta^3}{\beta^2+\beta+1} \leq \alpha < \frac{1}{\beta+1}$, *and set* $f(x) := \beta x + \alpha$. *Then* $(0,c) \to (c,1)$, $(c,1) \to (0,c)$, $(0,c) \to (T_f0,c)$, $({T_f}^2 0,1)$ *is the unique successor of* (T_f0,c), $(c,1) \to (c,T_f1)$ *and* $(c,T_f1) \to (0,c)$. *The interval* $(0,c)$ *is the unique successor of* (c,T_f1) *if* $\alpha = 1 - \frac{\beta^3}{\beta^2+\beta+1}$ *and* $(c,T_f1) \to (c,{T_f}^2 1)$ *otherwise. Moreover for every* $C \in \mathcal{D}$ *there exists a finite path* $C_0 := C \to C_1 \to \cdots \to C_k$ *with* $C_k \in \{(0,c),(T_f0,c),(c,{T_f}^2 1)\}$.

Proof. The properties $(0,c) \to (c,1)$, $(c,1) \to (0,c)$, $(0,c) \to (T_f0,c)$, $({T_f}^2 0,1)$ is the unique successor of (T_f0,c), $(c,1) \to (c,T_f1)$, $(c,T_f1) \to (0,c)$, $(0,c)$ is the unique successor of (c,T_f1) if $\alpha = 1 - \frac{\beta^3}{\beta^2+\beta+1}$ and $(c,T_f1) \to (c,{T_f}^2 1)$ otherwise follow immediately from Lemma 9.

We claim that for every $C \in \mathcal{D}$ having c as an endpoint there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$ with c is an endpoint of C_n and $C_n \in \{(0,c),(T_f0,c),(c,{T_f}^2 1)\}$ or $|C_n| \geq \beta|C|$. To this end we assume at first that $C = (d,c)$ for some d. In the case $T_f d \leq c$ we are done as $C \to (c,1) \to (0,c)$. Otherwise $C_1 := T_f C = (T_f d, 1)$ is the unique successor of $C_0 := C$. If ${T_f}^2 d \leq c$ we are done since $C_1 \to (c,T_f1) \to (0,c)$. Now suppose that ${T_f}^2 d > c$. Then $C_2 := {T_f}^2 C = ({T_f}^2 0, T_f1)$ is the unique successor of C_1. Moreover, either $C_2 \to (c,{T_f}^2 1)$ and we are done or $C_3 := {T_f}^3 C$ is the unique successor of C_2. The same argument as in the proof of Lemma 10 gives the existence of an $s \geq 3$ and of a finite path $C_0 := C \to C_1 \to \cdots \to C_{s+1}$ such that c is an endpoint of C_{s+1} and $|C_{s+1}| \geq \beta|C|$.

For $C = (c,d)$ for some d exactly the same proof as in the proof of Lemma 10 shows that either $C \to (0,c)$ or $C \to T_f C \to (T_f0,c)$ or there is an $s \geq 3$ and a finite path $C_0 := C \to C_1 \to \cdots \to C_{s+1}$ such that c is an endpoint of C_{s+1} and $|C_{s+1}| \geq \beta|C|$. Now the same arguments as in the proof of Lemma 10 show that for every $C \in \mathcal{D}$ there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_k$ with $C_k \in \{(0,c),(T_f 0,c),(c,{T_f}^2 1)\}$. □

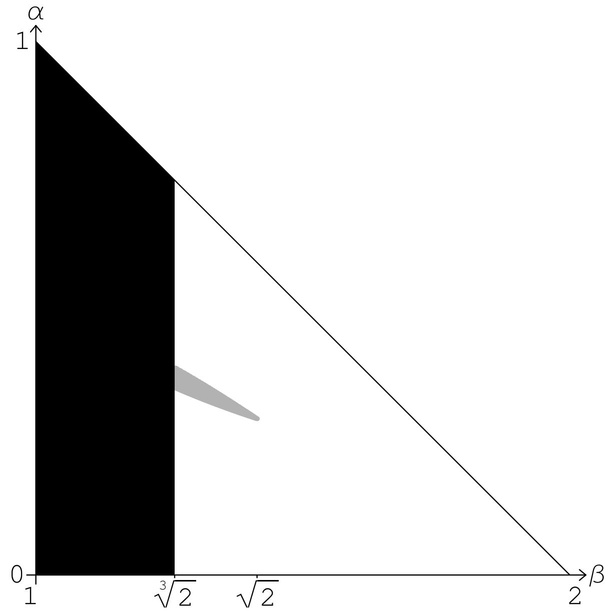

Figure 4: For (β,α) in the white region of this triangle the map $\beta x+\alpha \pmod 1$ is topologically transitive and in the gray region it is not topologically transitive. The black region is not completely classified.

Our next result classify those (β,α) with $\sqrt[3]{2} \le \beta < \sqrt{2}$ and $0 \le \alpha \le 2-\beta$ such that $\beta x+\alpha \pmod 1$ is topologically transitive. In Figure 4 the white region inside the triangle shows those (β,α) with $\sqrt[3]{2} \le \beta \le 2$ for which $\beta x+\alpha \pmod 1$ is topologically transitive. Recall that for $\beta \ge \sqrt{2}$ the map $\beta x+\alpha \pmod 1$ is topologically transitive by Theorem 1 of [10]. The gray region shows those (β,α) with $\sqrt[3]{2} \le \beta \le 2$ where $\beta x+\alpha \pmod 1$ is not topologically transitive. For $1 \le \beta < \sqrt[3]{2}$ the set of all (β,α) where $\beta x+\alpha \pmod 1$ is topologically transitive has not been described completely.

Theorem 4. *Let* $\sqrt[3]{2} \le \beta < \sqrt{2}$ *and let* $0 \le \alpha \le 2-\beta$. *Then* $\beta x+\alpha \pmod 1$ *is topologically transitive if and only if* $0 \le \alpha < \frac{1}{\beta^2+\beta}$ *or* $2-\beta-\frac{1}{\beta^2+\beta} < \alpha \le 2-\beta$.

Proof. From Lemma 9 we obtain that $\beta x+\alpha \pmod 1$ is not topologically transitive for $\frac{1}{\beta^2+\beta} \le \alpha \le 2-\beta-\frac{1}{\beta^2+\beta}$ (see the paragraph below the remark following Lemma 9).

At first we consider the case $\alpha > 2-\beta-\frac{1}{\beta^2+\beta}$. We start the proof investigating the case $\alpha < \frac{1}{\beta+1}$. Suppose at first that $\alpha < 1-\frac{\beta^3}{\beta^2+\beta+1}$. Set

$C_0 := (c, T_f 1)$. By Lemma 10 we get that $C_1 := (0, {T_f}^2 1)$ is the unique successor of C_0, $C_1 \to (T_f 0, c)$ and $C_1 \to C_2 := (c, {T_f}^3 1)$, and $|C_2| > |C_0|$. In particular this implies $C_0 \subseteq C_2$. Note that $|C_2| = \beta^2 |C_0| - |(T_f, c)|$. Now we prove by induction that for every $n \in \mathbb{N}$ there exists a $k \leq 2n$ and a path $C_0 \to C_1 \to \cdots \to C_k$ with $C_k = (0, c)$ or $C_0 \subseteq C_k$ and $|C_k| \geq |C_0| + \beta^{n-1}(|C_2| - |C_0|)$. For $n = 1$ we have obviously $|C_k| = |C_0| + \beta^0(|C_2| - |C_0|)$. Let $n > 1$ and assume that $l \leq 2n-2$, $C_0 \to C_1 \to \cdots \to C_l$ is a path with $C_l = (0, c)$ or $C_0 \subseteq C_l$ and $|C_l| \geq |C_0| + \beta^{n-2}(|C_2| - |C_0|)$. If $C_l = (0, c)$ set $k := l \leq 2n$ and we are done. Otherwise either $C_l \to (0, c)$ or $C_{l+1} := T_f C_l \supseteq C_1$ is the unique successor of C_l. In the first case we are done setting $k := l + 1 \leq 2n$ and $C_k := (0, c)$. Consider the second case. Then C_{l+1} has the two successors $(T_f 0, c)$ and $C_{l+2} := TfC_l \cap (c, 1) \supseteq C_2 \supseteq C_0$. Set $k := l + 2 \leq 2n$. We have that $|C_k| + |(T_f 0, c)| = \beta^2 |C_l| \geq \beta^2 |C_0| + \beta^n(|C_2| - |C_0|)$. Since $\beta^2 |C_0| - |(T_f 0, c)| = |C_2| \geq |C_0|$ and $\beta > 1$ this implies $|C_k| \geq |C_0| + \beta^{n-1}(|C_2| - |C_0|)$ finishing the induction. As $|C_0| + \beta^{n-1}(|C_2| - |C_0|)$ tends to infinity for $n \to \infty$ there exists a finite path $C_0 := (c, T_f 1) \to C_1 \to \cdots \to C_k$ with $C_k = (0, c)$.

Next assume that $d \leq T_f 0$ and set $C_0 := (d, c)$. If $T_f d \leq c$ then $C_0 \to C_1 := (c, 1) \to C_2 := (0, c)$. Otherwise $C_1 := T_f C_0 = (T_f d) \subseteq (c, 1)$ is the unique successor of C_0. In the case ${T_f}^2 d \leq c$ we have $C_1 \to (c, T_f 1)$. Then set $C_2 := (c, T_f 1)$ and as shown above there exists a finite path $C_0 \to C_1 \to C_2 = (c, T_f 1) \to \cdots \to C_k$ with $C_k = (0, c)$. Now we consider the case ${T_f}^2 d > c$. In this case $C_2 := TfC_1 = ({T_f}^2 d, T_f 1) \subseteq (c, T_f 1)$ is the unique successor of C_1 and by Lemma 10 $C_3 := T_f C_2 = ({T_f}^3 d, {T_f}^2 1) \subseteq (0, {T_f}^2 1)$ is the unique successor of C_2. Hence by Lemma 1 of [10] there is a minimal $s \geq 4$ such that $C_j := {T_f}^j C_0$ is the unique successor of C_{j-1} for $j = 1, 2, \dots, s-1$ and C_{s-1} has two different successors. Then either C_{s-1} has a successor C_s with $C_0 \subseteq C_s$ and $|C_s| > |C_0|$ or C_{s-1} has a successor $C_s = (c, \widetilde{d})$ with $|C_s| \geq (\beta^s - 1)|C_0|$. Consider the latter case. If $s = 4$ then $C_4 = (c, {T_f}^3 1)$ and we have shown above that there exists a finite path $C_4 \to C_5 \to \cdots \to C_k$ with $C_k = (0, c)$. Otherwise $s \geq 5$ and therefore $\beta^s \geq 2\beta^2 > 2$. In the case $T_f \widetilde{d} \geq c$ we get $C_s \to C_{s+1} := (0, c)$. Now suppose that $T_f \widetilde{d} < c$. Then $C_{s+1} := (0, T_f \widetilde{d}) \subseteq (0, c)$ is the unique successor of C_s. As $\beta^2(\beta^s - 1) > 1(2-1) = 1$ we get that $|T_f C_{s+1}| \geq \beta^2(\beta^s - 1)|C_0| > |C_0| \geq |(T_f 0, c)|$. Hence C_{s+1} has two successors, $(T_f 0, c)$ and $C_{s+2} := (c, {T_f}^2 \widetilde{d})$. Because of $\beta^2 \geq \sqrt[3]{4} > \frac{3}{2}$ we obtain $2\beta^2 - 3 > 0$ which implies $(\beta^2 - 1)(2\beta^2 - 1) > 1$. Hence $(\beta^2 - 1)|C_s| \geq (\beta^2 - 1)(\beta^s - 1)|C_0| \geq (\beta^2 - 1)(2\beta^2 - 1)|C_0| > |C_0|$. Therefore $|C_{s+2}| \geq \beta^2 |C_s| - |C_0| > |C_s|$. In particular this implies $C_s \subseteq C_{s+2}$. Now an analogous proof as above in the case starting with $(c, T_f 1)$ shows that there exists a finite path $C_s \to C_{s+1} \to \cdots \to C_k$ with $C_k = (0, c)$.

This means that we have shown for $d \leq T_f 0$ that there exists a finite path starting in (d,c) and ending in $(0,c)$ or there is a path $C_0 = (d,c) \to C_1 \to \cdots \to C_s$ with C_j is the unique successor of C_{j-1} for $j = 1,2,\ldots,s-1$, C_{s-1} has two different successors, $s \geq 4$, $C_0 \subseteq C_s$ and $|C_s| > |C_0|$. Now we set $C_0 := (T_f 0, c)$. Using induction we get that either there exists a finite path $C_0 \to C_1 \to \cdots \to C_k$ with $C_k = (0,c)$ or there is an infinite path $C_0 \to C_1 \to C_2 \to \cdots$ and a sequence $(s_n)_{n\in\mathbb{N}}$ of natural numbers with $s_n \geq 4$ for all n such that C_j is the unique successor of C_{j-1} for $j = S_{n-1}, S_{n-1}+1, \ldots, S_{n-1}+s_n-1$, $C_{S_{n-1}+s_n-1}$ has two different successors, $C_{S_{n-1}} \subseteq C_{S_n}$ and $|C_{S_n}| > |C_{S_{n-1}}|$, where $S_0 := 0$ and $S_n := \sum_{j=1}^n s_j$ for $n \geq 1$. Consider the second case. Because of $C_{S_{n-1}} \subseteq C_{S_n}$ we get that $s_{n+1} \leq s_n$ for all n. Hence there exists an n_0 and an $s \geq 4$ with $s_n = s$ for all $n \geq n_0$. For $n \geq n_0$ and $j = 0,1,\ldots,s-1$ the intervals $C_{S_{n-1}+j}$ and C_{S_n+j} have the same right endpoint. Analogous to the proof for the starting interval $(c, T_f 1)$ one proves by induction that $|C_{S_n}| \geq |C_{S_{n_0-1}}| + \beta^{n-n_0}(|C_{S_{n_0}}| - |C_{S_{n_0-1}}|)$. As the right hand side of this inequality tends to infinity for $n \to \infty$ this contradicts $|C_{S_n}| \leq 1$ for all n.

Hence we have proved that there exists a finite path starting in $(T_f 0, c)$ and ending in $(0,c)$ and there exists a finite path starting in $(c, T_f 1)$ and ending in $(0,c)$. Using Lemma 10 one obtains that for every $C \in \mathcal{D}$ there exists a finite path $C_0 \to C_1 \to \cdots \to C_k$ with $C_0 = C$ and $C_k = (0,c)$. Now the topological transitivity of T_f follows from Lemma 2 of [10].

Next we investigate the case $1 - \frac{\beta^3}{\beta^2+\beta+1} \leq \alpha < \frac{1}{\beta+1}$. By the remark after Lemma 9 this implies $\beta > \beta_0$ where β_0 is the largest zero of the polynomial $x^3 - x - 1$. This implies $\beta^3 - \beta - 1 > 0$ and therefore $\beta^3 - 1 > \beta$. Set $C_0 := (T_f 0, c)$. From Lemma 11 we get that $C_1 := ({T_f}^2 0, 1)$ is the unique successor of C_0. In the case ${T_f}^3 0 \leq c$ using Lemma 11 one obtains that $C_0 \to C_1 \to C_2 := (c, T_f 1) \to C_3 := (0,c)$. Otherwise $C_2 := T_f C_1$ is the unique successor of C_1. By Lemma 1 of [10] there is a minimal $s \geq 3$ such that $C_j := {T_f}^j C_0$ is the unique successor of C_{j-1} for $j = 1,2,\ldots,s-1$ and C_{s-1} has two different successors. We obtain that either C_{s-1} has a successor C_s with $C_0 \subseteq C_s$ and $|C_s| > |C_0|$ or C_{s-1} has a successor $C_s = (c, \widetilde{d})$ with $|C_s| \geq (\beta^s - 1)|C_0|$. In the second case $|C_s| > \beta|C_0|$ since $s \geq 3$ and $\beta^3 - 1 > \beta$. If $T_f \widetilde{d} \geq c$ we get $C_s \to C_{s+1} := (0,c)$. Otherwise $C_{s+1} := (0, T_f \widetilde{d})$ is the unique successor of C_s, $T_f C_{s+1} = (T_f 0, {T_f}^2 \widetilde{d})$ and $|T_f C_{s+1}| = \beta^2 |C_s| > |C_0|$. Therefore C_{s+1} has two successors, $(T_f 0, c)$ and $C_{s+2} := (c, {T_f}^2 \widetilde{d})$. Since $\beta^3 - \beta > 1$ we obtain $(\beta^2 - 1)|C_s| > (\beta^3 - \beta)|C_0| > |C_0|$. This implies $|C_{s+2}| = \beta^2 |C_s| - |C_0| > |C_s|$ and in particular $C_s \subseteq C_{s+2}$. A proof analogous as in the case $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ shows the existence of a finite path $C_s \to C_{s+1} \to \cdots \to C_k$ with $C_k = (0,c)$. Now assume that C_{s-1} has a successor C_s with $C_0 \subseteq C_s$ and $|C_s| > |C_0|$. We can repeat the argument and obtain analogous to the proof in the case $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ the existence of a finite path $C_0 \to C_1 \to \cdots \to C_k$ with $C_k = (0,c)$.

Suppose that $\alpha > 1 - \frac{\beta^3}{\beta^2+\beta+1}$ and set $C_0 := (c, {T_f}^2 1)$. If ${T_f}^3 1 \geq c$ we get $C_0 \to C_1 := (0,c)$. Now assume that ${T_f}^3 1 < c$. Then $C_1 := (0, {T_f}^3 1)$ is the unique successor of C_0. If ${T_f}^4 1 \geq c$ then $C_2 := (T_f 0, c)$ is a successor of C_1 and as shown above there is a finite path $C_2 \to C_3 \to \cdots \to C_k$ with $C_k = (0,c)$. Otherwise $C_2 := T_f C_1 \subseteq (T_f 0, c)$ is the unique successor of C_1 and by Lemma 11 $C_3 := T_f C_2$ is the unique successor of C_2. It follows from Lemma 1 of [10] that there is a minimal $s \geq 4$ such that $C_j := {T_f}^j C_0$ is the unique successor of C_{j-1} for $j = 1, 2, \ldots, s-1$ and C_{s-1} has two different successors. Then either C_{s-1} has a successor C_s with $C_0 \subseteq C_s$ and $|C_s| > |C_0|$ or C_{s-1} has a successor $C_s = (\widetilde{d}, c)$ with $|C_s| \geq (\beta^s - 1)|C_0|$. At first we consider the second case. As $s \geq 4$ we get $|C_s| > |C_0|$. If $T_f \widetilde{d} \leq c$ Lemma 11 implies that $C_s \to C_{s+1} := (c,1) \to C_{s+2} := (0,c)$. In the case $T_f \widetilde{d} > c$ the interval $C_{s+1} := (T_f \widetilde{d}, 1)$ is the unique successor of C_s. By Lemma 11 we obtain $C_s \to C_{s+1} \to C_{s+2} := (c, T_f 1) \to C_{s+3} := (0,c)$ if ${T_f}^2 \widetilde{d} \leq c$. Otherwise $C_{s+2} := ({T_f}^2 \widetilde{d}, T_f 1)$ is the unique successor of C_{s+1}. Moreover, $T_f C_{s+2} = ({T_f}^3 \widetilde{d}, {T_f}^2 1)$ and $|T_f C_{s+2}| = \beta^3 |C_s| > |C_0|$ imply that C_{s+2} has the two successors C_0 and $C_{s+3} := ({T_f}^3 \widetilde{d}, c)$. As $\beta \geq \beta_0 > \sqrt[3]{2}$ we get that $|C_{s+3}| = \beta^3 |C_s| - |C_0| > 2|C_s| - |C_0| > |C_s|$. Now we get analogous to the proof in the case $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ that there is a finite path $C_s \to C_{s+1} \to \cdots \to C_k$ with $C_k = (0,c)$. Next assume that C_{s-1} has a successor C_s with $C_0 \subseteq C_s$ and $|C_s| > |C_0|$. Then one obtains analogous to the proof in the case $\alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ that there exits a finite path $C_s \to C_{s+1} \to \cdots \to C_k$ with $C_k = (0,c)$.

We have proved that there exists a finite path starting in $(T_f 0, c)$ and ending in $(0,c)$ and in the case $\alpha > 1 - \frac{\beta^3}{\beta^2+\beta+1}$ there exists a finite path starting in $(c, {T_f}^2 1)$ and ending in $(0,c)$ (note that $(c, {T_f}^2 1) = \emptyset$ for $\alpha = 1 - \frac{\beta^3}{\beta^2+\beta+1}$). Therefore Lemma 11 implies that for every $C \in \mathcal{D}$ there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_k$ with $C_k = (0,c)$. By Lemma 2 of [10] the map T_f is topologically transitive.

In the case $\alpha \geq \frac{1}{\beta+1}$ it follows from Theorem 1 that T_f is topolologically transitive. Hence we have proved the topological transitivity of T_f for all $\alpha \in \left(2 - \beta - \frac{1}{\beta^2+\beta}, 2 - \beta\right]$.

Finally let $\alpha \in \left(0, \frac{1}{\beta^2+\beta}\right]$. The conjugation $h(x) := 1 - x$ conjugates $\beta x + \alpha \pmod 1$ to $\beta x + 2 - \beta - \alpha \pmod 1$. By the assumptions for α we obtain $2 - \beta - \frac{1}{\beta^2+\beta} < 2 - \beta - \alpha \leq 2 - \beta$. Hence $\beta x + 2 - \beta - \alpha \pmod 1$ is topologically transitive and therefore $\beta x + \alpha \pmod 1$ is topologically transitive. □

References

[1] K. Brucks, H. Bruin, *Topics from One-dimensional Dynamics*, Cambridge University Press, Cambridge, 2004.

[2] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math. **34** (1979), 213–237; *Part 2*, Israel J. Math. **38** (1979), 107–115.

[3] F. Hofbauer, *Monotonic mod one transformations*, Studia Math. **80** (1984), 17–40.

[4] F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Theory Relat. Fields **72** (1986), 359–386.

[5] S. Kolyada, Ľ. Snoha, *Some aspects of topological transitivity – a survey*, in *Proceedings of the European Conference on Iteration Theory (ECIT 94), Opava 1994* (eds.: L. Reich, J. Smítal, Gy. Targoński), Grazer Math. Ber. **334** (1997), 3–35.

[6] A. Nagar, V. Kannan, K. Srinivas, *Some simple conditions implying topological transitivity for interval maps*, Aequationes Math. **67** (2004), 201–204.

[7] P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps*, Israel J. Math. **80** (1992), 97–133.

[8] P. Raith, *Continuity of the entropy for monotonic mod one transformations*, Acta Math. Hungar. **77** (1997), 247–262.

[9] P. Raith, *Topological transitivity for expanding piecewise monotonic maps on the interval*, Aequationes Math. **57** (1999), 303–311.

[10] P. Raith, *Topological transitivity for expanding monotonic mod one transformations with two monotonic pieces*, Ann. Math. Sil. **13** (1999), 233–241.

[11] P. Walters, *Equilibrium states for β-transformations and related transformations*, Math. Z. **159** (1978), 65–88.

[12] P. Walters, *An Introduction to Ergodic Theory*, Springer, New York, 1982.

Peter Raith
Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, 1090 Wien, Austria
e-mail address: `peter.raith@univie.ac.at`

Angela Stachelberger
Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, 1090 Wien, Austria
e-mail address: `angela.stachelberger@univie.ac.at`