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Mixing properties in expanding Lorenz maps

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Abstract. Let $T_f\colon [0,1] \to [0,1]$ be an expanding Lorenz map, this means $T_f x := f(x) \pmod 1$ where $f\colon [0,1] \to [0,2]$ is a strictly increasing map satisfying $\inf f' > 1$. Then T_f has two pieces of monotonicity. In this paper, sufficient conditions when T_f is topologically mixing are provided. For the special case $f(x) = \beta x + \alpha$ with $\beta \geq \sqrt[3]{2}$ a full characterization of parameters (β, α) leading to mixing is given. Furthermore relations between renormalizability and T_f being locally eventually onto are considered, and some gaps in classical results on the dynamics of Lorenz maps are corrected.

Introduction

Lorenz attractor is one of the most recognized mathematical models which had very strong influence on mathematical understanding of idea of chaos and unpredictability in dynamics. It was obtained as a solution to a system of differential equations in $\mathbb{R}^3$ and later was extended to a plethora of Lorenz-like attractors and models (e. g. see [28], [10] or [19]; it is worth mentioning that it turned out to be extremely difficult to prove that set detected in numerical simulations is an attractor [27]).

Very quickly it was realized that some interval maps may serve as models for Poincaré map in Lorenz-like systems (see e. g. [10], [11] and [1]). A class

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of such models is formed by so-called *expanding Lorenz maps*, that is maps $T\colon [0,1] \to [0,1]$ satisfying the following three conditions:

1. there is a *critical point* $c \in (0,1)$ such that T is continuous and strictly increasing on $[0,c)$ and $(c,1]$;
2. $\lim_{x\to c^-} T(x) = 1$ and $\lim_{x\to c^+} T(x) = 0$;
3. T is differentiable for all points not belonging to a finite set $F \subseteq [0,1]$ and $\inf_{x\notin F} T'(x) > 1$.

There is a continuous interest in dynamics of these maps for more than last 20 years. It was discovered long time ago that kneading sequences of these maps can easily be characterized [16] and so ideas of Milnor and Thurston [18] work very well in this context. In particular, many authors were interested in fully characterizing these maps in terms of kneading sequences and renormalization (e.g. see [8], [9] and [2]). An important related question was asking when Lorenz map can be presented (in terms of conjugacy) as a map with constant slope. One of the most recent advances can be found in [6] together with historical comments. Kneading theory was also used as a tool in characterization of transitivity in Lorenz maps. One of the first classes of Lorenz maps studied in the literature, were the maps which are locally eventually onto (as defined in [9]). This includes works by Williams [28, 29], Guckenheimer and Williams [11] or Rand [26]. Characterization of locally eventually onto maps in terms of properties of the kneading sequence of critical point are stated in [8] and [9].

In some sense complementary approach for description of transitivity of piecewise monotone, piecewise continuous maps was developed by Hofbauer [12, 13, 14, 15] who developed and popularized approach using Markov diagrams. While the approach using kneading theory concentrates mainly on the (one-sided) trajectories of the critical point c, Hofbauer's controls evolution of the intervals $(0,c)$ and $(c,1)$.

We show in the present paper (see Theorem 4.1) that every locally eventually onto Lorenz map is mixing, however converse is not true (see Example 4.1), so the standard relation between these two properties known from topological dynamics of continuous interval maps holds also here (see [3]). Among other things, it shows that there may be no natural condition on kneading sequence (or renormalization) distinguishing between transitivity but not mixing, and mixing but not locally eventually onto ([8] proposes some conditions on kneading invariant which can help distinguishing transitivity from locally eventually onto; as we will see these conditions do not always work). To deal with this difficulty we extend Hofbauer's ideas and apply them to detect topological mixing in some expanding Lorenz maps. A useful tool in our considerations is provided by a partial description of Markov diagrams of some of these maps in [22, 24, 25], which saves us some hard work.

As a by-product of our study we make a surprising observation. The cases when Lorenz map is transitive but not mixing or mixing but not locally eventually onto trace borders where general results of kneading theory fail. We provide an example of a renormalizable expanding Lorenz map which is locally eventually onto (see Example 5.1). Hence for expanding Lorenz maps being prime is not equivalent to locally eventually onto. Motivated by this example, we introduce another property, called strongly eventually onto, which is equivalent to being prime (Corollary 6.3). Moreover, in Example 6.1 there is a transitive but not mixing map having primary 2(1)-cycle as defined in [8], showing a slight mistake in [8, Proposition 1]. Fortunately, this is a kind of border case and so in most cases the approach from [8] works properly (see Theorem 6.5). In Section 6 we also show that connections between strong transitivity and locally eventually onto condition may not be that tide as was claimed in [7, Proposition 1].

Except some particular (negative) examples, we develop in the paper numerous techniques helping to detect if an expanding Lorenz map is topologically mixing. As a particular application of our approach we provide a full characterization of topological mixing in linear Lorenz maps T_f induced by $f(x) = \beta x + \alpha$ with $\beta \geq \sqrt[3]{2}$ (see Theorem 7.1).

1. Preliminaries

1.1. Topological dynamics

We say that a continuous map $T\colon X \to X$ acting on a compact metric space is *(topologically) transitive* if for every two nonempty open sets $U, V \subseteq X$ there is an integer $n > 0$ such that $T^n U \cap V \neq \emptyset$. It is called *(topologically) mixing* if for every two nonempty open sets $U, V \subseteq X$ there is an $N > 0$ such that for every $n > N$ we have $T^n U \cap V \neq \emptyset$. The above definitions are among the most fundamental properties studied in dynamics (see [17, 4]).

Note that the expanding Lorenz map is *positively expanding*, this means there is $\varepsilon > 0$ such that for any distinct $x, y \in [0, 1]$ there is $n \geq 0$ such that $|T^n(x) - T^n(y)| > \varepsilon$. To each Lorenz map we can associate a strictly increasing continuous function such that

$$Tx := f(x) \pmod 1 = f(x) - \lfloor f(x) \rfloor$$

for $x \in [0,1]$ where $\lfloor y \rfloor = \max\{k \in \mathbb{Z} : k \leq y\}$. To emphasize connection between these two functions we write T_f instead of T. Maps of this form are also called *monotonic mod one transformations* (see e. g. [14], [23], [24] and [25]).

Since T_f has discontinuities it is not a topological dynamical system, this means a continuous map on a compact metric space. However, using a standard doubling points construction (see e. g. [22] for details) one can create a topological dynamical system from T_f. In this construction all elements in $\left(\bigcup_{n=0}^{\infty} T_f^{-n} E\right) \setminus \{0,1\}$ are doubled, where E is the set of discontinuities of T_f (we perform a kind of Denjoy extension). We easily see that this new space differs from the original interval $[0,1]$ by at most countably many points. Since it is always possible to perform the above identification, we will apply the standard definitions from topological dynamics to T_f without any further reference.

Following [9] (see also in [8]), we present a standard definition of a locally eventually onto Lorenz map T_f.

Definition 1.1. Suppose that T_f is an expanding Lorenz map. Then T_f is said to be *locally eventually onto* if for every nonempty open subset $U \subseteq [0,1]$ there exist open intervals $J_1, J_2 \subseteq U$ and $n_1, n_2 \in \mathbb{N}$ such that ${T_f}^{n_1}$ maps J_1 homeomorphically to $(0,c)$ and ${T_f}^{n_2}$ maps J_2 homeomorphically to $(c,1)$.

Below in Section 1.2 we will define renormalizable Lorenz maps. It was believed that essentially an expanding Lorenz map is locally eventually onto if and only if it is not renormalizable. However, we will show in Example 5.1 that this is not true if one uses the definition above. Therefore we give the following definition of strongly locally eventually onto Lorenz maps.

Definition 1.2. An expanding Lorenz map T_f is said to be *strongly locally eventually onto* if for every nonempty open subset $U \subseteq [0,1]$ there exist open intervals $J_1, J_2 \subseteq U$ and $n_1, n_2 \in \mathbb{N}$ such that:

1. ${T_f}^{n_1}$ maps J_1 homeomorphically to $(0,c)$,
2. the restriction of ${T_f}^k$ to J_1 is continuous for all $k \in \{0,1,\dots,n_1\}$,
3. ${T_f}^{n_2}$ maps J_2 homeomorphically to $(c,1)$,
4. the restriction of ${T_f}^k$ to J_2 is continuous for all $k \in \{0,1,\dots,n_2\}$.

Obviously every strongly locally eventually onto Lorenz map is locally eventually onto. The converse is not true, as we will see in Example 5.1, which is a locally eventually onto Lorenz map which is not strongly eventually onto.

1.2. Kneading theory

Let T_f be an expanding Lorenz map. For each $x \in [0,1]$ we can define the *kneading sequence* $k(x) \in \{0,*,1\}^{\mathbb{N}}$ putting

$$k(x)_0 = \begin{cases} 0, & \text{if } x < c \\ *, & \text{if } x = c \\ 1, & \text{if } x > c \end{cases}$$

and then recursively $k(x)_j = k({T_f}^j x)_0$ for $j \in \mathbb{N}$. The *kneading invariant* k_f is the pair (k_+, k_-) where $k_+ = \lim_{x \to c^+} k(x)$ and $k_- = \lim_{x \to c^-} k(x)$, where the limits are calculated through points which are not preimages of c. Note that both k_+, k_- are sequences consisting only of symbols 0 and 1. A kneading invariant $k_f = (k_+, k_-)$ is *renormalizable* if there exist a pair of finite words $(w_+, w_-) \neq (1,0)$ such that we can write

$$\begin{aligned} k_+ &= w_+ w_-^{p_1} w_+^{p_2} \dots \\ k_- &= w_- w_+^{m_1} w_-^{m_2} \dots \end{aligned}$$

where lengths of these words satisfy $|w_+| + |w_-| \geq 3$. We allow that one or both m_1 and p_1 can be infinite. The kneading invariant k_f is *minimally renormalizable* with words (w_+, w_-) if they are the shortest possible such words. If the kneading invariant is not renormalizable, then we say it is *prime*. One calls the kneading invariant k_f *trivially renormalizable* if the sum of the lengths of the words w_+, w_- is exactly three, i. e. $(w_+, w_-) = (1, 01)$ or $(w_+, w_-) = (10, 0)$. The kneading invariant is called *special trivial renormalizable (STR)* if it is trivially renormalizable with $p_1 = +\infty$ or $m_1 = +\infty$. In [9] it is related to the case when $T_f 1 = 1$ or $T_f 0 = c$, however it seems necessary to include also the symmetric case. Hence, for our further investigations we will use the following definition.

Definition 1.3. If T_f is an expanding Lorenz map such that at least one of the following conditions hold:

$$T_f 0 = 0 \quad \text{or} \quad T_f 1 = 1 \quad \text{or} \quad T_f 0 = c \quad \text{or} \quad T_f 1 = c$$

then we say that T_f is *special trivial renormalizable* (*STR* for short).

We will also need the following definition, which we repeat after [6] (see also [9]).

Definition 1.4. An expanding Lorenz map T_f is called *renormalizable* if there are $0 \leq u < c < v \leq 1$ and $l, r \geq 1$ with $l + r \geq 3$ such that

$$G(x) = \begin{cases} {T_f}^l x, & \text{if } x \in [u, c), \\ {T_f}^{r-1} 0, & \text{if } x = c, \\ {T_f}^r x, & \text{if } x \in (c, v], \end{cases}$$

is itself an expanding Lorenz map (after linear change of domain from $[u, v]$ to $[0, 1]$). Note that this definition implies that $u = {T_f}^{r-1} 0$. If T_f is not renormalizable, then we say it is *prime*.

Definition 1.5. We say that expanding Lorenz map is *trivially renormalizable* if it is renormalizable with constants $l + r = 3$ in the definition.

Remark 1.1. Note that some special trivial renormalizable maps are prime. For example it is the case when $f(x) = 2x$.

Remark 1.2. Observe that Definition 1.4 is slightly different form the definition of renormalizability given in [6] and [7], where both $l \geq 2$ and $r \geq 2$ are required instead of $l + r \geq 3$. Therefore the results in this paper cannot be compared directly with those in [6] and [7].

2. Markov diagrams of expanding Lorenz maps T_f

Let $f \colon [0, 1] \to [0, 2]$ be a *piecewise differentiable* function, this means that there exists a finite set $F \subseteq [0, 1]$ such that f is differentiable on $(0, 1) \setminus F$. Put $\inf f' := \inf\{f'(x) : x \in (0, 1) \setminus F\}$.

Moreover, suppose that $\inf f' > 1$. Then there exists a unique $c \in (0, 1)$ such that $f(c) = 1$ (note that $f(0) < 1 < f(1)$). This point c is the critical point of the associated Lorenz map T_f. Define $\mathcal{Z} := \{(0, c), (c, 1)\}$ and observe that on each $Z \in \mathcal{Z}$ the restriction $T_f|_Z$ is continuous and strictly increasing. Obviously the image $T_f Z$ is always an interval which may or may not contain the critical point. If I is an interval denote by $|I|$ its length.

It will be important in our considerations to know where are the endpoints of iterates of $Z \in \mathcal{Z}$. For that purpose for $n \in \mathbb{N}$ we set ${T_f}^n 0 := \lim_{x \to 0^+} {T_f}^n x$ and ${T_f}^n 1 := \lim_{x \to 1^-} {T_f}^n x$.

Let $Z \in \mathcal{Z}$ and let $D \subseteq Z$ be an open interval. An open interval C is a *successor* of D, denoted $D \to C$, if there exists a $Y \in \mathcal{Z}$ such that $C = T_f D \cap Y$. Now let $\mathcal{D}$ be the smallest set consisting of open intervals such that $\mathcal{Z} \subseteq \mathcal{D}$ and if $D \in \mathcal{D}$ has a successor C then also $C \in \mathcal{D}$. We can view $\mathcal{D}$ as a possibly infinite, directed graph with arrows given by successor relation. Then $\mathcal{D}$ is called the *Markov diagram* of T_f. Any element of $\mathcal{D}$ is called *vertex* and $C_1 \to C_2 \to \cdots \to C_n$ is called a *path* from C_1 to C_n in $\mathcal{D}$ if $C_j \in \mathcal{D}$ for all $j \in \{1, 2, \ldots, n\}$ and $C_j \to C_{j+1}$ for all $j \in \{1, 2, \ldots, n-1\}$. One calls this path $C_1 \to C_2 \to \cdots \to C_n$ a *path of length* $n - 1$.

We say that $\mathcal{C} \subseteq \mathcal{D}$ is *irreducible* if for any $C, D \in \mathcal{C}$ there is a path from C to D in $\mathcal{C}$, this means there are $C_1, C_2, \ldots, C_n \in \mathcal{C}$ such that

$$C \to C_1 \to \cdots \to C_n \to D$$

is a path. A subset $\mathcal{C} \subseteq \mathcal{D}$ is called *closed* if it has the property that if $C \in \mathcal{C}$, $D \in \mathcal{D}$ and there is a path from C to D (in $\mathcal{D}$) then $D \in \mathcal{C}$.

Lemma 2.1 ([24, Lemma 1])**.** *Assume that T_f is an expanding Lorenz map. Then for every $D \in \mathcal{D}$ there exists a finite path $C_0 \to C_1 \to \cdots \to C_n$ in $\mathcal{D}$ such that $C_0 = D$ and C_n has two different successors in $\mathcal{D}$.*

The following simple fact will be used in several places of this paper.

Lemma 2.2. *If $C, D \in \mathcal{D}$, $C \subseteq D$, $D \in \mathcal{Z}$ and there is a path of length q from C to D then there is also a path of length q from D to D.*

Proof. Let $C_0 \to C_1 \to C_2 \to \cdots \to C_q$ be the path with $C_0 = C$ and $C_q = D$, and for $j \in \{0, 1, \ldots, q\}$ let $Z_j \in \mathcal{Z}$ be so that $C_j \subseteq Z_j$. As $C_q = D$ and $D \in \mathcal{Z}$ we get that $Z_q = D$. Define $D_0 := D$ and $D_j := T_f D_{j-1} \cap Z_j$. Then $D_0 \to D_1 \to D_2 \to \cdots \to D_q$ is a path with $D_0 = D$ and $C_j \subseteq D_j$ for all $j \in \{0, 1, \ldots, q\}$. Hence $D = C_q \subseteq D_q \subseteq Z_q = D$ which implies $D_q = D$. □

Lemma 2.3 ([25, Lemma 2])**.** *Assume that T_f is an expanding Lorenz map and suppose that there is an integer $n \geq 2$ such that $c \leq {T_f}^{n-2}0 < \cdots < {T_f}^2 0 < T_f 0$. Let $C \in \mathcal{D}$ be such that c is an endpoint of C.*

(1) *if $C \subseteq (c, 1)$, denote $C_0 := C$, $C_1 := T_f C \cap (0, c)$ and $C_j := T_f C_{j-1} \cap (c, 1)$ for $j \in \{2, 3, \ldots, n\}$.*

(2) *if $C \subseteq (0, c)$, denote $C_0 := C$, $C_j := T_f C_{j-1} \cap (c, 1)$ for $j \in \{1, 2, \ldots, n-1\}$ and $C_n := T_f C_{n-1} \cap (0, c)$.*

Suppose that $C_0 \subseteq C_n$ and $|C_0| < |C_n|$. Then there exists a path $C \to C_1 \to \cdots \to C_k$ in $\mathcal{D}$ such that $C_k \in \{(0, c), (c, {T_f}^{n-2}1)\}$.

Remark 2.1. Observe that in the case $T_f 1 < 1$ we get

$${T_f}^n 1 < {T_f}^{n-1} 1 < \cdots < {T_f}^2 1 < T_f 1 < 1$$

for any n such that ${T_f}^k 1 \geq c$ for all $k \in \{1, 2, \ldots, n-1\}$, since T_f is strictly increasing on $(c, 1)$. Otherwise one has ${T_f}^n 1 = 1$ for all n.

Combining [25, Lemma 3] with Remark 2.1 we obtain the following.

Lemma 2.4. *Assume that T_f is an expanding Lorenz map.*

(1) *there exists an $r \in \mathbb{N}$ such that ${T_f}^r 0 < c$.*

(2) *denote by $r(f)$ the smallest $r \in \mathbb{N}$ with ${T_f}^r 0 < c$. Then*

$$ {T_f}^{r(f)} 0 < c \leq {T_f}^{r(f)-1} 0 < \cdots < {T_f}^2 0 < T_f 0 \ , $$
$$ {T_f}^{r(f)} 1 \leq {T_f}^{r(f)-1} 1 \leq \cdots \leq {T_f}^2 1 \leq T_f 1 \ , $$

and ${T_f}^j 0 < {T_f}^j 1$ for $j \in \{1, 2, \dots, r(f)\}$.

From now on let $r(f)$ be always as in Lemma 2.4. Next set $A(0,0) := (0,c)$ and $A(1,0) := (c,1)$. For $n \in \mathbb{N}$ let $A(0,n)$ be the successor of $A(0,n-1)$ with $\inf A(0,n) = {T_f}^n 0$ and let $A(1,n)$ be the successor of $A(1,n-1)$ with $\sup A(1,n) = {T_f}^n 1$. Then $\mathcal{D} = \{A(j,n) : j \in \{0,1\}, n \in \mathbb{N}_0\}$, and we have $A(j,n-1) \to A(j,n)$ for all $j \in \{0,1\}$ and all $n \in \mathbb{N}$. If $A(j,n-1)$ has two successors then the other one is of the form $A(1-j,k)$ for some $k < n-1$.

Lemma 2.5. *Suppose that T_f is an expanding Lorenz map and $r(f) \geq 2$. Put*

$$ \mathcal{E} := \Big\{ (0,c), (T_f 0, 1), ({T_f}^2 0, T_f 1), \dots, ({T_f}^{r(f)-1} 0, {T_f}^{r(f)-2} 1), (c, {T_f}^{r(f)-1} 1) \Big\} \, . $$

Then for every $C \in \mathcal{D} \setminus \big\{ (c, {T_f}^j 1) : j \in \{0,1,\dots,r(f)-2\} \big\}$ there is a $D \in \mathcal{E}$ with $C \subseteq D$. Moreover, if $D \in \mathcal{E}$ and $\inf D = c$ then $D = (c, {T_f}^{r(f)-1} 1)$ or ${T_f}^{r(f)-1} 0 = c$.

Proof. By Lemma 2.4 we get that ${T_f}^k 0 > c$ for all $k \in \{1, 2, \dots, r(f)-1\}$, provided that ${T_f}^{r(f)-1} 0 \neq c$. Hence $(c, {T_f}^{r(f)-1} 1)$ is the only element in $\mathcal{E}$ having c as a left endpoint.

Note that by Lemma 2.4 we have

$$ \big\{ (c, {T_f}^j 1) : j \in \{0,1,\dots,r(f)-2\} \big\} = \{ A(1,j) : j \in \{0,1,\dots,r(f)-2\} \} \, . $$

We are going to prove by induction that for every $n \geq 0$ there is a $D \in \mathcal{E}$ with $A(0,n) \subseteq D$, and for every $n \geq r(f)-1$ there is a $D \in \mathcal{E}$ with $A(1,n) \subseteq D$. By definition $A(0,0) = (0,c) \in \mathcal{E}$, and by Lemma 2.4 we get that $A(1, r(f)-1) = (c, {T_f}^{r(f)-1} 1) \in \mathcal{E}$. Now assume that $j = 0$ and $n > 0$, or $j = 1$ and $n > r(f)-1$. Then by induction hypothesis there is an $E \in \mathcal{E}$ with $A(j,n-1) \subseteq E$. If $E = (0,c)$ then $T_f E = (T_f 0, 1) \subseteq (c,1)$, hence $A(j,n) \subseteq (T_f 0, 1) \in \mathcal{E}$ is the unique successor of $A(j,n-1)$. Next we consider the case $E = ({T_f}^k 0, {T_f}^{k-1} 1)$ for some $k \in \{1, 2, \dots, r(f) - 2\}$. Then Lemma 2.4 implies that $T_f E = ({T_f}^{k+1} 0, {T_f}^k 1) \subseteq (c, 1)$, and therefore $A(j, n) \subseteq ({T_f}^{k+1} 0, {T_f}^k 1) \in \mathcal{E}$ is the unique successor of $A(j, n-1)$. In the case $E = ({T_f}^{r(f)-1} 0, {T_f}^{r(f)-2} 1)$ we have $T_f E = ({T_f}^{r(f)} 0, {T_f}^{r(f)-1} 1)$. Hence $A(j, n-1)$ may have two successors, where $({T_f}^{r(f)} 0, c) \subseteq (0, c) \in \mathcal{E}$ is always a successor, and the second one is $(c, {T_f}^{r(f)-1} 1) \in \mathcal{E}$, provided that ${T_f}^{r(f)-1} 1 > c$. Again $A(j, n)$ is contained in an element of $\mathcal{E}$. Finally, it remains to consider the case $E = (c, {T_f}^{r(f)-1} 1)$. Here $T_f E = (0, {T_f}^{r(f)} 1)$ and ${T_f}^{r(f)} 1 \leq {T_f}^{r(f)-1} 1$ by Lemma 2.4. If $A(j, n) \subseteq (0, c) \in \mathcal{E}$ we are done, so it remains to consider the case $A(j, n) \subseteq (c, {T_f}^{r(f)} 1)$. Observe that ${T_f}^{r(f)} 1 \leq {T_f}^{r(f)-1} 1$, hence we obtain $(c, {T_f}^{r(f)} 1) \subseteq (c, {T_f}^{r(f)-1} 1) \in \mathcal{E}$, completing the proof. □

Remark 2.2. Let $r(f) \geq 2$. Note that if $C \in \mathcal{D}$ is contained in an element of $\mathcal{E}$, then also every successor D of C is contained in an element of $\mathcal{E}$. In particular this implies that if $T_f 1 \neq 1$ then $\mathcal{D} \setminus \big\{(c, {T_f}^j 1) : j \in \{0, 1, \dots, r(f) - 2\}\big\}$ is closed (see Remark 2.1).

Lemma 2.6. *Assume that T_f is an expanding Lorenz map, that $r(f) \geq 2$ and that ${T_f}^{r(f)-1} 0 = c$. Then for every $C \in \mathcal{D}$ there exists $C \to C_1 \to \cdots \to C_k$ in $\mathcal{D}$ such that $C_k = (0, c)$.*

Proof. Put $\beta := \inf f' > 1$ and note that $|T_f A| \geq \beta |A|$ for every interval A which is a subset of an interval of monotonicity.

First, we claim that for $C_0 := (c, {T_f}^{r(f)-1} 1)$ there is a path $C_0 \to C_1 \to \cdots \to C_k$ in $\mathcal{D}$ with $C_k := (0, c)$. Put $Z_0 := (c, 1)$, $C_1 := T_f C_0 \cap (0, c)$ and $Z_1 := (0, c)$. If $C_1 = (0, c)$ we are done. Otherwise $C_1 = (0, {T_f}^{r(f)} 1)$ is the unique successor of C_0, and $|C_1| \geq \beta |C_0|$. For $j \in \{2, 3, \dots, r(f)\}$ put $C_j := T_f C_{j-1} \cap (c, 1)$ and $Z_j := (c, 1)$. By Lemma 2.4, C_j is the unique successor of C_{j-1}, hence $|C_j| \geq \beta |C_{j-1}|$. Moreover, the left endpoint of C_j is ${T_f}^{j-1} 0$. Therefore $C_{r(f)}$ has $c = T^{r(f)-1} 0$ as the left endpoint and $|C_{r(f)}| \geq \beta^{r(f)} |C_0|$.

Now define $C_{tr(f)+j} := T_f C_{tr(f)+j-1} \cap Z_j$ for $t \in \mathbb{N}$ and $j \in \{1, 2, \dots, r(f)\}$. We prove by induction that either $C_{(t-1)r(f)+1} = (0, c)$ or $C_{tr(f)}$ has c as the left endpoint and $|C_{tr(f)}| \geq \beta^{tr(f)} |C_0|$. If $C_{(t-1)r(f)+1} \neq (0, c)$ then $C_{(t-1)r(f)+j}$ is the unique successor of $C_{(t-1)r(f)+j-1}$, ${T_f}^{j-1} 0$ is its left endpoint and $|C_{(t-1)r(f)+j}| \geq \beta |C_{(t-1)r(f)+j-1}|$ for $j \in \{1, \dots, r(f)\}$. Therefore c is the left endpoint of $C_{tr(f)}$ and $|C_{tr(f)}| \geq \beta^{tr(f)} |C_0|$.

As $\lim_{t \to \infty} \beta^{tr(f)} = +\infty$ there must exist $t \in \mathbb{N}$ such that $C_{(t-1)r(f)+1} = (0, c)$ and therefore there exists a path $C_0 \to C_1 \to \cdots \to C_k$ in $\mathcal{D}$ with $C_k := (0, c)$. The claim is proved.

Now take any $C \in \mathcal{D}$. By Lemma 2.1 there exists a path $C \to C_1 \to \cdots \to C_{s-1}$ in $\mathcal{D}$ such that C_{s-1} has two successors. Denote by $C_s = (c, a)$

one of these successors, where $c < a \leq 1$. If C_s has two successors then $C \to C_1 \to \cdots \to C_s \to (0, c)$ is a path in $\mathcal{D}$ and we are done. In the other case $C_{s+1} = (0, T_f a)$ is the unique successor of C_s and this extends to the path

$$\begin{aligned} C_s &\to (0, T_f a) \to (T_f 0, {T_f}^2 a) \to \\ &\to \ldots \to ({T_f}^{r(f)-2} 0, {T_f}^{r(f)-1} a) \to (c, {T_f}^{r(f)} a) = C_{s+r(f)}. \end{aligned}$$

Since each successor on this path is unique, we clearly have $|C_s| < |C_{s+r(f)}|$ and $C_s \subseteq C_{s+r(f)}$. Therefore, by (1) of Lemma 2.3 there exists a path $C \to C_1 \to \cdots \to C_u$ in $\mathcal{D}$ with $C_u := (0, c)$ or $C_u := (c, {T_f}^{r(f)-1} 1)$. If $C_u = (0, c)$ then we are done, so assume that $C_u = (c, {T_f}^{r(f)-1} 1)$. We have already proved that in such a case, there exists in $\mathcal{D}$ a path $C_u \to C_{u+1} \to \cdots \to C_p$ with $C_p := (0, c)$, hence there always exists a path $C \to C_1 \to \cdots \to C_p$ in $\mathcal{D}$ with $C_p := (0, c)$, which completes the proof. $\square$

Lemma 2.7. *Let T_f be an expanding Lorenz map and $r(f) \geq 2$. Moreover, assume that ${T_f}^{r(f)-1} 0 \neq c$. Suppose that $C, D \in \mathcal{D}$, $C \subseteq D$, and there is a path $C_0 := C \to C_1 \to \cdots \to C_n$ of length n in $\mathcal{D}$ with $C_n \in \Big\{(0, c), (c, {T_f}^{r(f)-1} 1)\Big\}$. Then there exists a path $D_0 := D \to D_1 \to \cdots \to D_n$ of length n in $\mathcal{D}$ with $D_n = C_n$ or $D_n \in \big\{(c, {T_f}^j 1) : j \in \{0, 1, \ldots, r(f) - 2\}\big\}$.*

Proof. For $j \in \{0, 1, \ldots, n\}$ let $Z_j \in \mathcal{Z}$ be so that $C_j \subseteq Z_j$. Then $C_j = T_f C_{j-1} \cap Z_j$ for $j \in \{1, 2, \ldots, n\}$. Now define $D_0 := D$ and $D_j = T_f D_{j-1} \cap Z_j$ for $j \in \{1, 2, \ldots, n\}$. Then we obviously obtain that $C_j \subseteq D_j$ for $j \in \{0, 1, \ldots, n\}$. In particular $C_n \subseteq D_n$ and so $D_n = C_n = Z_n$, provided that $C_n \in \mathcal{Z}$. Otherwise $T_f 1 \neq 1$ and $(c, {T_f}^{r(f)-1} 1) = C_n \subseteq D_n$. If $D_n \in \big\{(c, {T_f}^j 1) : j \in \{0, 1, \ldots, r(f) - 2\}\big\}$ then we are done, and if $D_n \in \mathcal{D} \setminus \big\{(c, {T_f}^j 1) : j \in \{0, 1, \ldots, r(f) - 2\}\big\}$ then, since $\inf D_n = c$, Lemma 2.5 implies that $D_n = (c, {T_f}^{r(f)-1} 1)$ completing the proof. $\square$

Lemma 2.8 ([25, Lemma 8])**.** *Suppose that T_f is an expanding Lorenz map with $\inf f' = \beta \geq \sqrt[3]{2}$ and $f(0) \geq \frac{1}{\beta+1}$. Moreover, assume that $f(x) \neq \sqrt[3]{2} x + \frac{2 + \sqrt[3]{4} - 2\sqrt[3]{2}}{2}$ for some x and fix any $C \in \mathcal{D}$.*

1. *If $r(f) = 2$, then there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$, where $C_n \in \{(0, c), (c, T_f 1)\}$.*
2. *In the case $r(f) \geq 3$ there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$, where $C_n = (0, c)$. Moreover ${T_f}^{r(f)} 1 > c$.*

Lemma 2.9 ([25, Lemma 5])**.** *Let T_f be an expanding Lorenz map and denote $\inf f' = \beta$ and $\alpha = f(0)$. If $\alpha \geq \frac{1}{\beta+1}$ then $c \leq \frac{1}{\beta+1} \leq \alpha$.*

Remark 2.3. From Lemma 2.9 we obtain immediately that $T_f 0 \geq c$, hence $r(f) \geq 2$, if the assumptions of Lemma 2.9 are satisfied.

Lemma 2.10 ([24, Lemma 4])**.** *Assume that T_f is an expanding Lorenz map with* $\inf f' \geq \sqrt{2}$ *such that $f(x) \neq \sqrt{2}x + \frac{2-\sqrt{2}}{2}$ for an $x \in [0,1]$. Let $C \in \mathcal{D}$ having c as an endpoint. Then there exists a finite path $C_0 := C \to C_1 \to \cdots \to C_n$ in $\mathcal{D}$ with $c \in \overline{C_n}$, such that $C_n \in \mathcal{Z}$ or $|C_n| \geq \sqrt{2}|C|$.*

Two Lorenz maps T_f, T_g are *conjugate* if there is a homeomorphism $h\colon [0,1] \to [0,1]$ such that $h \circ T_f \circ h^{-1} = T_g$ on $[0,1] \setminus g^{-1}(\mathbb{Z})$. Note that one obtains that $(h \circ {T_f}^n \circ h^{-1})(x) = {T_g}^n x$ for all but finitely many $x \in [0,1]$.

Proposition 2.11. *Let $f\colon [0,1] \to [0,2]$ be continuous and strictly increasing. Define $g\colon [0,1] \to [0,2]$ by $g(x) := 2 - f(1-x)$. Then T_g is conjugated to T_f by the conjugacy $h(x) := 1 - x$. Moreover, $T_f x < y$ if and only if $T_g(1-x) > 1-y$, $T_f x > y$ if and only if $T_g(1-x) < 1-y$, and $T_f x < T_f y$ if and only if $T_g(1-x) > T_g(1-y)$. In particular, $T_f 1 < {T_f}^2 1 < \cdots < {T_f}^{r-1} 1 \leq c < {T_f}^r 1$ is equivalent to $T_g 0 > {T_g}^2 0 > \cdots > {T_g}^{r-1} 0 \geq \widetilde{c} > {T_g}^r 0$, where $\widetilde{c} := 1 - c$.*

Proof. Obviously $h^{-1}(x) = 1-x$. If $x < \widetilde{c}$ then $1-x > c$ and $(h \circ T_f \circ h^{-1})(x) = 1 - T_f(1-x) = 1 - (f(1-x) - 1) = 2 - f(1-x) = g(x) = T_g x$. In the case $x > \widetilde{c}$ we obtain $1 - x < c$ and $(h \circ T_f \circ h^{-1})(x) = 1 - T_f(1-x) = 1 - f(1-x) = 2 - f(1-x) - 1 = g(x) - 1 = T_g x$.

We have that $T_f x < y$ is equivalent to $1 - y < 1 - T_f x = (h \circ T_f \circ h^{-1})(1-x) = T_g(1-x)$. Analogously we get that $T_f x > y$ is equivalent to $T_g(1-x) < 1-y$. Furthermore, this implies $T_f x < T_f y$ if and only if $T_g(1-x) > 1 - T_f y = (h \circ T_f \circ h^{-1})(1-y) = T_g(1-y)$. As $(h \circ {T_f}^n \circ h^{-1})(x) = {T_g}^n x$ for any $n \in \mathbb{N}$ and all but finitely many x, one obtains that $T_f 1 < {T_f}^2 1 < \cdots < {T_f}^{r-1} 1 \leq c < {T_f}^r 1$ is equivalent to $T_g 0 > {T_g}^2 0 > \cdots > {T_g}^{r-1} 0 \geq \widetilde{c} > {T_g}^r 0$. □

3. Properties of locally eventually onto Lorenz maps

Note that in the definition of locally eventually onto Lorenz maps the intervals J_1 and J_2 need not be disjoint, and it may be that $n_1 \neq n_2$. Next we show that in practice we may assume that $n_1 = n_2$ and $J_1 \cap J_2 = \emptyset$.

Proposition 3.1. *Assume that T_f is an expanding Lorenz map which is locally eventually onto. Then for every nonempty open set $U \subseteq [0,1]$ there exist disjoint open intervals $J_1 \subseteq U$ and $J_2 \subseteq U$ and there exists an $n \in \mathbb{N}$ such that ${T_f}^n$ maps J_1 homeomorphically to $(0,c)$ and ${T_f}^n$ maps J_2 homeomorphically to $(c,1)$.*

Proof. Without loss of generality we may assume that U is an open interval, as any nonempty open set contains a nontrivial open interval. As T_f is locally eventually onto there exist open intervals $V_1, V_2 \subseteq U$ and $n_1, n_2 \in \mathbb{N}$ such that ${T_f}^{n_1}$ maps V_1 homeomorphically to $(0, c)$ and ${T_f}^{n_2}$ maps V_2 homeomorphically to $(c, 1)$. First we assume that $T_f 0 < c$. In this case ${T_f}^{n_1+1}$ maps V_1 homeomorphically to $(T_f 0, 1)$ and both $(T_f 0, 1) \cap (0, c)$ and $(T_f 0, 1) \cap (c, 1)$ are nonempty. Hence there are disjoint open intervals $K_1, K_2 \subseteq V_1$ such that ${T_f}^{n_1+1}$ maps K_1 homeomorphically to $L_1 := (T_f 0, c) \subseteq (0, c)$ and K_2 homeomorphically to $L_2 := (c, 1) \subseteq (c, 1)$. Set $V := V_1$ and $m := n_1 + 1$ in this case.

Otherwise $c \leq T_f 0 < T_f 1$. In this case ${T_f}^{n_2+1}$ maps V_2 homeomorphically to $(0, T_f 1)$ and both $(0, T_f 1) \cap (0, c)$ and $(0, T_f 1) \cap (c, 1)$ are nonempty. Therefore there exist disjoint open intervals $K_1, K_2 \subseteq V_2$ such that ${T_f}^{n_2+1}$ maps K_1 homeomorphically to $L_1 := (0, c) \subseteq (0, c)$ and K_2 homeomorphically to $L_2 := (c, T_f 1) \subseteq (c, 1)$. Set $V := V_2$ and $m := n_2 + 1$ in this case.

In any case we have an open interval $V \subseteq U$, disjoint open intervals $K_1, K_2 \subseteq V$, open intervals $L_1 \subseteq (0, c)$ and $L_2 \subseteq (c, 1)$, and an $m \in \mathbb{N}$ such that ${T_f}^m$ maps K_1 homeomorphically to L_1 and K_2 homeomorphically to L_2. Since T_f is locally eventually onto there is an open interval $W_1 \subseteq L_1$ and a $k_1 \in \mathbb{N}$ such that ${T_f}^{k_1}$ maps W_1 homeomorphically to $(0, c)$. Analogously there exists an open interval $W_2 \subseteq L_2$ and a $k_2 \in \mathbb{N}$ such that ${T_f}^{k_2}$ maps W_2 homeomorphically to $(c, 1)$. Because of $W_2 \subseteq L_2 \subseteq (c, 1)$ there exists an open interval $I_2 \subseteq W_2 \subseteq L_2$ such that ${T_f}^{k_1 k_2}$ maps I_2 homeomorphically to $(c, 1)$. Using $W_1 \subseteq L_1 \subseteq (0, c)$ we obtain also the existence of an open interval $I_1 \subseteq W_1 \subseteq L_1$ such that ${T_f}^{k_1 k_2}$ maps I_1 homeomorphically to $(0, c)$. As ${T_f}^m$ maps K_1 homeomorphically to L_1 and K_2 homeomorphically to L_2 there are open intervals $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$ such that ${T_f}^m$ maps J_1 homeomorphically to I_1 and J_2 homeomorphically to I_2. Since K_1 and K_2 are disjoint also J_1 and J_2 are disjoint. Setting $n := m + k_1 k_2$ we get that ${T_f}^n$ maps J_1 homeomorphically to $(0, c)$ and J_2 homeomorphically to $(c, 1)$ which completes the proof. $\square$

More or less the same proof works also for strongly locally eventually onto Lorenz maps. One has only to observe that the restriction of ${T_f}^j$ to V_k is continuous for any $k \in \{1, 2\}$ and any $j \in \{0, 1, \ldots, n_k\}$, hence both restrictions of ${T_f}^j$ to K_1 and K_2 are continuous for every $j \in \{0, 1, \ldots, m\}$. Again one obtains that the restriction of ${T_f}^j$ to W_1 is continuous for all $j \in \{0, 1, \ldots, k_1\}$ and the restriction of ${T_f}^j$ to W_2 is continuous for all $j \in \{0, 1, \ldots, k_2\}$, implying that both restrictions of ${T_f}^j$ to I_1 and I_2 are continuous for any $j \in \{0, 1, \ldots, k_1 k_2\}$. This implies that both restrictions of ${T_f}^j$ to J_1 and J_2, respectively, are continuous for all $j \in \{0, 1, \ldots, n\}$.

Therefore we have proved the following result.

Proposition 3.2. *Suppose that T_f is an expanding Lorenz map which is strongly locally eventually onto. Then for every nonempty open set $U \subseteq [0,1]$ there exist disjoint open intervals $J_1 \subseteq U$ and $J_2 \subseteq U$ and there exists an $n \in \mathbb{N}$ such that*

(1) *${T_f}^n$ maps J_1 homeomorphically to $(0,c)$,*

(2) *the restriction of ${T_f}^k$ to J_1 is continuous for every $k \in \{0,1,\dots,n\}$,*

(3) *${T_f}^n$ maps J_2 homeomorphically to $(c,1)$, and*

(4) *the restriction of ${T_f}^k$ to J_2 is continuous for every $k \in \{0,1,\dots,n\}$.*

Our next result shows that for a strongly locally eventually onto Lorenz map every interval of monotonicity must be contained in the image of an interval of monotonicity.

Proposition 3.3. *Let T_f be an expanding Lorenz map. If T_f is strongly locally eventually onto then for every $Z \in \mathcal{Z} := \{(0,c),(c,1)\}$ there exists $Y \in \mathcal{Z}$ with $Z \subseteq T_f Y$.*

Proof. Assume that $Z \in \mathcal{Z}$. Since T_f is strongly locally eventually onto there exists an open interval J and an n such that ${T_f}^n$ maps J homeomorphically to Z and ${T_f}^k$ restricted to J is continuous for all $k \in \{0,1,\dots,n\}$. Hence ${T_f}^{n-1}J$ must be an interval. As ${T_f}^n$ is continuous on J the map T_f must be continuous on ${T_f}^{n-1}J$. Therefore there is a $Y \in \mathcal{Z}$ with ${T_f}^{n-1}J \subseteq Y$ implying $Z = {T_f}^n J = T_f({T_f}^{n-1}J) \subseteq T_f Y$. □

4. Mixing in expanding Lorenz maps

Now we show that every locally eventually onto Lorenz map is mixing.

Theorem 4.1. *Let T_f be an expanding Lorenz map which is locally eventually onto. Then T_f is topologically mixing.*

Proof. Fix any two nonempty open sets U, V. By Proposition 3.1 there exists an $N \in \mathbb{N}$ and open intervals $J_1, J_2 \subseteq U$ such that ${T_f}^N$ maps J_1 homeomorphically to $(0,c)$ and J_2 homeomorphically to $(c,1)$. Hence ${T_f}^N U \supseteq (0,1) \setminus \{c\}$. Since $T_f(0,c) \cup T_f(c,1) \supseteq (0,1)$ we obtain that ${T_f}^n U \supseteq (0,1) \setminus \{c\}$ for any $n \geq N$. Therefore for every $n \geq N$ we have that ${T_f}^n U \cap V \neq \emptyset$ which shows that T_f is topologically mixing. □

As every strongly locally eventually onto Lorenz map is locally eventually onto Theorem 4.1 immediately implies the following result.

Corollary 4.2. *If T_f is a strongly locally eventually onto expanding Lorenz then T_f is topologically mixing.*

The next example shows that topologically mixing Lorenz maps need not be locally eventually onto. To prove mixing of this example we will need some tools developed later in this paper. Nevertheless, we decided to present this example here to highlight differences between considered notions of (strong) mixing.

Example 4.1. Define $f(x) := \frac{3}{2}x + \frac{1}{16}$, and let T_f be the corresponding Lorenz map. Then $c = \frac{5}{8}$. In Figure 1 the graph of T_f is shown. We claim that for any $n \in \mathbb{N}$ there are odd natural numbers a_n, b_n such that ${T_f}^n 0 = \frac{a_n}{2^{n+3}}$ and ${T_f}^n 1 = \frac{b_n}{2^{n+3}}$. Obviously $T_f 0 = \frac{1}{16} = \frac{1}{2^{1+3}}$ and $T_f 1 = \frac{9}{16} = \frac{9}{2^{1+3}}$. Now let $n > 1$, $x \in \{0, 1\}$ and suppose that ${T_f}^{n-1}x = \frac{k}{2^{(n-1)+3}} = \frac{k}{2^{n+2}}$ for some odd k. Note that this implies that ${T_f}^{n-1}x \neq c$. If ${T_f}^{n-1}x < c$ then ${T_f}^n x = \frac{3}{2}{T_f}^{n-1}x + \frac{1}{16} = \frac{3k+2^{n-1}}{2^{n+3}}$ and $3k + 2^{n-1}$ is odd. Otherwise ${T_f}^n x = \frac{3}{2}{T_f}^{n-1}x - \frac{15}{16} = \frac{3k-15\times 2^{n-1}}{2^{n+3}}$ and $3k - 15 \times 2^{n-1}$ is odd, completing the proof of our claim.

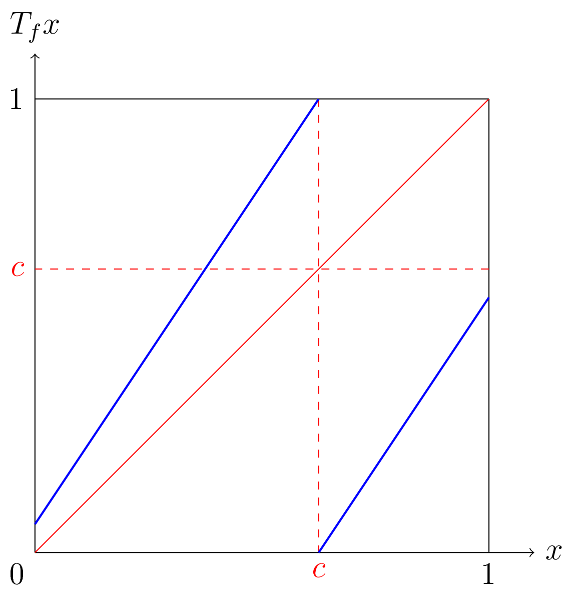

Figure 1: The graph of T_f for f from Example 4.1.

Next we claim that for $x_1, x_2 \in \{0, 1\}$ and $n_1, n_2 \in \mathbb{N}$ we have ${T_f}^{n_1}x_1 \neq {T_f}^{n_2}x_2$ if $x_1 \neq x_2$ or $n_1 \neq n_2$. For $n_1 \neq n_2$ this is obvious, because then

$\frac{k_1}{2^{n_1+3}} = \frac{k_2}{2^{n_2+3}}$ cannot hold for odd k_1, k_2. Hence it remains to prove that ${T_f}^n x_1 \neq {T_f}^n x_2$ for $x_1 \neq x_2$. Suppose that $x_1 \neq x_2$ and ${T_f}^n x_1 = {T_f}^n x_2$ for some $n \in \mathbb{N}$. Let n be the smallest positive integer with this property. Because of $T_f 0 \neq T_f 1$ we must have $n \geq 2$. Then there are different odd numbers k_1, k_2 such that ${T_f}^{n-1} x_1 = \frac{k_1}{2^{n+2}}$ and ${T_f}^{n-1} x_2 = \frac{k_2}{2^{n+2}}$. Without loss of generality we may assume $k_1 < k_2$. This implies ${T_f}^{n-1} x_1 < c < {T_f}^{n-1} x_2$. Therefore ${T_f}^n x_1 = \frac{3k_1 + 2^{n-1}}{2^{n+3}}$ and ${T_f}^n x_2 = \frac{3k_2 - 15 \times 2^{n-1}}{2^{n+3}}$. Since ${T_f}^n x_1 = {T_f}^n x_2$ we obtain $3k_1 + 2^{n-1} = 3k_2 - 15 \times 2^{n-1}$, which implies $3(k_2 - k_1) = 2^{n+3}$. Obviously this is a contradiction (3 does not divide 2^{n+3}), hence our claim is proved.

In order to show that T_f is not locally eventually onto, we assume on the contrary that T_f has this property. Then there exists an open interval J and an $n \geq 1$ such that ${T_f}^n$ maps J homeomorphically to $(0, c)$. Since both $(0, c) \setminus T_f(0, c) \neq \emptyset$ and $(0, c) \setminus T_f(c, 1) \neq \emptyset$ (the first set contains $\big(0, \frac{1}{16}\big)$, the second one contains $\big(\frac{9}{16}, \frac{5}{8}\big)$) we have that ${T_f}^{n-1} J \cap (0, c) \neq \emptyset$ and ${T_f}^{n-1} J \cap (c, 1) \neq \emptyset$. Note that $c \notin {T_f}^{n-1} J$ because $0 \notin {T_f}^n J$. Therefore there must be a $p \in J$ and a $k \in \{0, 1, \ldots, n-2\}$ with ${T_f}^k p = c$. Setting $r := n - k - 1$ we see that $r \geq 1$. Moreover, $\lim_{x \to p^-} {T_f}^n x = {T_f}^r 1$ and $\lim_{x \to p^+} {T_f}^n x = {T_f}^r 0$. As we have shown above that ${T_f}^r 0 \neq {T_f}^r 1$ we see that ${T_f}^n$ is not continuous at $p \in J$ which contradicts the fact that ${T_f}^n$ maps J homeomorphically to $(0, c)$. Hence T_f is not locally eventually onto.

Observe that $f' = \frac{3}{2}$, hence $\inf f' = \frac{3}{2} > \sqrt{2}$. By Theorem 4.6 below (or by Theorem 7.1) this implies that T_f is topologically mixing.

Before we can prove mixing in some Lorenz maps, let us start by recalling a classical result by Hofbauer [15]. We will need only its simplified version as presented in [25].

Lemma 4.3 ([25, Lemma 1])**.** *Assume that T_f is an expanding Lorenz map and let $\mathcal{D}$ be the Markov diagram of T_f. Suppose that $\mathcal{C} \subseteq \mathcal{D}$ is an irreducible and closed graph and that there are $C_1, \ldots, C_n \in \mathcal{C}$ such that $\bigcup_{i=1}^n \overline{C_i} = [0, 1]$. Then $([0, 1], T_f)$ is topologically transitive.*

Now we are ready to prove the following.

Theorem 4.4. *Assume that $f\colon [0, 1] \to [0, 2]$ is continuous and strictly increasing, suppose that $\inf f' > 1$, and let $\mathcal{D}$ be the Markov diagram of T_f. Suppose that $\mathcal{C} \subseteq \mathcal{D}$ is an irreducible and closed graph and that there are $C_1, C_2, \ldots, C_n \in \mathcal{C}$ such that $\bigcup_{j=1}^n \overline{C_j} = [0, 1]$. Furthermore, assume that there are $C \in \mathcal{C}$, $k \geq 2$ and coprime integers $p_1, p_2, \ldots, p_k \geq 1$ such that for every $j \in \{1, 2, \ldots, k\}$ there is a path of length p_j from C to C. Then $([0, 1], T_f)$ is topologically mixing.*

Proof. By Lemma 4.3 the dynamical system $([0,1], T_f)$ is topologically transitive. Let (X, g) be a continuous map on a Cantor set X obtained from $([0,1], T_f)$ by the standard doubling points procedure. Clearly (X, g) is transitive so it contains a residual set R of points with dense orbit. Since at most countably many points are doubled, by a natural identification we may assume that $R \subseteq [0,1]$, for each $x \in R$ we have ${T_f}^r(x) \neq c$ for every $r \geq 0$ and the orbit of x under T_f is dense.

As $([0,1], T_f)$ is transitive, to prove mixing, it suffices to show that for every nonempty open set U there is N such that ${T_f}^r(U) \cap U \neq \emptyset$ for every $r > N$. Fix any open set U. Without loss of generality, we may assume that U is a subset of some element of $\{C_1, C_2, \ldots, C_n\}$, say $U \subseteq C_1$.

Take any integer K such that there is a path $D_0 \to D_1 \to \cdots \to D_K$ in $\mathcal{D}$ with $D_0 \in \mathcal{Z}$ and $D_K = C_1$. Set $A := \bigcap_{j=0}^{K} {T_f}^{-j}(D_j)$, which is obviously open, and fix any $z \in A \cap R$.

Define $\mathcal{Z}_r = \left\{\bigcap_{j=0}^{r} {T_f}^{-j}(Z_j) \colon Z_j \in \mathcal{Z}\right\}$ and observe that ${T_f}^r$ is one-to-one and expanding on each element of $\mathcal{Z}_r$. Let $V_r(z)$ denote the element Z of $\mathcal{Z}_r$ with $z \in Z$. For $r \in \mathbb{N}_0$ let $Z_r \in \mathcal{Z}$ be the element satisfying ${T_f}^r z \in Z_r$. As $z \in A$ we have $D_j \subseteq Z_j$ for $j = 0, 1, \ldots, K$. If $r > K$ then define $D_r := T_f D_{r-1} \cap Z_r$. Then $D_0 \to D_1 \to D_2 \to \cdots$ is an infinite path in $\mathcal{D}$, and for every $r \geq 0$ we have ${T_f}^r z \in D_r \subseteq Z_r$ and $V_r(z) = \bigcap_{j=0}^{r} {T_f}^{-j}(Z_j)$. By [15, Lemma 1] we obtain that ${T_f}^r(V_r(z)) = D_r$. Since T_f is expanding we see that $\bigcap_{j=0}^{\infty} V_j(z) = \{z\}$.

As the orbit of z is dense in $[0,1]$, there is $m > K$ such that ${T_f}^m(z) \in U$. But then, there is $M > m$ such that $V_M(z)$ has sufficiently small diameter to imply ${T_f}^m(V_M(z)) \subseteq U$. Note that by the definition of A we have ${T_f}^K(V_K(z)) = C_1$. But since $\mathcal{C}$ is closed and irreducible, and because of the fact that ${T_f}^M(V_M(z)) \in \mathcal{D}$ we obtain that ${T_f}^M(V_M(z)) \in \mathcal{C}$, say ${T_f}^M(V_M(z)) = D$.

There exists an $L \in \mathbb{N}$ such that for every $r \geq L$ there exists a path of length r from C to C, as there exist paths of coprime length from C to C. Because of the irreducibility of $\mathcal{C}$ there is a path of length q_1 from D to C and a path of length q_2 from C to C_1. Set $N := q_1 + q_2 + L + M - m$. Now let $r \geq N$. Then $r - M + m \geq q_1 + q_2 + L$, hence there exists a path of length $r - M + m$ from D to C_1, which implies $C_1 \subseteq {T_f}^{r-M+m} D$. From this we obtain

$$\begin{aligned} {T_f}^r(U) &\supseteq {T_f}^r\left({T_f}^m\left(V_M(z)\right)\right) = {T_f}^{r-M+m}\left({T_f}^M\left(V_M(z)\right)\right) = \\ &= {T_f}^{r-M+m}(D) \supseteq C_1 \supseteq U \end{aligned}$$

and therefore ${T_f}^r(U) \cap U \neq \emptyset$ completing the proof. □

Theorem 4.5. *If $f\colon [0,1] \ni x \mapsto \sqrt{2}x + \frac{2-\sqrt{2}}{2}$ then $([0,1], T_f)$ is transitive but not mixing.*

Proof. Notice that $T_f 0 = f(0) = \frac{2-\sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} - \sqrt{2}$ and $T_f 1 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$. From Theorem 1 in [24] we get directly that T_f is topological transitive. We set $J := (T_f 0, T_f 1)$, $J_1 := (T_f 0, c)$, $J_2 := (c, T_f 1)$. In this setting we have (see Figure 2):

$$T_f{}^2(J_1) = T_f{}^2(J_2) = J,$$

which implies $\overline{T_f{}^2(J)} = \overline{J}$. For that reason T_f cannot be mixing. □

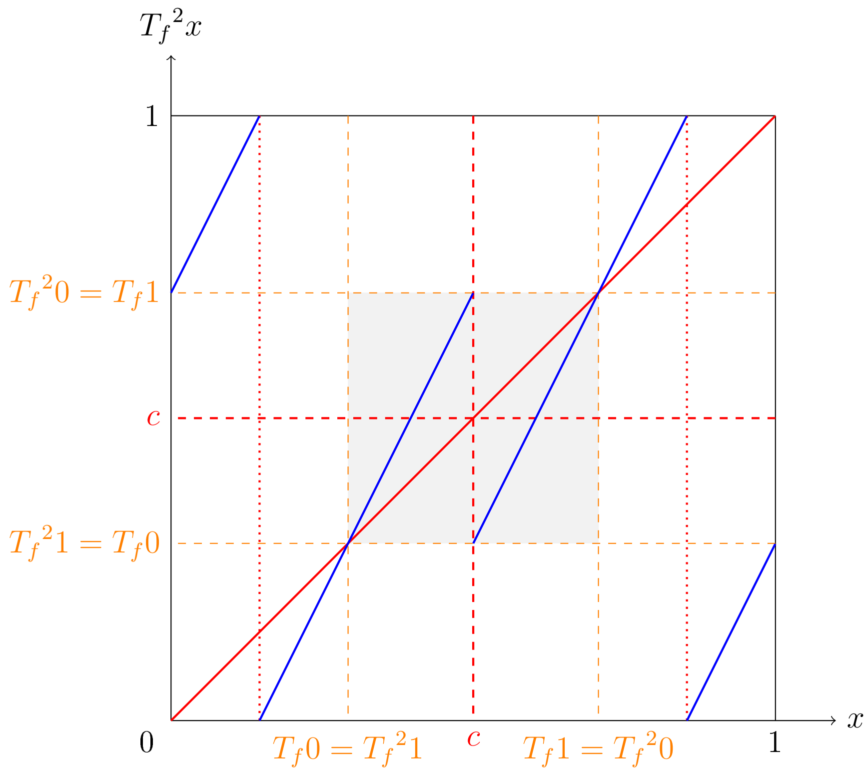

Figure 2: Graph of $T_f{}^2 x$ in the case $f(x) = \sqrt{2}x + \frac{2-\sqrt{2}}{2}$.

Theorem 4.6. *Let T_f be an expanding Lorenz map and assume that $\sqrt{2} \leq \beta \leq 2$, $\inf f' \geq \beta$ and $f(x) \neq \sqrt{2}x + \frac{2-\sqrt{2}}{2}$ for an $x \in [0,1]$. Then T_f is topologically mixing.*

Proof. First we claim that for every $C \in \mathcal{D}$ there is a finite path $C_0 \to C_1 \to \cdots \to C_n$ with $C_0 = C$ and $C_n \in \mathcal{Z}$. Denote $C_0 = C$. By Lemma 2.1 there is a path $C_0 = C \to C_1 \to \cdots \to C_{l_0-1} \to C_{l_0}$ in $\mathcal{D}$ such that C_{l_0-1} has two successors, and therefore $c \in \overline{C_{l_0}}$. Now Lemma 2.10 implies that there is a path $C_{l_0} \to C_{l_0+1} \to \cdots \to C_{l_1}$ in $\mathcal{D}$ such that $c \in \overline{C_{l_1}}$ and either $C_{l_1} \in \mathcal{Z}$ or $|C_{l_1}| \geq \sqrt{2}|C_0|$. In the latter case, we can apply Lemma 2.10 again obtaining a path $C_{l_1} \to C_{l_1+1} \to \cdots \to C_{l_2}$ such that $c \in \overline{C_{l_2}}$ and either $C_{l_2} \in \mathcal{Z}$ or $|C_{l_2}| \geq \sqrt{2}|C_{l_1}| \geq (\sqrt{2})^2|C_0|$. Since the diameter of any element of $\mathcal{D}$ is bounded from the above by 1, applying Lemma 2.10 a finite number of times we eventually construct a path from C to an element of $\mathcal{Z}$. The claim is proved.

Let $r = r(f)$ be provided by Lemma 2.4. We will consider a few cases.

Case 1. $\mathbf{T_f 0 \geq c}$. Directly by the definition we see that $r \geq 2$. By Lemma 2.4 we have $c \leq T_f 0 < T_f 1$ and

$$c \leq {T_f}^{r-1}0 < \cdots < {T_f}^2 0 < T_f 0$$

and ${T_f}^j 0 < {T_f}^j 1$ for $j \in \{1, \ldots, r\}$. Therefore $(0, c)$ is a successor of $(c, 1)$, $(0, c)$ has the unique successor $(T_f 0, 1)$, and $({T_f}^j 0, {T_f}^{j-1} 1)$ has the unique successor $({T_f}^{j+1} 0, {T_f}^j 1)$ for $j \in \{1, 2, \ldots, r-2\}$. Moreover, $(c, {T_f}^j 1)$ has the successors $(0, c)$ and $(c, {T_f}^{j+1} 1)$ for $j \in \{0, 1, \ldots, r-2\}$. Define $A := (0, {T_f}^r 1) \cap (0, c)$. Then $(c, {T_f}^{r-1} 1)$ has A as a successor, and $({T_f}^{r-1} 0, {T_f}^{r-2} 1)$ has $(c, {T_f}^{r-1} 1)$ as a successor. Furthermore, because there is a path $(c, 1) \to (0, c)$, by previous considerations for every $C \in \mathcal{D}$ there is a finite path $C_0 \to C_1 \to \cdots \to C_n$ with $C_0 = C$ and $C_n = (0, c)$.

We will consider two cases depending on the value of r.

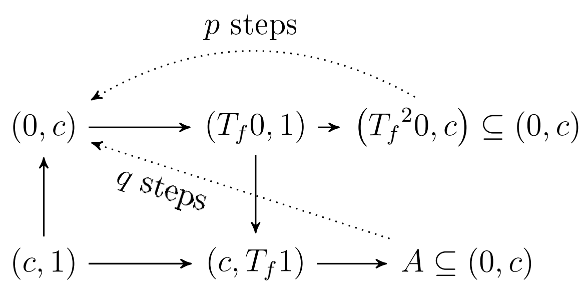

Figure 3: Part of the graph $\mathcal{D}$ in the case 1(a) in Theorem 4.6

(a) $\mathbf{r = 2}$. In this case ${T_f}^2 0 < c < T_f 0$. Then either $A = (0, c)$ is a successor of $(c, T_f 1)$ or ${T_f}^2 1 \leq c$ and $A = (0, {T_f}^2 1)$ is the unique successor of

$(c, T_f 1)$. By the above observations and Lemma 2.2 there are path in $\mathcal{D}$ from $(0, c)$ to itself of lengths (see Figure 3):

$$p, p+2, q, q+3 \ .$$

Clearly these numbers are coprime. Furthermore

$$[0,1] = [0,c] \cup [c, T_f 1] \cup [T_f 0, 1]$$

which by Theorem 4.4 implies that $([0,1], T_f)$ is mixing, completing proof of this case.

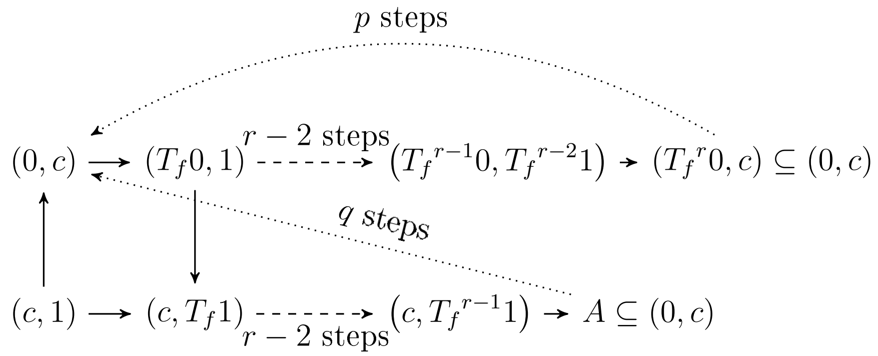

Figure 4: Part of the graph $\mathcal{D}$ in the case 1(b) in Theorem 4.6

(b) $\mathbf{r > 2}$. Similarly to the previous case, $A = (0, c)$ is a successor of $(c, {T_f}^{r-1} 1)$ or ${T_f}^{r-1} 1 \leq c$ and $A = (0, {T_f}^r 1)$ is the unique successor of $(c, {T_f}^{r-1} 1)$. This shows that graph presented on Figure 4 is a subgraph of Markov diagram for T_f. Then using Lemma 2.2 we obtain paths from $(0, c)$ to itself of lengths:

$$p, p+r, q, q+r+1 \ ,$$

which again are coprime numbers and we also have that

$$[0,1] = [0,c] \cup \left[c, {T_f}^{r-1} 1\right] \cup [T_f 0, 1] \cup \bigcup_{j=3}^{r} \left[{T_f}^{j-1} 0, {T_f}^{j-2} 1\right] \ .$$

This completes the proof of case 1(b).

Case 2. $\mathbf{T_f 0 < c < T_f 1}$. It is not hard to see that in this case Markov diagram of T_f contains the paths $(0, c) \to (c, 1)$ and $(c, 1) \to (0, c)$. Using

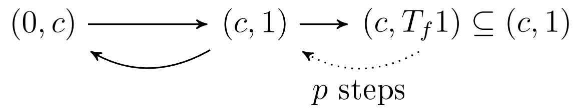

Figure 5: Part of the graph $\mathcal{D}$ in the case 2 in Theorem 4.6

Lemma 2.10 we obtain that there is $p > 0$ and a path of length p from $(c, T_f 1)$ to $(c, 1)$. But $(c, T_f 1) \subseteq (c, 1)$, hence by Lemma 2.2 there is also a path of length p from $(c, 1)$ to itself, as depicted on Figure 5. Clearly $[0, 1] = [0, c] \cup [c, 1]$ and by the above arguments we see that there are paths from $(c, 1)$ to itself of lengths:

$$2, p, p+1 \ ,$$

which ends the proof of this case by Theorem 4.4.

Case 3. $\mathbf{T_f 1 \leq c}$**.** By Proposition 2.11 in this case T_f is conjugate to a map T_g satisfying the assumptions of Case 1. Therefore we obtain that T_f is topologically mixing from Case 1. $\square$

Theorem 4.7. *If* $f\colon [0, 1] \ni x \mapsto \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ *then* $([0, 1], T_f)$ *is topologically transitive but not topologically mixing.*

Proof. The transitivity of T_f has been shown in [25, Lemma 4]. Denote $J := \left({T_f}^2 0, T_f 0\right)$ and observe that $T_f 0 > c$ and there is a unique point $\hat{c} \in (c, 1)$ such that $T_f \hat{c} = c$. Denote $J_1 := \left({T_f}^2 0, c\right)$, $J_2 := (c, T_f 0)$ and $K := (T_f 0, T_f 1)$, $K_1 := (T_f 0, \hat{c})$, $K_2 := (\hat{c}, T_f 1)$. Then we obtain (see Figure 6):

$$\begin{aligned} {T_f}^3 (J_1) &= {T_f}^3 (J_2) = J \subseteq J_1 \cup J_2, \\ {T_f}^3 (K_1) &= {T_f}^3 (K_2) = K \subseteq K_1 \cup K_2. \end{aligned}$$

Moreover, $\overline{{T_f}^3(K)} = \overline{K}$ and $\overline{{T_f}^3(J)} = \overline{J}$. Hence T_f is not mixing. $\square$

Theorem 4.8. *Let* T_f *be an expanding Lorenz map such that* $\inf f' \geq \beta$ *and* $f(0) \geq \frac{1}{\beta+1}$*, where* $\sqrt[3]{2} \leq \beta < \sqrt{2}$*, and* $f(x) \neq \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ *for an* $x \in [0, 1]$*. Then* T_f *is topologically mixing.*

Proof. Let $r = r(f)$ be provided by Lemma 2.4. We will consider three cases depending on values of ${T_f}^2$ at endpoints 0 and 1. Since $f(c) = 1 \geq f(0) + \beta c$, it is not hard to verify that $c \leq \frac{1}{1+\beta}$. Observe that $r \geq 2$ (see Remark 2.3) and ${T_f}^2 0 < {T_f}^2 1$ (see Lemma 2.4).

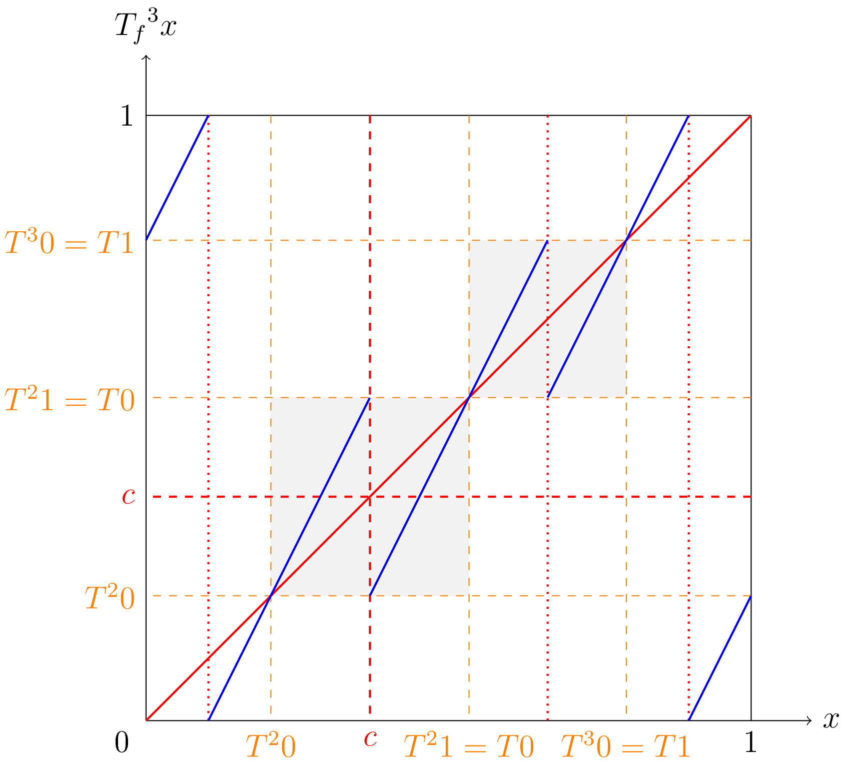

Figure 6: Graph of $T_f{}^3x$ in the case $f(x) = \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$.

Case 1. $\mathbf{T_f{}^21 < c}$. Our assumptions imply that $T_f{}^20 < T_f{}^21 < c \leq T_f0 < T_f1$ and $r = 2$. Note that $f(x) = T_f(x)$ for each $x \in (0, c)$, thus:

$$
\begin{aligned}
T_f{}^31 &= T_f\left(T_f{}^21\right) = f\left(T_f{}^21\right) > f(0) = T_f0 \geq c, \\
T_f{}^30 &= T_f\left(T_f{}^20\right) = f\left(T_f{}^20\right) \geq f(0) = T_f0 \geq c.
\end{aligned} \tag{4.1}
$$

As $T_f{}^20 < T_f{}^21$ and T_f is strictly increasing on $(0, c)$ we get $T_f{}^30 < T_f{}^31$. Moreover, from (4.1) we have $c \leq T_f{}^30 < T_f{}^31$. Note that $f(0) = T_f0$ and $f(1) = T_f1 + 1$, hence $T_f1 + 1 - T_f0 = f(1) - f(0) \geq \beta$ which gives $|(c, T_f1)| \geq |(T_f0, T_f1)| \geq \beta - 1$. Since (c, T_f1) and $(0, T_f{}^21)$ have uniques successors, we obtain that

$$
\left|\left(T_f0, T_f{}^31\right)\right| = \left|T_f{}^2\left(c, T_f1\right)\right| \geq \beta^2(\beta - 1),
$$

$$
T_f{}^41 = \left|\left(0, T_f{}^41\right)\right| \geq \left|T_f\left(T_f0, T_f{}^31\right)\right| \geq \beta^3(\beta - 1) > \frac{1}{\beta + 1} \geq c
$$

because $x^5 - x^3 - 1 > 0$ for $x \geq \sqrt[3]{2}$. Furthermore, since $c < T_f{}^31 < 1$ and T_f is monotone on $(c, 1)$ we obtain $c < T_f{}^41 < T_f1$.

To finish the proof of this case, let us consider two possible values of $T_f{}^40$.

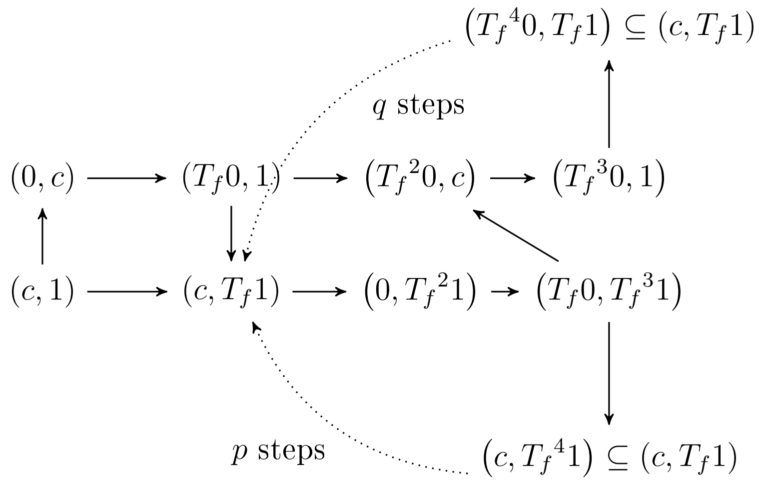

Figure 7: Part of the graph $\mathcal{D}$ in the case 1(a) in Theorem 4.8

(a) $\mathbf{T_f{}^4 0 \geq c}$. In this case $\left(T_f{}^4 0, T_f 1\right)$ is the unique successor of $(T_f{}^3 0, 1)$ and $\left(T_f{}^4 0, T_f 1\right) \subseteq (c, T_f 1)$. By Lemma 2.8 there exists a finite path from any $C \in \mathcal{D}$ to a vertex $C_n \in \{(0, c), (c, T_f 1)\}$. In particular there are integers $p, q > 0$ such that there is a path from $(c, T_f{}^4 1)$ to $(c, T_f 1)$ of length p, and a path of length q form $(T_f{}^4 0, T_f 1)$ to $(c, T_f 1)$. Then the graph presented on Figure 7 is a subgraph of $\mathcal{D}$. In particular starting from vertex $(c, T_f 1)$ we can return to it following paths of length:

$$p + 3, q + 5.$$

Observe that $(c, T_f{}^4 1) \subseteq (c, T_f 1)$ hence either there is a path of length p from $(c, T_f 1)$ to $(c, T_f 1)$ or there is an a with $T_f 1 \leq a \leq 1$ and a path of length p from $(c, T_f 1)$ to (c, a). This vertex has as a successor $(0, c)$ or $(0, T_f a)$ depending whether $T_f a > c$ or not. In any case we see that there is a path of length $p + 3$ from $(c, T_f 1)$ to $(T_f{}^2 0, c)$ and so we have a path of length $p + q + 5$ from $(c, T_f 1)$ to $(c, T_f 1)$. By an analogous argument we see that either we have a path of length q from $(c, T_f 1)$ to itself, or there is a path of length $q + 3$ from $(c, T_f 1)$ to $(T_f{}^2 0, c)$ and as a consequence, there is a path from $(c, T_f 1)$ to itself of length $2q + 5$.

Then we have the following four possible combinations of lengths of paths from $(c, T_f 1)$ to itself:

$$(p, q, p + 3, q + 5),\ (p + 3, q, p + q + 5, q + 5),$$
$$(p, q + 5, p + 3, 2q + 5),\ (p + 3, q + 5, p + q + 5, 2q + 5).$$

Clearly in any of the above four cases these lengths are coprime numbers and furthermore

$$[0,1] = \left[0, {T_f}^2 1\right] \cup \left[{T_f}^2 0, c\right] \cup [c, T_f 1] \cup [T_f 0, 1] \,.$$

The proof of this case is finished by Theorem 4.4.

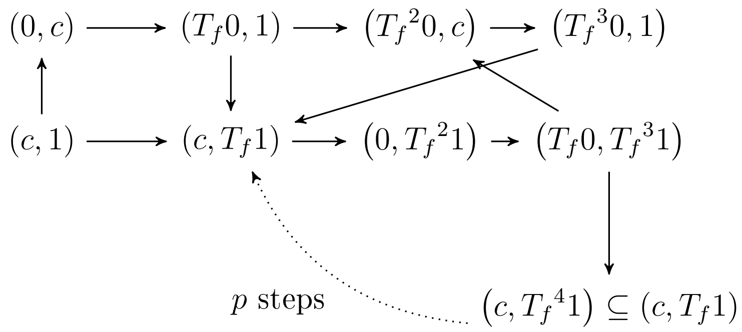

Figure 8: Part of the graph $\mathcal{D}$ in the case 1(b) in Theorem 4.8

(b) $\mathbf{{T_f}^4 0 < c}$. In this case $({T_f}^3 0, 1)$ has the two successors $(c, T_f 1)$ and $\left({T_f}^4 0, c\right)$. By Lemma 2.8 there is an integer $p > 0$ and path from $(c, {T_f}^4 1)$ to $(c, T_f 1)$ of length p. The graph obtained in this case is presented in Figure 8. Note that in this case we have paths from $(c, T_f 1)$ to itself of lengths:

$$5, p+3.$$

Furthermore, since $(c, {T_f}^4 1) \subseteq (c, T_f 1)$, repeating the arguments used in the previous case, we see that either there is a path from $(c, T_f 1)$ to itself of length p or there is a path from $(c, T_f 1)$ to $({T_f}^2 0, c)$ of length $p+3$ which easily extends to a path from $(c, T_f 1)$ to itself of length $p+5$. Then we have two possible sets of lengths of paths from $(c, T_f 1)$ to itself:

$$(5, p+3, p) \quad \text{or} \quad (5, p+3, p+5).$$

In both cases these three lengths are coprime numbers and therefore the proof of this case follows by Theorem 4.4, because

$$[0,1] = \left[0, {T_f}^2 1\right] \cup \left[{T_f}^2 0, c\right] \cup [c, T_f 1] \cup [T_f 0, 1] \,.$$

Case 2. $\mathbf{{T_f}^2 0 \geq c}$. In this case we clearly have $r \geq 3$, which by Lemma 2.4 implies that ${T_f}^r 0 < c \leq {T_f}^{r-1} 0 < \cdots < {T_f}^2 0 < T_f 0$ and ${T_f}^j 0 < {T_f}^j 1$ for $j \in \{1, 2, \ldots, r\}$. Observe that $T_f 0 < T_f 1 \leq 1$, T_f is strictly

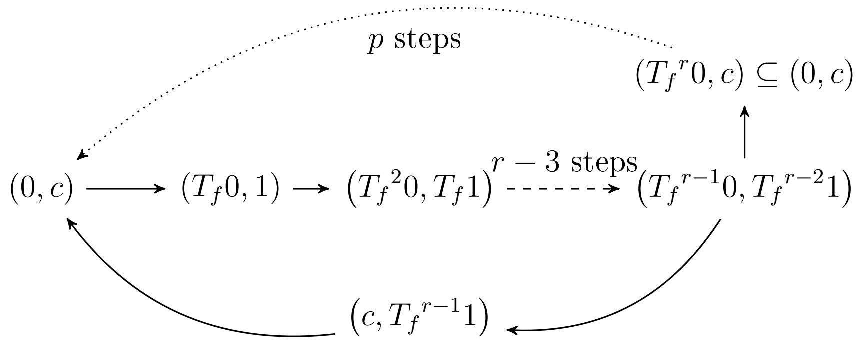

Figure 9: Part of the graph $\mathcal{D}$ in the case 2 in Theorem 4.8

increasing on $(c,1)$ and $c \le T_f{}^2 0$, which gives $c \le T_f{}^2 0 < T_f 1$. Repeating these arguments we obtain that $c \le T_f{}^j 0 < T_f{}^{j-1} 1$ for $j = 2, \dots, r-1$ and we also have $T_f{}^r 0 < c < T_f{}^{r-1} 1$. Applying Lemma 2.8 to the interval $(T_f{}^r 0, c)$ we obtain a path from $(T_f{}^r 0, c)$ to $(0,c)$ of length $p > 0$. By Lemma 2.2 there is also a path from $(0,c)$ to $(0,c)$ of length p. Moreover, Lemma 2.8 gives that $T_f{}^r 1 > c$. Hence $(0,c)$ is a successor of $(c, T_f{}^{r-1} 1)$ and the Markov diagram of T_f contains the graph presented on Figure 9. Therefore we have paths from $(0,c)$ to itself of lengths:

$$p, p+r, r+1$$

which are clearly coprime numbers. To complete the proof in this case 2 it is enough to note that we have

$$[0,1] = [0,c] \cup \left[c, T_f{}^{r-1} 1\right] \cup [T_f 0, 1] \cup \bigcup_{j=1}^{r-2} \left[T_f{}^{j+1} 0, T_f{}^j 1\right]$$

and Theorem 4.4 implies that T_f is mixing.

Case 3. $\mathbf{T_f{}^2 0 < c \le T_f{}^2 1}$**.** We have $r = 2$ and by Lemma 2.4 we additionally know that $T_f{}^2 0 < c \le T_f 0 < T_f 1$. Observe that $(T_f 0, 1)$ has two successors: $(T_f{}^2 0, c)$ and $(c, T_f 1)$. By our assumptions $(0,c)$ is a successor of $(c, T_f 1)$ (not necessarily unique), and by Lemma 2.8 there exists a finite path of length $p > 0$ from $(T_f{}^2 0, c)$ to $C \in \{(0,c), (c, T_f 1)\}$, so in fact to $(0,c)$ as easily seen on Figure 10 (we rename $p+1$ by p if necessary). Using Lemma 2.2 we obtain also a path of length p from $(0,c)$ to itself. Starting at $(0,c)$ we can return to this vertex along paths with length:

$$3, p, p+2$$

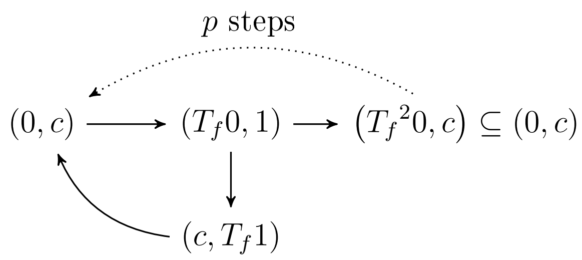

Figure 10: Part of the graph $\mathcal{D}$ in the case 3 in Theorem 4.8

which are coprime numbers. Now it is enough to observe

$$[0,1] = [0,c] \cup [c, T_f 1] \cup [T_f 0, 1]$$

and apply Theorem 4.4 to complete the proof also in this case.

All three cases considered above exhaust all possibilities. The proof of Theorem 4.8 is completed. $\square$

5. A renormalizable locally eventually onto expanding Lorenz map

The aim of this section is to show that there is an expanding Lorenz map which is at the same time:

1. renormalizable,
2. locally eventually onto,
3. not strongly locally eventually onto.

As we will see, all these three properties are satisfied by the map defined in Example 5.1.

Example 5.1. Let T_f be the expanding Lorenz map induced by $f(x) = \beta x + \alpha$ satisfying:

1. $f^4(0) = 1$,
2. $f(1) - 1 = f^2(0)$.

This leads to the equations

$$\alpha(\beta^3+\beta^2+\beta+1)=1 \quad \text{and} \quad \beta-1=\beta\alpha$$

which can be reduced to

$$\alpha = 1-\frac{1}{\beta} \quad \text{and} \quad \beta^4-\beta-1=0.$$

Hence β is the largest zero of the polynomial x^4-x-1, which means

$$\begin{aligned}\beta &= \frac{1}{2}\sqrt{-\frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{849}\right)}}{3^{2/3}}+4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}+\frac{2}{\sqrt{\frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{849}\right)}}{3^{2/3}}-4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}}}} \\ &\quad +\frac{1}{2}\sqrt{\frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{849}\right)}}{3^{2/3}}-4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}} \\ &\approx 1.2207440846\,.\end{aligned}$$

Moreover, we obtain that $c=\frac{1}{\beta^2}$. In Figure 11 the graph of T_f and in Figure 14 the Markov diagram of T_f are shown.

First we will show that T_f is locally eventually onto. For $j\in\{0,1,\ldots,12\}$ set

$$\begin{aligned} Z(0,j) &:= \begin{cases}(c,1)\,, & \text{if } j\equiv 2 \pmod 3,\\ (0,c)\,, & \text{otherwise,}\end{cases} \\ Z(1,j) &:= \begin{cases}(c,1)\,, & \text{if } j\equiv 1 \pmod 4,\\ (0,c)\,, & \text{otherwise.}\end{cases}\end{aligned} \tag{5.1}$$

Furthermore for $p\in\{0,1\}$ define

$$\begin{aligned} C(p,0) &:= Z(p,0) = (0,c)\,,\\ C(p,j) &:= T_f C(p,j-1)\cap Z(p,j) \quad \text{for } j\in\{1,2,\ldots,12\}, \text{ and}\\ V(p,j) &:= \bigcap_{k=0}^{j} {T_f}^{-k} Z(p,k) \quad \text{for } j\in\{0,1,\ldots,12\}.\end{aligned} \tag{5.2}$$

In order to prove the above statement we need the following.

Lemma 5.1. *The following properties are satisfied.*

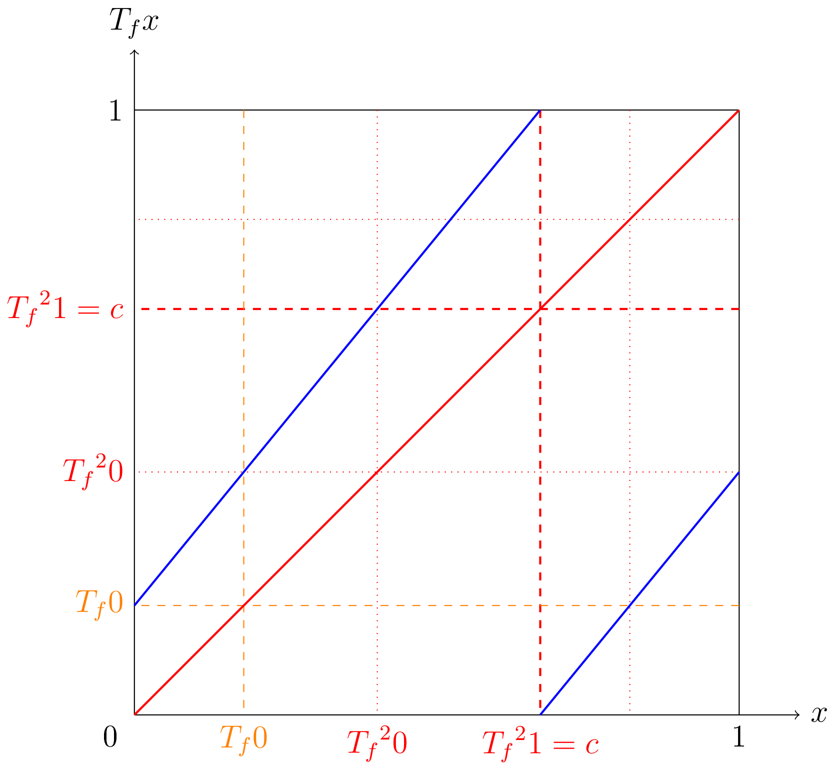

Figure 11: Graph of $T_f x$ in the case $f(x)$ from Example 5.1.

(1) *$V(p,j) \neq \emptyset$ for any $p \in \{0,1\}$ and $j \in \{0,1,\dots,12\}$.*

(2) *${T_f}^j$ maps $V(p,j)$ homeomorphically to $C(p,j)$ for any $p \in \{0,1\}$ and $j \in \{0,1,\dots,12\}$.*

(3) *For $j \in \{1,2,\dots,12\}$ we have* $C(0,j) = \begin{cases} (T_f 0, c)\ , & \textit{if } j \equiv 1 \pmod 3, \\ (c,1)\ , & \textit{if } j \equiv 2 \pmod 3, \\ (0, {T_f}^2 0)\ , & \textit{if } j \equiv 0 \pmod 3. \end{cases}$

(4) *For $j \in \{1,2,\dots,12\}$ we have* $C(1,j) = \begin{cases} (c,1)\ , & \textit{if } j \equiv 1 \pmod 4, \\ (0, {T_f}^2 0)\ , & \textit{if } j \equiv 2 \pmod 4, \\ (T_f 0, c)\ , & \textit{if } j \equiv 3 \pmod 4, \\ ({T_f}^2 0, c)\ , & \textit{if } j \equiv 0 \pmod 4. \end{cases}$

(5) *If $j \in \{1,2,\dots,12\}$ then $\sup V(0,j) = \inf V(1,j) = {T_f}^2 0$.*

Proof. First we deal with the case $j = 0$. It is obviously true that $V(p,0) = Z(p,0) = C(p,0) = (0,c) \neq \emptyset$ for $p \in \{0,1\}$.

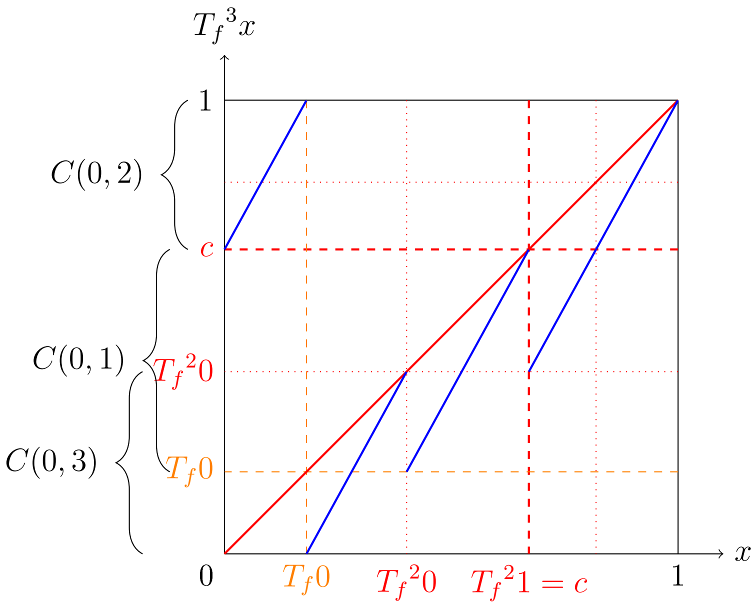

Figure 12: Graph of $T_f{}^3x$ in the case $f(x)$ from Example 5.1.

Now let $j \in \{1, 2, \ldots, 12\}$. This part of the lemma we prove by induction. First we consider the case $j = 1$. In this case $T_f C(p, 0) = T_f(0, c) = (T_f 0, 1)$ and T_f maps $V(p, 0)$ homeomorphically to $(T_f 0, 1)$. As $Z(0, 1) = (0, c)$ we get $C(0, 1) = (T_f 0, c)$, $V(0, 1) = (0, T_f{}^2 0) \neq \emptyset$, $\sup V(0, 1) = T_f{}^2 0$ and the restriction of T_f to $V(0, 1)$ is continuous and strictly increasing. Because of

$$T_f\left(V(0,1)\right) = T_f\left(C(0,0) \cap T_f{}^{-1} Z(0,1)\right) = T_f C(0,0) \cap Z(0,1) = C(0,1)$$

we get that T_f maps $V(0, 1)$ homeomorphically to $C(0, 1)$. Similarly $Z(1, 1) = (c, 1)$ implies that $C(1, 1) = (c, 1)$, $V(1, 1) = (T_f{}^2 0, c) \neq \emptyset$, $\inf V(1, 1) = T_f{}^2 0$ and T_f restricted to $V(1, 1)$ is continuous and strictly increasing. Now

$$T_f\left(V(1,1)\right) = T_f\left(C(1,0) \cap T_f{}^{-1} Z(1,1)\right) = T_f C(1,0) \cap Z(1,1) = C(1,1)$$

implies that T_f maps $V(1, 1)$ homeomorphically to $C(1, 1)$.

Assume that $j > 1$. By induction hypothesis $V(p, j-1) \neq \emptyset$ and $T_f{}^{j-1}$ maps $V(p, j-1)$ homeomorphically to $C(p, j-1)$ for $p \in \{0, 1\}$. We have to consider different cases depending on values of p and j. To start assume that $p = 0$.

If $j \equiv 2 \pmod 3$ then T_f maps $C(0, j-1) = (T_f 0, c)$ homeomorphically to $(T_f{}^2 0, 1)$ which has nonempty intersection with $Z(0, j) = (c, 1)$. Hence $V(0, j) = V(0, j-1) \cap T_f{}^{-j} Z(0, j) \neq \emptyset$ and the restriction of $T_f{}^j$ to $V(0, j)$ is continuous and strictly increasing. As $\lim_{x \to T_f{}^2 0-} T_f{}^{j-1} x = c$ we

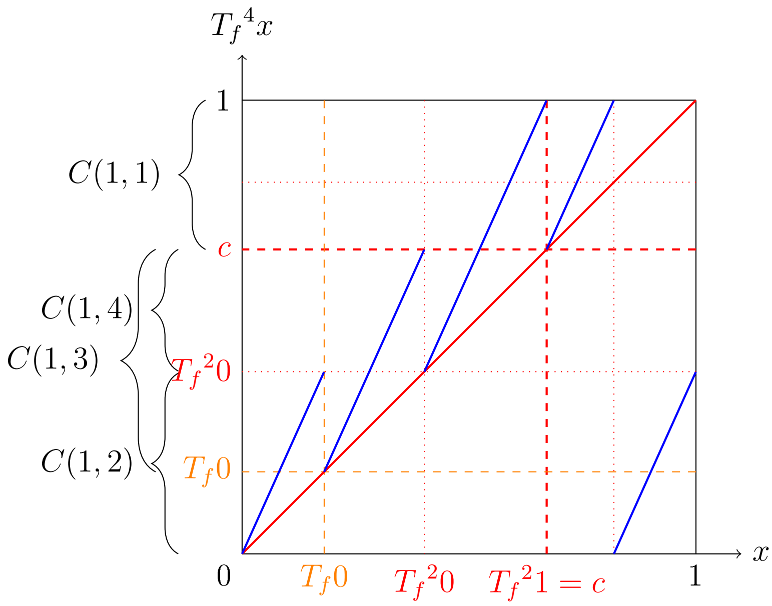

Figure 13: Graph of $T_f{}^4 x$ in the case $f(x)$ from Example 5.1.

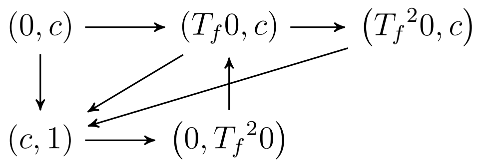

Figure 14: The Markov diagram $\mathcal{D}$ of T_f where $f(x)$ is from Example 5.1.

get $\lim_{x \to T_f{}^2 0^-} T_f{}^j x = 1$ which implies that the right endpoint of $V(0,j)$ is $T_f{}^2 0$. Observing that

$$\begin{aligned} T_f{}^j V(0,j) &= T_f{}^j \big(V(0,j-1) \cap T_f{}^{-j} Z(0,j)\big) = T_f \big(T_f{}^{j-1} V(0,j-1)\big) \cap Z(0,j) \\ &= T_f C(0,j-1) \cap Z(0,j) = C(0,j) = (c,1) \end{aligned}$$

this completes the proof in this case. Next suppose that $j \equiv 0 \pmod 3$ then T_f maps $C(0,j-1) = (c,1)$ homeomorphically to $(0, T_f{}^2 0)$ which is contained in $Z(0,j) = (0,c)$. Therefore $V(0,j) = V(0,j-1) \neq \emptyset$, $\sup V(0,j) = T_f{}^2 0$ and $T_f{}^j$ restricted to $V(0,j)$ is continuous and strictly increasing. Furthermore

$$\begin{aligned} T_f{}^j V(0,j) &= T_f{}^j \big(V(0,j-1) \cap T_f{}^{-j} Z(0,j)\big) = T_f \big(T_f{}^{j-1} V(0,j-1)\big) \cap Z(0,j) \\ &= T_f C(0,j-1) \cap Z(0,j) = C(0,j) = (0, T_f{}^2 0) \end{aligned}$$

proving the lemma in this case. In order to finish the case $p = 0$ it remains to assume that $j \equiv 1 \pmod 3$. Then T_f maps $C(0, j-1) = (0, {T_f}^2 0)$ homeomorphically to $(T_f 0, c)$ which is contained in $Z(0, j) = (0, c)$. We obtain that $V(0, j) = V(0, j-1) \neq \emptyset$, $\sup V(0, j) = {T_f}^2 0$ and ${T_f}^j$ restricted to $V(0, j)$ is continuous and strictly increasing. Since

$$\begin{aligned} {T_f}^j V(0,j) &= {T_f}^j \left(V(0,j-1) \cap {T_f}^{-j} Z(0,j) \right) = T_f \left({T_f}^{j-1} V(0,j-1) \right) \cap Z(0,j) \\ &= T_f C(0,j-1) \cap Z(0,j) = C(0,j) = (T_f 0, c) \end{aligned}$$

the lemma is proved also in this case.

Finally, we have to consider the case $p = 1$. Assume at first that $j \equiv 2 \pmod 4$. Then T_f maps $C(1, j-1) = (c, 1)$ homeomorphically to $(0, {T_f}^2 0)$ which is contained in $Z(1, j) = (0, c)$. Hence $V(1, j) = V(1, j-1) \neq \emptyset$, $\inf V(1, j) = {T_f}^2 0$ and ${T_f}^j$ restricted to $V(1, j)$ is continuous and strictly increasing. Observing that

$$\begin{aligned} {T_f}^j V(1,j) &= {T_f}^j \left(V(1,j-1) \cap {T_f}^{-j} Z(1,j) \right) = T_f \left({T_f}^{j-1} V(1,j-1) \right) \cap Z(1,j) \\ &= T_f C(1,j-1) \cap Z(1,j) = C(1,j) = (0, {T_f}^2 0) \end{aligned}$$

this finishes the proof in this case. Next suppose that $j \equiv 3 \pmod 4$. Then T_f maps $C(1, j-1) = (0, {T_f}^2 0)$ homeomorphically to $(T_f 0, c)$ which is contained in $Z(1, j) = (0, c)$. We get that $V(1, j) = V(1, j-1) \neq \emptyset$, $\inf V(1, j) = {T_f}^2 0$ and ${T_f}^j$ restricted to $V(1, j)$ is continuous and strictly increasing. As

$$\begin{aligned} {T_f}^j V(1,j) &= {T_f}^j \left(V(1,j-1) \cap {T_f}^{-j} Z(1,j) \right) = T_f \left({T_f}^{j-1} V(1,j-1) \right) \cap Z(1,j) \\ &= T_f C(1,j-1) \cap Z(1,j) = C(1,j) = (T_f 0, c) \end{aligned}$$

the lemma is shown in this case. Assume that $j \equiv 0 \pmod 4$. In this case T_f maps $C(1, j-1) = (T_f 0, c)$ homeomorphically to $({T_f}^2 0, 1)$ which has nonempty intersection with $Z(1, j) = (0, c)$. Therefore $V(1, j) = V(1, j-1) \cap {T_f}^{-j} Z(1, j) \neq \emptyset$ and ${T_f}^j$ restricted to $V(1, j)$ is continuous and strictly increasing. As $\lim_{x \to {T_f}^2 0^+} {T_f}^{j-1} x = T_f 0$ we get $\lim_{x \to {T_f}^2 0^+} {T_f}^j x = {T_f}^2 0$ implying that the left endpoint of $V(1, j)$ is ${T_f}^2 0$. Moreover

$$\begin{aligned} {T_f}^j V(1,j) &= {T_f}^j \left(V(1,j-1) \cap {T_f}^{-j} Z(1,j) \right) = T_f \left({T_f}^{j-1} V(1,j-1) \right) \cap Z(1,j) \\ &= T_f C(1,j-1) \cap Z(1,j) = C(1,j) = ({T_f}^2 0, c) \end{aligned}$$

proving the lemma in this case. It remains to consider the case $j \equiv 1 \pmod 4$. Then T_f maps $C(1, j-1) = ({T_f}^2 0, c)$ homeomorphically to $(c, 1)$ which is contained in $Z(1, j) = (c, 1)$. Therefore $V(1, j) = V(1, j-1) \neq \emptyset$,

$\inf V(1,j) = {T_f}^2 0$ and the restriction of ${T_f}^j$ to $V(1,j)$ is continuous and strictly increasing. Since

$$\begin{aligned} {T_f}^j V(1,j) &= {T_f}^j \left(V(1,j-1) \cap {T_f}^{-j} Z(1,j)\right) = T_f \left({T_f}^{j-1} V(1,j-1)\right) \cap Z(1,j) \\ &= T_f C(1,j-1) \cap Z(1,j) = C(1,j) = (c,1) \end{aligned}$$

this completes the proof. □

Setting $a := \inf V(0,12)$ and $b := \sup V(1,12)$ we get that $V(0,12) = (a, {T_f}^2 0) \neq \emptyset$, $V(1,12) = ({T_f}^2 0, b) \neq \emptyset$, ${T_f}^{12}$ maps $V(0,12)$ homeomorphically to $(0, {T_f}^2 0)$ and ${T_f}^{12}$ maps $V(1,12)$ homeomorphically to $({T_f}^2 0, c)$. As ${T_f}^{12}({T_f}^2 0) = {T_f}^2 0$ this implies that ${T_f}^{12}$ maps (a,b) homeomorphically to $(0,c)$. Furthermore $(a,b) = V(0,12) \cup V(1,12) \cup \left\{{T_f}^2 0\right\}$. Assume that $a < T_f 0$. Then $T_f a < {T_f}^2 0$ and ${T_f}^2 a < c$ which implies ${T_f}^2 V(0,12) \cap (0,c) \neq \emptyset$ contradicting ${T_f}^2 V(0,12) \subseteq (c,1)$, moreover, we have

$$V(1,12) = Z(1,0) \cap \bigcap_{k=1}^{12} {T_f}^{-k} Z(1,k) = (0,c) \cap \bigcap_{k=1}^{12} {T_f}^{-k} Z(1,k) \subseteq (0,c).$$

Hence $a \geq T_f 0$ and $b \leq c$, so $(a,b) \subseteq (T_f 0, c)$.

Let U be a nonempty open set. Then there exists an $x \in U \setminus \left(\bigcup_{k=0}^{\infty} {T_f}^{-k}\{c\}\right)$. For $k \in \mathbb{N}_0$ let $Z_k(x) \in \mathcal{Z} := \{(0,c),(c,1)\}$ be so that ${T_f}^k x \in Z_k(x)$ and set $V_k(x) := \bigcap_{j=0}^{k} {T_f}^{-j} Z_j(x)$ (We have already used this notation in proof of Theorem 4.4). Note that $V_k(x)$ is an interval for all $k \in \mathbb{N}_0$. Defining $D_0(x) := Z_0(x)$ and $D_k(x) := T_f D_{k-1}(x) \cap Z_k(x)$ for $k \in \mathbb{N}$ we see that $D_0(x) \to D_1(x) \to \cdots$ is a path in the Markov diagram of T_f. By Lemma 1 of [15] we get that ${T_f}^k$ maps $V_k(x)$ homeomorphically to $D_k(x)$ for every $k \in \mathbb{N}_0$ (observe that ${T_f}^k$ is continuous and strictly increasing on $V_k(x)$ as ${T_f}^j V_k(x) \subseteq D_j(x) \subseteq Z_j(x)$ for all $j \in \{0,1,\ldots,k\}$). Since $\inf |f'| > 1$ one obtains that $\bigcap_{k=0}^{\infty} V_k(x) = \{x\}$, and therefore there exists a k_1 such that $V_{k_1}(x) \subseteq U$. Then $V_{k_1}(x)$ is an interval, and ${T_f}^{k_1}$ maps it homeomorphically to $D_{k_1}(x)$. As $D_{k_1}(x) \in \mathcal{D}$ there exists a path $D_0 = D_{k_1}(x) \to D_1 \to \cdots \to D_{k_2}$ in the Markov diagram with $D_{k_2} = (T_f 0, c)$ (see Figure 14). Hence there exists a $y \in V_{k_1}(x) \setminus \left(\bigcup_{k=0}^{\infty} {T_f}^{-k}\{c\}\right)$ with ${T_f}^k V_k(y) = D_{k-k_1}$ for $k \in \{k_1, k_1+1, \ldots, k_1+k_2\}$ (obviously ${T_f}^k V_k(y) = D_k(x)$ for $k \in \{0,1,\ldots,k_1\}$). Then $V_{k_1+k_2}(y) \subseteq V_{k_1}(x) \subseteq U$ is an interval which is mapped homeomorphically to $D_{k_2} = (T_f 0, c)$ by ${T_f}^{k_1+k_2}$. From this we obtain that there exists an open interval $J_1 \subseteq V_{k_1+k_2}(y) \subseteq U$ such that ${T_f}^{k_1+k_2}$ maps J_1 homeomorphically to (a,b). Setting $n_1 := k_1 + k_2 + 12$ we get that ${T_f}^{n_1}$ maps J_1 homeomorphically to $(0,c)$. As $T_f(0,c) \supseteq (c,1)$ there exists an open interval $J_2 \subseteq J_1 \subseteq U$ such that ${T_f}^{n_1+1}$ maps J_2 homeomorphically to $(c,1)$ completing the proof that T_f is locally eventually onto.

Since T_f is locally eventually onto it is also topologically mixing by Theorem 4.1. Therefore it is also topologically transitive.

Note that $(0, c)$ is neither contained in $T_f(0, c)$ nor in $T_f(c, 1)$. By Proposition 3.3 this implies that T_f is not strongly locally eventually onto.

Finally, we show that T_f is renormalizable. To this end set $u := {T_f}^2 0$, $v := 1$, $l := 1$ and $r := 3$. One can see that

$$G(x) = \begin{cases} T_f x, & \text{if } x \in \left({T_f}^2 0, c\right), \\ {T_f}^2 0, & \text{if } x = c, \\ {T_f}^3 x, & \text{if } x \in (c, 1), \end{cases}$$

is itself an expanding Lorenz map. Hence T_f is renormalizable. Nonetheless T_f is neither trivially renormalizable nor special trivially renormalizable (STR).

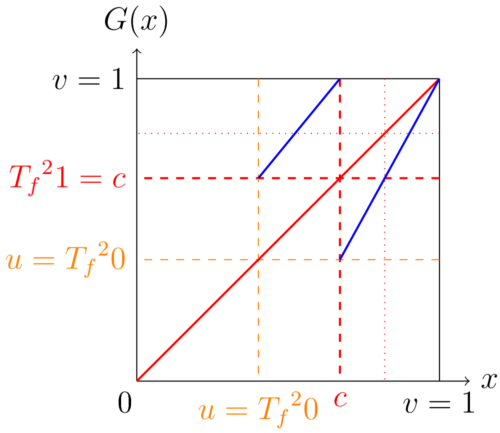

Figure 15: Graph of $G(x)$ in the case $f(x)$ from Example 5.1.

Remark 5.1. Consider the map T_f from Example 5.1. Setting $u := {T_f}^2 0$, $v := 1$, $l := 4$ and $r := 3$ one obtains that

$$\widetilde{G}(x) = \begin{cases} {T_f}^4 x, & \text{if } x \in \left({T_f}^2 0, c\right), \\ {T_f}^2 0, & \text{if } x = c, \\ {T_f}^3 x, & \text{if } x \in (c, 1), \end{cases}$$

is itself an expanding Lorenz map. Therefore T_f is also renormalizable in the sense defined in [6] and [7] (see Remark 1.2).

6. Locally eventually onto, mixing and $n(k)$-cycles

Following [8], we say that a periodic orbit of minimal period n of an expanding Lorenz map T_f is an *$n(k)$-cycle* if the points of the orbit $\{z_j : j \in \{0, \dots, n-1\}\}$ can be ordered so that

$$z_0 < z_1 < \cdots < z_{n-k-1} < c < z_{n-k} < \cdots < z_{n-1}$$

An $n(k)$-cycle is called a *primary $n(k)$-cycle* if it satisfies the following conditions

1. $T_f(z_j) = z_{j+k(\text{mod } n)}$ for all $j \geq 0$;
2. the integers k and n are coprime;
3. $z_{k-1} \leq T_f 0$ and $T_f 1 \leq z_k$.

Note that the order of the points of the $n(k)$-cycle is the same as that of the periodic orbits of a rotation $R(x) = x + k/n(\text{mod } 1)$.

Example 6.1. Consider the expanding Lorenz map induced by the function $x \mapsto \sqrt{2}x + \frac{2-\sqrt{2}}{2}$. In Theorem 4.5 we have seen that it is transitive but not mixing. Notice that $T_f 0 = \frac{2-\sqrt{2}}{2}$ and $T_f 1 = \frac{\sqrt{2}}{2}$, hence the orbit of $T_f 0$ can be written as $z_0 = T_f 0$ and $z_1 = T_f 1$. Therefore it forms a primary 2(1)-cycle for T_f.

In [8, Proposition 1] it is claimed that an expanding Lorenz map with primary $n(k)$-cycle cannot be transitive. Example 6.1 shows that this statement is wrong. One should mention that the notion of $n(k)$-cycles was first introduced by Palmer in [20] as a notion characterizing the weak Bernoulli property of invariant measures in Lorenz maps.

Note that if we take $u := z_0$, $v := z_1$ and $l = r = 2$ in Example 6.1 then T_f satisfies Definition 1.4, hence it is renormalizable. One could think that for Lorenz maps being renormalizable will prevent the map to be locally eventually onto. It was first observed in [9] that expanding Lorenz maps T_f satisfying STR may be locally eventually onto.

Example 6.2. Let T_f be the expanding Lorenz map induced by $f(x) = \frac{1+\sqrt{5}}{2}x$. We have $c = \frac{1}{\beta} = \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} - 1$, $T_f 1 = c$ and $T_f 0 = 0$. This implies that

$$G(x) = \begin{cases} T_f x, & \text{if } x \in [0, c), \\ 0, & \text{if } x = c, \\ {T_f}^2 x, & \text{if } x \in (c, 1], \end{cases}$$

is an expanding Lorenz map. Therefore T_f is renormalizable and STR. Since for every nonempty open set U there is $n > 0$ such that $0 \in {T_f}^n U$, it is also clear that T_f is locally eventually onto.

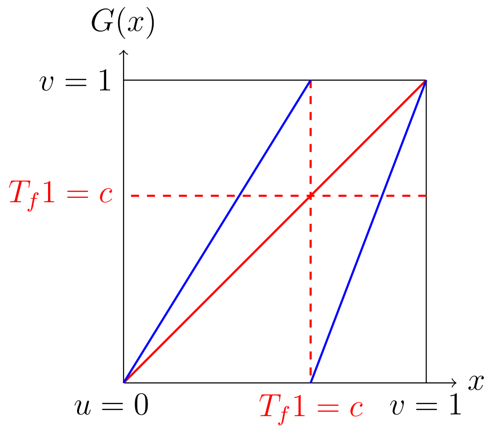

Figure 16: Graph of $G(x)$ in the case $f(x)$ from Example 6.2.

Remark 6.1. Consider the map T_f be from Example 6.2. Note that because of $T_f 0 = 0$ in order to obtain a renormalized map one must have $u = 0$ and $l = 1$. Hence T_f cannot be renormalizable in the sense defined in [6] and [7] (see Remark 1.2).

It is then a little bit surprising that a Lorenz map can be renormalizable with $T_f 0 \neq 0$, $T_f 1 \neq 1$ and mixing (in fact locally eventually onto) at the same time, as shown by the next example.

Example 6.3. Set $f(x) := \sqrt{2}x + \frac{1}{1+\sqrt{2}}$, and let T_f be the associated expanding Lorenz map. By Theorem 4.6 we obtain that T_f is topologically mixing. Observe that $c = \frac{1}{1+\sqrt{2}}$, $T_f 0 = c$ and $T_f 1 = \frac{2}{1+\sqrt{2}}$. Setting $v := T_f 1$ we see that the map

$$G(x) = \begin{cases} {T_f}^2 x = 2x, & \text{if } x \in [0, c), \\ 0, & \text{if } x = c, \\ T_f x, & \text{if } x \in (c, v], \end{cases}$$

is an expanding Lorenz map, hence T_f is renormalizable. Since $T_f 0 = c$ this map is STR (by Theorem 6.1 below T_f is locally eventually onto).

Remark 6.2. For the map T_f from Example 6.3 the points 0 and c form a periodic orbit of period 2. Hence one must have $u = 0$ if one wants to construct a renormalization. Moreover $l \in \{1, 2\}$ must hold, since ${T_f}^3$

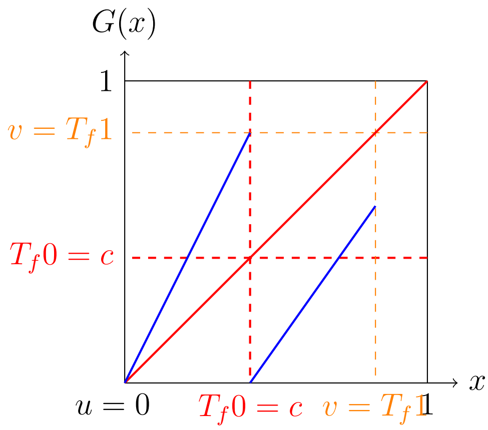

Figure 17: Graph of $G(x)$ in the case $f(x)$ from Example 6.3.

is not continuous on $[0, c]$, and therefore $v \geq T_f 1$. As ${T_f}^2$ is not continuous on $[c, T_f 1]$ one must have $r = 1$. This shows that T_f cannot be renormalizable in the sense defined in [6] and [7] (see Remark 1.2).

Suppose that T_f is transitive in the following stronger sense: for every open set U there exists m such that $\bigcup_{j=1}^{m} {T_f}^j U = [0, 1]$. Some authors call this property *strong transitivity* (e.g. see [5] and [21]) and it is clear that some piecewise monotone (and continuous) interval maps can satisfy this condition without being mixing.

Remark 6.3. In [7, Proposition 1] the author claims that strong transitivity is equivalent to locally eventually onto in the context of expanding Lorenz maps, when there exists a periodic orbit of period $\kappa \leq 2$. Unfortunately, this statement is incorrect, which is clear by simple analysis of the map in Theorem 4.5 (see also Example 6.1). Additionally, note that this map satisfies both the definition of l.e.o. and renormalization from [7] (see Remark 1.2), showing some gap in arguments of [7, Corollary 2].

To justify the above statement about the map from Theorem 4.5, observe that for any open set U there is $n > 0$ such that $c \in {T_f}^n U$. But then (see the graph of ${T_f}^2$ in Figure 2) there is $k > 0$ such that ${T_f}^{n+2k} U \supseteq [T_f 0, T_f 1]$ and then ${T_f}^{n+2k+1} U \supseteq [0, {T_f}^2 1] \cup [{T_f}^2 0, 1] = [0, T_f 0] \cup [T_f 1, 1]$. Indeed T_f induced by $f(x) = \sqrt{2}x + \frac{2-\sqrt{2}}{2}$ is strongly transitive and has a unique periodic orbit of period $\kappa = 2$, while it is not even mixing (hence cannot be locally eventually onto by Theorem 4.1; in our particular case it is not hard to see it directly from the graph).

The following theorem is a combination of statements in [7, Corollary 2] and [9, Theorem 1], correcting slight gaps in reasoning of proofs contained in these papers.

Theorem 6.1. *Let T_f be an expanding Lorenz map and assume that one of the following conditions holds:*

(1) *T_f is prime, or*

(2) *T_f is special trivial renormalizable (STR).*

Then T_f is strongly locally eventually onto.

Proof. We start with the easier case of STR. Let $U \subseteq [0,1]$ be a nonempty open set. Observe that there exists an $n \in \mathbb{N}$ such that $c \in {T_f}^{n-1}U$. We may assume that n is minimal with this property. First assume that $T_f 0 = 0$. As $c \in {T_f}^{n-1}U$ there is an open interval $L \subseteq U$ such that ${T_f}^n L = (0,a)$ for some $a \in (0,c)$, ${T_f}^n|_L$ is a homeomorphism and ${T_f}^j|_L$ is continuous for all $j \in \{0,1,\ldots,n\}$. Then $c \in {T_f}^k(0,a)$ for some $k \geq 1$, and again we suppose that k is minimal with this property. We obtain that ${T_f}^j|_L$ is continuous for all $j \in \{0,1,\ldots,n+k\}$ and ${T_f}^{n+k}L = \big(0,{T_f}^k a\big) \supsetneq (0,c)$. Therefore there exists an open interval $J_1 \subseteq L$ such that ${T_f}^j|_{J_1}$ is continuous for all $j \in \{0,1,\ldots,n+k\}$ and ${T_f}^{n+k}|_{J_1}$ is a homeomorphism from J_1 to $(0,c)$. Because of $T_f(0,c) = (0,1)$ there is an open interval $J_2 \subseteq J_1$ such that ${T_f}^j|_{J_2}$ is continuous for all $j \in \{0,1,\ldots,n+k+1\}$ and ${T_f}^{n+k+1}|_{J_2}$ is a homeomorphism from J_2 to $(c,1)$ showing that T_f is strongly locally eventually onto in this case. An analogous proof shows that T_f is strongly locally eventually onto if $T_f 1 = 1$.

Next suppose that $T_f 0 = c$ which implies $T_f 1 > c$. This shows that $T_f(0,c) = (c,1)$ and $T_f(c,1) \supseteq (0,c)$. Again $c \in {T_f}^{n-1}(U)$ implies the existence of an open interval $L \subseteq U$ such that ${T_f}^n L = (0,a)$ for some $a \in (0,c)$, ${T_f}^n|_L$ is a homeomorphism and ${T_f}^j|_L$ is continuous for all $j \in \{0,1,\ldots,n\}$. Because of $T_f(0,c) = (c,1)$ and ${T_f}^2 0 = 0$, and using that T_f is expanding, we get that there exists a $k \in \mathbb{N}$ with $c \in {T_f}^{2k}(0,a)$. We assume that k is minimal with this property. Then ${T_f}^j|_L$ is continuous for all $j \in \{0,1,\ldots,n+2k\}$ and ${T_f}^{n+2k}L = \big(0,{T_f}^{2k}a\big) \supsetneq (0,c)$, which implies that there exists an open interval $J \subseteq L$ such that ${T_f}^j|_J$ is continuous for all $j \in \{0,1,\ldots,n+2k\}$ and ${T_f}^{n+2k}|_J$ is a homeomorphism from J to $(0,c)$. Since $T_f(0,c) = (c,1)$ also ${T_f}^{n+2k+1}|_J$ is a homeomorphism from J to $(c,1)$, hence T_f is strongly locally eventually onto. By an analogous proof one shows that T_f is strongly locally eventually onto in the case $T_f 1 = c$ completing the proof of T_f is strongly locally eventually onto if (2) is satisfied.

It remains to consider the case in that (1) is satisfied and T_f is not STR. Observing that $T_f 0 \geq c$ implies that

$$G(x) = \begin{cases} {T_f}^2 x\,, & \text{if } x \in [0,c), \\ 0\,, & \text{if } x = c, \\ T_f x\,, & \text{if } x \in (c, T_f 1], \end{cases}$$

is an expanding Lorenz map, which contradicts the fact that T_f is prime, we see that $T_f 0 < c$. As T_f is not STR we have $0 < T_f 0$. Using analogous arguments we also get $c < T_f 1 < 1$, hence

$$0 < T_f 0 < c < T_f 1 < 1\,. \tag{6.1}$$

In particular we have $T_f(0,c) \supseteq (c,1)$ and $T_f(c,1) \supseteq (0,c)$.

Now assume that T_f is not strongly locally eventually onto. Then there exists a nonempty open set $U \subseteq [0,1]$ which does not contain any two open subintervals J_1, J_2 such that for some $n_1, n_2 \in \mathbb{N}$ one has that ${T_f}^k|_{J_1}$ is continuous for every $k \in \{0,1,\ldots,n_1\}$, ${T_f}^{n_1}|_{J_1}$ is a homeomorphism from J_1 to $(0,c)$, ${T_f}^k|_{J_2}$ is continuous for every $k \in \{0,1,\ldots,n_2\}$ and ${T_f}^{n_2}|_{J_2}$ is a homeomorphism from J_2 to $(c,1)$. Without loss of generality we may assume that U is a nonempty open interval. If for some $r \in \mathbb{N}$ one has $c \notin U$, $c \notin T_f U$, …, $c \notin {T_f}^{r-1} U$, then ${T_f}^r U$ is again an open interval having the same property as described above. Since T_f is expanding there is an $r \geq 0$ with $c \in {T_f}^r U$. Hence there exist $a_1 < c < b_1$ such that (a_1, b_1) does not contain any two open subintervals J_1, J_2 satisfying that for some $n_1, n_2 \in \mathbb{N}$ one has that ${T_f}^k|_{J_1}$ is continuous for every $k \in \{0,1,\ldots,n_1\}$, ${T_f}^{n_1}|_{J_1}$ is a homeomorphism from J_1 to $(0,c)$, ${T_f}^k|_{J_2}$ is continuous for every $k \in \{0,1,\ldots,n_2\}$ and ${T_f}^{n_2}|_{J_2}$ is a homeomorphism from J_2 to $(c,1)$. Note that (6.1) implies that $0 < a_1 < c < b_1 < 1$.

Define A as the set of all $t \in (0,c)$ satisfying that (t,c) does not contain any open subinterval J such that for some $n \in \mathbb{N}$ one has that ${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(0,c)$, and set $a := \inf A$. Obviously $a_1 \in A$, and by (6.1) we get $0 < a \leq a_1 < c$. Furthermore A is obviously an interval. Suppose that there exists an open interval $J \subseteq (a,c)$ and an $n \in \mathbb{N}$ such that ${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(0,c)$. As ${T_f}^n|_J$ is a homeomorphism J contains an $a_2 > a$ with ${T_f}^n a_2 = a$. Hence $a_2 \in A$ and there exists an open interval $J_1 \subseteq J \cap (a_2, c)$ with ${T_f}^n J_1 = J$. But then ${T_f}^k|_{J_1}$ is continuous for every $k \in \{0,1,\ldots,2n\}$ and ${T_f}^{2n}|_{J_1}$ is a homeomorphism from J_1 to $(0,c)$ contradicting $a_2 \in A$. Therefore (a,c) does not contain any open subinterval J such that for some $n \in \mathbb{N}$ one has that

${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(0,c)$.

Analogous let B be the set of all $t \in (c,1)$ satisfying that (c,t) does not contain any open subinterval J such that for some $n \in \mathbb{N}$ one has that ${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(c,1)$, and define $b := \sup B$. Using a proof analogous as above we get that $c < b_1 \leq b < 1$, and that (c,b) does not contain any open subinterval J such that for some $n \in \mathbb{N}$ one has that ${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(c,1)$.

Since T_f is expanding there exists an $l \in \mathbb{N}$ with $c \in {T_f}^l(a,c)$. We may assume that l is minimal with this property. Because of (6.1) $l = 1$ would imply the existence of an open interval $J \subseteq (a,c)$ such that $T_f|_J$ and ${T_f}^2|_J$ are continuous and ${T_f}^2|_J$ maps J homeomorphically to $(0,c)$, which is a contradiction to the property proved above for (a,c). As $c \notin {T_f}^j(a,c)$ for $j \in \{0,1,\ldots,l-1\}$ we get that ${T_f}^l(a,c) = \left({T_f}^l a, {T_f}^{l-1}1\right)$ is an open interval, and it does not contain any open subinterval J such that for some $n \in \mathbb{N}$ one has that ${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(0,c)$. Then also $({T_f}^l a, c)$ has this property, hence ${T_f}^l a \in A$ implying ${T_f}^l a \geq a$. If $(c, {T_f}^{l-1}1)$ would contain an open subinterval J such that for some $n \in \mathbb{N}$ one has that ${T_f}^k|_J$ is continuous for every $k \in \{0,1,\ldots,n\}$ and ${T_f}^n|_J$ is a homeomorphism from J to $(c,1)$, then there would be an open interval $J_1 \subseteq J \subseteq {T_f}^l(a,c)$ such that ${T_f}^k|_{J_1}$ is continuous for every $k \in \{0,1,\ldots,n+1\}$ and ${T_f}^{n+1}|_{J_1}$ is a homeomorphism from J_1 to $(0,c)$. Since this contradicts the property of (a,c) proved above we obtain ${T_f}^{l-1}1 \in B$ and therefore $c < {T_f}^{l-1}1 \leq b$.

Using an analogous proof we find a minimal $r \in \mathbb{N}$ with $c \in {T_f}^r(c,b)$, and we obtain that $r \geq 2$, ${T_f}^r(c,b) = \left({T_f}^{r-1}0, {T_f}^r b\right)$ is an open interval, and $a \leq {T_f}^{r-1}0 < c < {T_f}^r b \leq b$. Now ${T_f}^j\left({T_f}^{r-1}0, c\right)$ is an open subinterval of ${T_f}^j(a,c)$ for every $j \in \{0,1,\ldots,l\}$ and ${T_f}^j\left(c, {T_f}^{l-1}1\right)$ is an open subinterval of ${T_f}^j(c,1)$ for every $j \in \{0,1,\ldots,r\}$. As T_f is expanding also ${T_f}^l$ is expanding, and therefore ${T_f}^l\left({T_f}^{r-1}0\right) - {T_f}^l a \geq {T_f}^{r-1}0 - a$. This gives ${T_f}^l\left({T_f}^{r-1}0\right) \geq {T_f}^{r-1}0 + {T_f}^l a - a$, and because of ${T_f}^l a \geq a$ we obtain ${T_f}^l\left({T_f}^{r-1}0\right) \geq {T_f}^{r-1}0$ and

$$
{T_f}^l\left({T_f}^{r-1}0, c\right) = \left({T_f}^l\left({T_f}^{r-1}0\right), {T_f}^{l-1}1\right) \subseteq \left({T_f}^{r-1}0, {T_f}^{l-1}1\right) .
$$

One proves analogously that ${T_f}^r\left({T_f}^{l-1}1\right) \leq {T_f}^{l-1}1$ and

$$
{T_f}^r\left(c, {T_f}^{l-1}1\right) = \left({T_f}^{r-1}0, {T_f}^r\left({T_f}^{l-1}1\right)\right) \subseteq \left({T_f}^{r-1}0, {T_f}^{l-1}1\right) .
$$

Hence we obtain that $l+r \geq 4 > 3$ and

$$G(x) = \begin{cases} {T_f}^l x\,, & \text{if } x \in \big[{T_f}^{r-1}0, c\big), \\ {T_f}^{r-1}0\,, & \text{if } x = c, \\ {T_f}^r x\,, & \text{if } x \in \big(c, {T_f}^{l-1}1\big], \end{cases}$$

is an expanding Lorenz map. But then T_f would be renormalizable contradicting (1). Therefore T_f is strongly locally eventually onto completing the proof. □

Theorem 6.2. *Suppose that T_f is an expanding Lorenz map which is renormalizable but not special trivially renormalizable. Then T_f is not strongly locally eventually onto.*

Proof. By Proposition 3.3 and the fact that T_f is not special trivially renormizable we get that $0 < T_f 0 < c < T_f 1 < 1$. Moreover there are $0 \leq u < c < v \leq 1$ and $l, r \geq 1$ with $l + r \geq 3$ such that

$$G(x) = \begin{cases} {T_f}^l x\,, & \text{if } x \in [u, c), \\ u\,, & \text{if } x = c, \\ {T_f}^r x\,, & \text{if } x \in (c, v], \end{cases}$$

is an expanding Lorenz map. Suppose that ${T_f}^k(u,c) \supseteq Z$ for a $Z \in \mathcal{Z} := \{(0,c),(c,1)\}$ and a $k \in \{0,1,\dots,l-2\}$. As $T_f(0,c) \supseteq (c,1)$ and $T_f(c,1) \supseteq (0,c)$ we get that ${T_f}^{l-2}(u,c) \supseteq Y$ for some $Y \in \mathcal{Z}$. This implies that there is an $x \in (u,c)$ with ${T_f}^{l-1}x = c$, and therefore ${T_f}^l$ cannot be continuous at x. Obviously this contradicts the fact that G is a Lorenz map. Hence ${T_f}^k(u,c) \supseteq Z$ for a $Z \in \mathcal{Z}$ implies that $k \geq l-1$. An analogous argument shows that ${T_f}^k(c,v) \supseteq Z$ for a $Z \in \mathcal{Z}$ implies that $k \geq r-1$.

Next assume that ${T_f}^{l-1}(u,c) \supseteq Z$ for a $Z \in \mathcal{Z}$. Then ${T_f}^l(u,c)$ contains an element of $\mathcal{Z}$. If ${T_f}^l(u,c)$ contains an element of $\mathcal{Z}$ then $u = 0$ or $v = 1$. However by the above $u = 0$ implies that $l = 1$ and $v = 1$ implies that $r = 1$. In the case $l = 1$ we get $v = 1$ which implies $r = 1$ and contradicts $l + r \geq 3$. Similarly, $r = 1$ implies $u = 0$ and therefore $l = 1$ which also contradicts $l + r \geq 3$. Analogously we get that ${T_f}^{r-1}(c,v)$ and ${T_f}^r(c,v)$ cannot contain an element of $\mathcal{Z}$.

In particular we have also shown that $u > 0$ and $v < 1$. Assume that $J \subseteq (u,c)$ is an nonempty open interval and that $n \in \mathbb{N}$ such that ${T_f}^k$ restricted to J is continuous for all $k \in \{0,1,\dots,n\}$. By the above ${T_f}^k J$ cannot contain an element of $\mathcal{Z}$ for $k \in \{0,1,\dots,\min\{l,n\}\}$. As G is a Lorenz map ${T_f}^l J \subseteq (u,v)$. If $n > l$ then the continuity of ${T_f}^{l+1}$ on J implies that either ${T_f}^l J \subseteq (0, c)$ or ${T_f}^l J \subseteq (c, 1)$. Then ${T_f}^k J$ cannot contain any element of $\mathcal{Z}$ for $k \in \{0, 1, \dots, \min\{2l, n\}\}$ in the first case, and ${T_f}^k J$ cannot contain any element of $\mathcal{Z}$ for $k \in \{0, 1, \dots, \min\{l + r, n\}\}$. Iterating this argument we obtain that ${T_f}^n J$ cannot contain any element of $\mathcal{Z}$. Therefore T_f cannot be strongly locally eventually onto. □

Remark 6.4. Recall that in Example 5.1 we have seen a renormalizable expanding Lorenz map which is also locally eventually onto. Hence Theorem 6.2 and Corollary 6.3 do not hold if "strongly locally eventually onto" is replaced by "locally eventually onto".

Combining Theorem 6.2 and Theorem 6.1 we immediately obtain the following result.

Corollary 6.3. *Let T_f be an expanding Lorenz map. Then the following conditions are equivalent.*

(1) *The map T_f is prime or T_f is special trivial renormalizable.*

(2) *The map T_f is strongly locally eventually onto.*

In a certain sense the above Corollary 6.3 is a kind of combination and improvement of statements in [8] and [9]. However, as in these papers the authors deal only with locally eventually onto Lorenz maps we see from Example 5.1 that they could not obtain an equivalence result similar to Corollary 6.3. It is also worth to mention that [8] and [9] do not contain complete proofs of statements analogous to Corollary 6.3. Unfortunately, some references in [9] have never been published (e. g. ref. 3 and 13 in [9]). Since [20] has been defended at University of Warwick, it is hardly available, but possible to obtain.[1] Nonetheless, Corollary 6.3 is not a direct consequence of any result contained in [20].

As we mentioned before, the notion of $n(k)$-cycle was used in [8, Proposition 1] to find range of parameters where the map $x \mapsto \beta x + \alpha \pmod 1$ is not transitive. Unfortunately, the formulas describing these regions are not completely clear. Before, we have proven a special case of this fact in Theorem 4.5 and Theorem 4.7.

Theorem 6.4. *Suppose that T_f is an expanding Lorenz map with a primary $n(k)$-cycle $\{z_j : 0 \leq j < n\}$ and assume that $T_f 1 = z_k$ and $T_f 0 = z_{k-1}$. Then T_f is transitive but not mixing.*

[1]We are much obliged to British Library and Library of University of Warwick for providing us with an electronic copy of [20] free of charge.

Proof. Before we start, note that throughout this proof the indices are always meant modulo n. First, we exclude the case $n = 2$ and $k = 1$. Put $C_r = (z_r, z_{r+1})$ for $r \neq n-k-1$. Then $C_r \to C_{r+k}$ for $r \notin \{n-2k-1, n-k-1\}$ and $C_{n-2k-1} \to (z_{n-k-1}, c)$ and $C_{n-2k-1} \to (c, z_{n-k})$. Moreover, $(c, z_{n-k}) \to (0, z_0) \to (z_{k-1}, z_k) = C_{k-1}$ and $(z_{n-k-1}, c) \to (z_{n-1}, 1) \to (z_{k-1}, z_k) = C_{k-1}$. Define

$$\begin{aligned} \mathcal{C} := {} & \{(z_r, z_{r+1}) : r \in \{0, 1, \ldots, n-2\} \setminus \{n-k-1\}\} \cup \\ & \cup \{(0, z_0), (z_{n-k-1}, c), (c, z_{n-k}), (z_{n-1}, 1)\}. \end{aligned} \tag{6.2}$$

Observe that $\bigcup_{C \in \mathcal{C}} \overline{C} = [0,1]$ and that the elements of $\mathcal{C}$ are pairwise disjoint. As n and k are coprime we get that $\mathcal{C}$ is a subset of the Markov diagram of T_f, and by the properties derived above it is closed. Furthermore $C_{k-1} \to C_{2k-1} \to \cdots \to C_{(n-2)k-1} \to (z_{n-k-1}, c) \to (z_{n-1}, 1) \to C_{k-1}$ and $C_{k-1} \to C_{2k-1} \to \cdots \to C_{(n-2)k-1} \to (c, z_{n-k}) \to (0, z_0) \to C_{k-1}$ are paths of length n from C_{k-1} to C_{k-1}. Again using that n and k are coprime one sees that every element of $\mathcal{C}$ is at least in one of these two paths, hence $\mathcal{C}$ is irreducible. By Lemma 4.3 we get that T_f is transitive. However, the calculations above show also that ${T_f}^n C_{k-1} = C_{k-1}$, hence T_f is not mixing.

Finally, we consider the case $n = 2$ and $k = 1$. Here easy calculations show that $\mathcal{C} := \{(0, z_0), (z_0, c), (c, z_1), (z_1, 1)\}$ forms a closed and irreducible subset of the Markov diagram of T_f. Obviously $\bigcup_{C \in \mathcal{C}} \overline{C} = [0,1]$ and the elements of $\mathcal{C}$ are pairwise disjoint. From Lemma 4.3 we get that T_f is transitive, and as ${T_f}^2 [z_0, z_1] = [z_0, z_1]$ we see that T_f is not mixing. $\square$

Theorem 6.5. *Assume that T_f is an expanding Lorenz map with a primary $n(k)$-cycle $\{z_j : 0 \leq j < n\}$ and suppose that $T_f 1 < z_k$ or $T_f 0 > z_{k-1}$. Then T_f is not transitive.*

Proof. Throughout this proof the indices are always meant modulo n. As the proof for the case $T_f 1 < z_k$ is analogous we may assume without loss of generality that $T_f 0 > z_{k-1}$.

Note that for any $j \in \{0, 1, \ldots, n-3\}$ there is a $Z_j \in \mathcal{Z}$ with ${T_f}^{j+1} 0$ and ${T_f}^j z_{k-1}$ are in Z_j. Consider at first the case that ${T_f}^{n-1} 0 < c$. Then both ${T_f}^{n-1} 0$ and ${T_f}^{n-2} z_{k-1} = z_{n-k-1}$ are in $(0, c)$, and both ${T_f}^n 0$ and ${T_f}^n z_{k-1} = z_{n-1}$ are in $(c, 1)$. Since ${T_f}^n$ is expanding this implies that ${T_f}^{n+1} 0 - {T_f}^n z_{k-1} > T_f 0 - z_{k-1}$, and because of $T^n z_{k-1} = z_{k-1}$ this gives ${T_f}^{n+1} 0 > T_f 0$. Hence $T_f\left({T_f}^n 0, 1\right) \subseteq (T_f 0, z_k)$. Define $A := \bigcup_{j=1}^{n-2} \left[{T_f}^j 0, z_{jk}\right] \cup \left[{T_f}^{n-1} 0, c\right] \cup \left[{T_f}^n 0, 1\right] \cup [c, z_{n-k}] \cup [0, z_0]$. Then A is closed, $T_f A \subseteq A$ and A has nonempty interior. Because of $[0,1] \setminus A \supseteq (z_{k-1}, T_f 0)$ and $T_f 0 > z_{k-1}$ also $[0,1] \setminus A$ has nonempty interior, which proves that T_f is not transitive.

It remains to assume that ${T_f}^{n-1} 0 \geq c$. We get in this case $T_f\left({T_f}^{n-2} 0, z_{n-2k}\right) = \left({T_f}^{n-1} 0, z_{n-k}\right) \subseteq (c, 1)$ and $T_f\left({T_f}^{n-1} 0, z_{n-k}\right) = \left({T_f}^n 0, z_0\right)$, and therefore the restriction of ${T_f}^n$ to $(0, z_0)$ is a homeomorphism satisfying ${T_f}^n(0, z_0) \subseteq (0, z_0)$. As this contradicts the fact that ${T_f}^n$ is expanding we see that this case cannot occur, finishing the proof. □

With Theorem 6.5 at hand we can try to find regions of parameters α, β, $\alpha + \beta \leq 2$ where the expanding Lorenz map T_f induced by $f(x) = \beta x + \alpha$ is not transitive. Such an attempt was made in [8, Proposition 2] however there are some problems with the formulas presented there. For example, when $n = 2$ and $k = 1$ and $\beta < \sqrt{2}$ then [8, Proposition 2] claims that T_f is not transitive for α in the range:

$$\frac{1-\beta}{\beta(\beta+1)} \leq \alpha \leq \frac{-\beta^3+\beta^2+\beta-1}{\beta(\beta+1)} < 0$$

which would lead to conclusion that there is no 2(1)-cycle for $\beta < \sqrt{2}$. The example given in (2) of [24] (T_f induced by $f(x) := \beta x + \left(1 - \frac{\beta}{2}\right)$) obviously has $z_0 := \frac{\beta}{2(1+\beta)}$ and $z_1 := \frac{2+\beta}{2(1+\beta)}$ as a primary 2(1)-cycle for any $\beta \in \left(1, \sqrt{2}\right]$ which shows that 2(1)-cycles exist for $\beta < \sqrt{2}$. However despite these problems with calculations, the approach from [8] may lead to exact calculations of regions with lack of transitivity as shown below.

Fix any integer $n \geq 1$, assume that $\beta \in (1, 2^{1/n}]$ and consider the expanding Lorenz map T_f induced by $f(x) := \beta x + \alpha$. We will try to find parameters for which there is a primary $n(1)$-cycle. In this way we can describe regions which satisfies assumptions of Theorem 6.5. Set $\alpha_0 = 0$ and $\alpha_k = \alpha\left(\sum_{j=0}^{k-1} \beta^j\right)$.

Remark 6.5. Observe that if $z_0 < z_1 < \ldots < z_{n-2} < c < z_{n-1}$ is an $n(1)$-cycle, then it satisfies the following conditions:

1. $\alpha = T_f 0 \geq z_0$ and $\alpha + \beta - 1 = T_f 1 \leq z_1$,
2. $T_f(z_j) = \beta z_j + \alpha$ for $j \in \{0, 1, \ldots, n-2\}$, hence $z_j = \beta^j z_0 + \alpha_j$ for $j \in \{0, 1, \ldots, n-1\}$, and
3. $T_f(z_{n-1}) = \beta z_{n-1} + \alpha - 1$.

Furthermore $f(z_{n-1}) \leq 2$, so in particular $\beta \leq 2^{1/n}$.

Using the above conditions and ${T_f}^n(z_0) = z_0$ it is not hard to calculate that

$$z_0 = \frac{1}{\beta^n - 1} - \frac{\alpha}{\beta - 1} \tag{6.3}$$

and then using $z_1 = \beta z_0 + \alpha = \frac{\beta}{\beta^n - 1} - \frac{\alpha}{\beta - 1}$ and comparing it with the values $T_f 0$ and $T_f 1$ we obtain that

$$\frac{1}{\sum_{j=1}^{n} \beta^j} \leq \alpha \leq \frac{-\beta^{n+1} + \beta^n + 2\beta - 1}{\sum_{j=1}^{n} \beta^j}. \tag{6.4}$$

Note that the assumptions of Theorem 6.5 can be satisfied only when ${T_f}^2 0 \neq T_f 1$, which means $\beta\alpha + \alpha \neq \alpha + \beta - 1$, therefore $\beta \neq \frac{1}{1-\alpha}$ or equivalently $\alpha \neq \frac{\beta-1}{\beta}$. Hence for each β there is at most one "bad" value of α. In particular, for $\beta = \sqrt{2}$ we obtain the case presented in Example 6.1. Observe that the first inequality in (6.4) can be equivalently written as $\frac{\beta-1}{\beta(\beta^n-1)} \leq \alpha$. Together with $\beta \leq 2^{1/n}$ this implies that for $\beta < 2^{1/n}$ region of parameters described by (6.4) never intersects the curve $\alpha = \frac{\beta-1}{\beta}$ which implies that $T_f 0 > z_0$ or $T_f 1 < z_1$ in this case. On the other hand, for $\beta = 2^{1/n}$ equation (6.4) reduces to $\alpha = \frac{\beta-1}{\beta} = 1 - \frac{1}{\sqrt[n]{2}}$.

Now we are ready to state theorem summarizing the above considerations. It provides regions where there is lack of transitivity except exactly one "top" point on the boundary of these regions (see Figure 18). Note that the condition for α in (1) of Theorem 6.6 below is exactly the condition described in (6.4).

Theorem 6.6. *Let $n \geq 2$ be an integer, and let $\beta \in \left(1, \sqrt[n]{2}\right]$. Assume that T_f is the expanding Lorenz map induced by $f(x) := \beta x + \alpha$. Then the following assertions hold.*

(1) *If $\beta < \sqrt[n]{2}$ and*

$$\frac{1}{\sum_{j=1}^{n} \beta^j} \leq \alpha \leq \frac{-\beta^{n+1} + \beta^n + 2\beta - 1}{\sum_{j=1}^{n} \beta^j}$$

then T_f is not transitive.

(2) *For $\beta = \sqrt[n]{2}$ and $\alpha = 1 - \frac{1}{\sqrt[n]{2}}$ the map T_f is transitive but not mixing.*

Proof. First, let z_0 be as in (6.3) and define $z_1, z_2, \ldots, z_{n-1}$ as in (2) of Remark 6.5. Then $z_{n-1} = \beta^{n-1} z_0 + \alpha \frac{\beta^{n-1}-1}{\beta-1}$ and using (6.3) we obtain

$$z_{n-1} = \frac{\beta^{n-1}}{\beta^n - 1} - \frac{\alpha\beta^{n-1}}{\beta - 1} + \alpha \frac{\beta^{n-1} - 1}{\beta - 1} = \frac{\beta^{n-1}}{\beta^n - 1} - \frac{\alpha}{\beta - 1}.$$

As $\beta^{n+1} - \beta^n - \beta + 1 = (\beta^n - 1)(\beta - 1) > 0$ we get $1 > -\beta^{n+1} + \beta^n + \beta$. Because of (6.4) this gives $\alpha \geq \frac{1}{\sum_{j=1}^{n} \beta^j} > \frac{-\beta^{n+1}+\beta^n+\beta}{\sum_{j=1}^{n} \beta^j} = (\beta - 1)\frac{\beta^{n-1}}{\beta^n - 1} -$

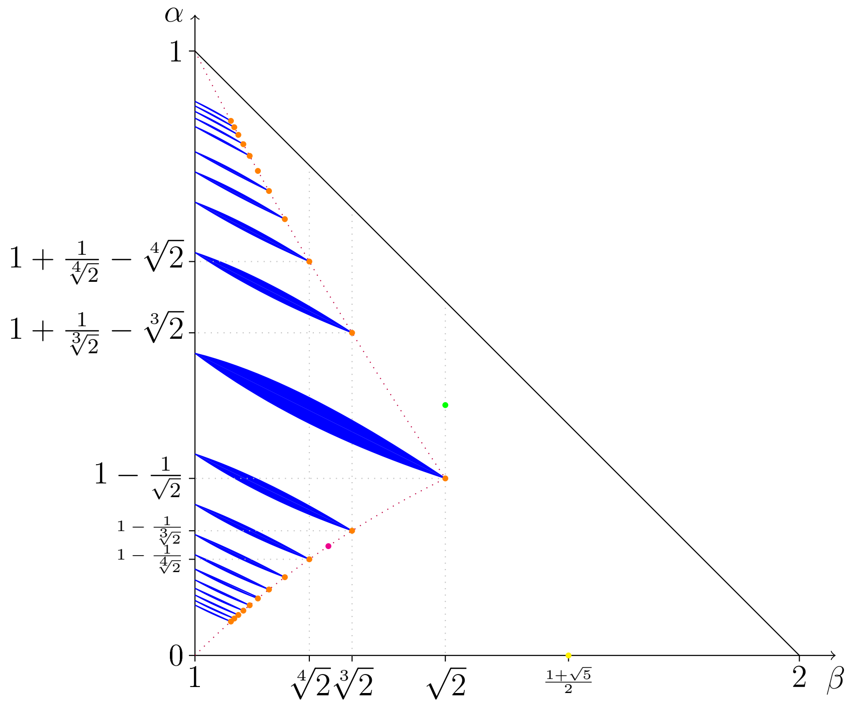

Figure 18: The upper purple dotted curve shows $1+\frac{1}{\beta}-\beta$, the lower one $1-\frac{1}{\beta}$. Moreover, the blue areas represent the parameters described in (1) from Theorem 6.6 and Remark 6.6 (for $n \in \{2,3,\dots,12\}$), and the orange dots represent the parameters described in (2) from Theorem 6.6 and Remark 6.6 (for $n \in \{2,3,\dots,12\}$). Furthermore, the pink dot is from Example 5.1, the yellow one from Example 6.2, and the green dot is from Example 6.3.

$\beta+1$ which implies $z_{n-1} = \frac{\beta^{n-1}}{\beta^n-1} - \frac{\alpha}{\beta-1} < 1$. From (6.4) one obtains that $\alpha \le \frac{-\beta^{n+1}+\beta^n+2\beta-1}{\sum_{j=1}^n \beta^j} = \frac{\beta-(\beta^n-1)(\beta-1)}{\sum_{j=1}^n \beta^j} < \frac{\beta}{\sum_{j=1}^n \beta^j} = \frac{\beta-1}{\beta^n-1} = (\beta-1)\frac{\beta^n-(\beta^n-1)}{\beta^n-1} = \frac{\beta^n(\beta-1)}{\beta^n-1}-\beta+1$. Dividing by $\beta(\beta-1)$ we get $\alpha\left(\frac{1}{\beta-1}-\frac{1}{\beta}\right) = \frac{\alpha}{\beta(\beta-1)} < \frac{\beta^{n-1}}{\beta^n-1}-\frac{1}{\beta}$ which implies $c = \frac{1-\alpha}{\beta} < \frac{\beta^{n-1}}{\beta^n-1} - \frac{\alpha}{\beta-1} = z_{n-1}$. Since $z_{n-2} \ge c$ would imply $z_{n-1} \ge 1$ we obtain $0 < z_0 < z_1 < \cdots < z_{n-2} < c < z_{n-1} < 1$. Obviously (2) of Remark 6.5 gives $T_f z_j = z_{j+1}$ for $j \in \{0,1,\dots,n-2\}$ and using also (6.3) we see that $T_f z_{n-1} = z_0$. As (6.4) implies that $T_f 0 \ge z_0$ and $T_f 1 \le z_1$ one obtains that T_f has a primary $n(1)$-cycle.

Suppose that $\beta < \sqrt[n]{2}$ and α satisfies (6.4). Now the arguments below Remark 6.5 imply that $T_f 0 > z_0$ or $T_f 1 < z_1$. Therefore Theorem 6.5 gives that T_f is not transitive.

It remains to consider the case $\beta = \sqrt[n]{2}$ and $\alpha = 1 - \frac{1}{\sqrt[n]{2}}$ (in Figure 19 the graph of T_f is shown for $f(x) = \sqrt[6]{2}x + 1 - \frac{1}{\sqrt[6]{2}}$). One easily calculates that

$T_f 0 = z_0$ and $T_f 1 = z_1 = {T_f}^2 0$ in this case. By Theorem 6.4 we get that T_f is transitive but not mixing completing the proof. $\square$

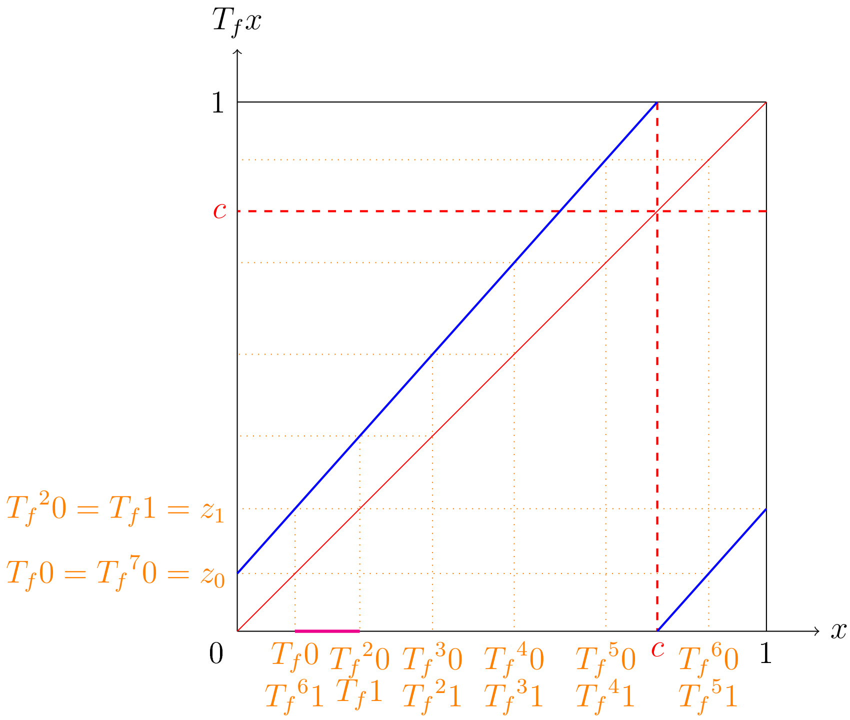

Figure 19: The graph of T_f for $f(x) = \sqrt[6]{2}x + 1 - \frac{1}{\sqrt[6]{2}}$. Here the pink interval is invariant under ${T_f}^6$.

Let us also consider the symmetric case.

Remark 6.6. From Proposition 2.11 and Theorem 6.6 we obtain that, if $n \geq 2$ is an integer and $\beta \in \left(1, \sqrt[n]{2}\right]$, then the expanding Lorenz map T_f induced by $f(x) := \beta x + \alpha$ satisfies the following properties.

(1) If $\beta < \sqrt[n]{2}$ and

$$2 - \beta + \frac{\beta^{n+1} - \beta^n - 2\beta + 1}{\sum_{j=1}^n \beta^j} \leq \alpha \leq 2 - \beta - \frac{1}{\sum_{j=1}^n \beta^j}$$

then T_f is not transitive.

(2) For $\beta = \sqrt[n]{2}$ and $\alpha = 1 + \frac{1}{\sqrt[n]{2}} - \sqrt[n]{2}$ the map T_f is transitive but not mixing (in Figure 20 the graph of T_f is shown for $f(x) = \sqrt[6]{2}x + 1 + \frac{1}{\sqrt[6]{2}} - \sqrt[6]{2}$).

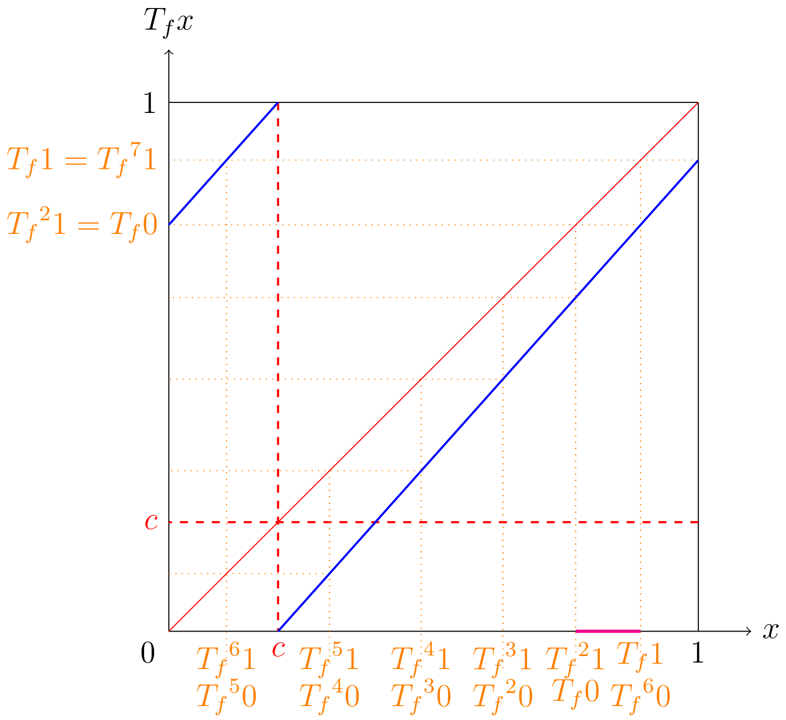

Figure 20: The graph of T_f for $f(x) = \sqrt[6]{2}x + 1 + \frac{1}{\sqrt[6]{2}} - \sqrt[6]{2}$. Here the pink interval is invariant under ${T_f}^6$.

7. Mixing in the case $\beta x + \alpha$

Theorem 7.1. *Let $\sqrt[3]{2} \le \beta \le 2$ and let $0 \le \alpha \le 2 - \beta$. Let T_f be an expanding Lorenz map induced by $f(x) := \beta x + \alpha$. Then T_f is topologically mixing if and only if one of the following conditions is satisfied:*

(1) *we have $\beta \ge \sqrt{2}$ and $f(x) \ne \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$.*

(2) *we have $\sqrt[3]{2} \le \beta < \sqrt{2}$, $0 \le \alpha < \frac{1}{\beta^2+\beta}$ or $2 - \beta - \frac{1}{\beta^2+\beta} < \alpha \le 2 - \beta$, and $f(x) \ne \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ and $f(x) \ne \sqrt[3]{2}x + \frac{2-\sqrt[3]{4}}{2}$.*

Proof. Set $T_f x := \beta x + \alpha \pmod 1$. From Theorem 4.6 we obtain that T_f is topologically mixing if $\beta \ge \sqrt{2}$ and $f(x) \ne \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$.

Now assume that $\sqrt[3]{2} \le \beta < \sqrt{2}$, and $f(x) \ne \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ and $f(x) \ne \sqrt[3]{2}x + \frac{2-\sqrt[3]{4}}{2}$. It follows from Theorem 4 in [25] that T_f is not topologically transitive (and therefore also not topologically mixing) in the case $\frac{1}{\beta^2+\beta} \le \alpha \le 2 - \beta - \frac{1}{\beta^2+\beta}$ (see also Theorem 6.6).

We consider the case $\alpha > 2 - \beta - \frac{1}{\beta^2+\beta}$. If $\alpha \geq \frac{1}{\beta+1}$ then Theorem 4.8 implies that T_f is topologically mixing. Hence it remains to consider the case $2 - \beta - \frac{1}{\beta^2+\beta} < \alpha < \frac{1}{\beta+1}$.

At first assume that $2 - \beta - \frac{1}{\beta^2+\beta} < \alpha < 1 - \frac{\beta^3}{\beta^2+\beta+1}$ and $\alpha < \frac{1}{\beta+1}$. In this case Lemma 10 of [25] gives that we have the arrows $(0, c) \to (c, 1)$, $(c, 1) \to (0, c)$ and $(0, c) \to (T_f 0, c)$ in the Markov diagram of T_f. Now the proof of [25, Theorem 4] shows that for any $C \in \mathcal{D}$ there is a path $C_0 = C \to C_1 \to \cdots \to C_n$ with $C_n = (0, c)$. Therefore $\mathcal{D} \supseteq \{(0, c), (c, 1)\}$ is irreducible, $\overline{(0, c) \cup (c, 1)} = [0, 1]$, and there is a path $C_0 = (T_f 0, c) \to C_1 \to \cdots \to C_p$ of length p with $C_p = (0, c)$. Using Lemma 2.7 there is a path of length p from $(0, c)$ to itself, and because of $(0, c) \to (T_f 0, c)$ there is also a path of length $p + 1$ from $(0, c)$ to itself. Obviously, p and $p + 1$ are coprime and therefore T_f is topologically mixing by Theorem 4.4.

Next assume that $1 - \frac{\beta^3}{\beta^2+\beta+1} \leq \alpha < \frac{1}{\beta+1}$. Applying Lemma 11 in [25] we have $(0, c) \to (c, 1)$, $(c, 1) \to (0, c)$ and $(0, c) \to (T_f 0, c)$. Again we can repeat argument from the proof of [25, Theorem 4] to obtain that for every $C \in \mathcal{D}$ there exists a path $C_0 = C \to C_1 \to \cdots \to C_n$ with $C_n = (0, c)$. Then is enough to repeat arguments from the previous case to see that T_f is topologically mixing.

Finally, let $\alpha < \frac{1}{\beta^2+\beta}$. Then $h(x) := 1 - x$ conjugates T_f to T_g, where $g(x) = \beta x + 2 - \beta - \alpha$ and $2 - \beta - \alpha > 2 - \beta - \frac{1}{\beta^2+\beta}$. Above we have shown that T_g is topologically mixing, and therefore also T_f is topologically mixing, completing the proof of this case.

If $f(x) = \sqrt[3]{2}x + \frac{2-\sqrt[3]{4}}{2} = \sqrt[3]{2}x + 1 - \frac{1}{\sqrt[3]{2}}$ then T_f is not mixing by Theorem 6.6 and the case $f(x) = \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2} = \sqrt[3]{2}x + 1 + \frac{1}{\sqrt[3]{2}} - \sqrt[3]{2}$ is covered by conjugacy argument presented above. □

Using Theorem 7.1 we can draw the region of (β, α) in the triangle defined by $\beta \geq \sqrt[3]{2}$, $\alpha \geq 0$ and $\beta + \alpha \leq 2$, where $\beta x + \alpha \pmod 1$ is topologically mixing. This is done in Figure 21.

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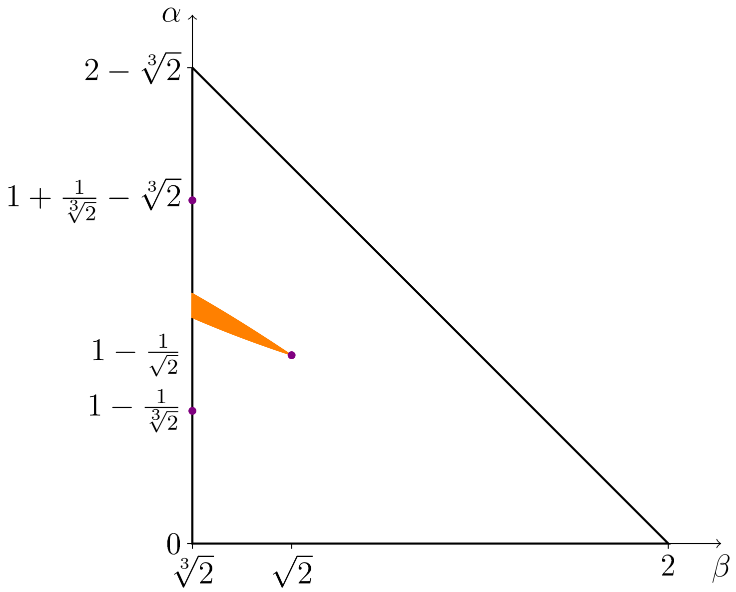

Figure 21: For β and α in white area of this triangle the map $\beta x+\alpha \pmod 1$ is topologically mixing. In the orange area it is not topologically transitive, and for the violet points it is topologically transitive but not topologically mixing.

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