ANALYTICAL VALIDATION OF THE YOUNG-DUPRÉ LAW FOR EPITAXIALLY-STRAINED THIN FILMS

ELISA DAVOLI AND PAOLO PIOVANO

Abstract. A variational model for epitaxially-strained thin films on substrates is derived both by Γ-convergence from a *transition-layer* setting, and by relaxation of a *sharp-interface* description. The model is characterized by a configurational energy that accounts for possibly different elastic properties for the film and the substrate, as well as for the surface tensions of all three involved interfaces: film/gas, substrate/gas, and film/substrate. Minimal configurations of this energy are then shown to exist and their regularity and geometrical properties are studied. The Young-Dupré law is shown to be satisfied by the angle that energetically-optimal profiles form at contact points with the substrate. This appears to be the first analytical validation of such relation, which was originally formulated in Fluid Mechanics, in the context of Continuum Mechanics for a thin-film model.

Contents

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1. Introduction

Originally formulated in the context of Fluid Mechanics and sessile liquid drops [12, 44], the *Young-Dupré law* characterizes the contact angle formed by drops at any touching point with their supporting surfaces (see Figure 1). As this condition involves both the tension of supporting surfaces and the contact angles of drops (see Subsection 1.1), the law is often used to determine the unknown surface tension of certain materials by measuring the contact angles formed by different probe liquids.

The use of this law is however not only restricted to liquid drops, but it has been naturally extended to *epitaxy*, i.e., to the deposition of crystalline films on crystalline substrates [35, Section 4.2.2]. Contact-angle conditions are in fact often essential for studying multiple-phase systems, as they represent the crucial boundary conditions for characterizing interface morphologies at (triple) *junctions* [4, 42]. A crucial difference between the setting of sessile drops and the one of thin-film deposition, though, is that in the latter elasticity has also to be taken into account as it might strongly affect the profile of the film. Indeed, the *mismatch* between the crystalline lattices of the film and the substrate can induce large stresses in the film. In order to release such energy the atoms of the film move from their crystalline equilibrium to reach more favorable arrangements [19].

Despite the applications of the Young-Dupré law to elastic solids, a mathematical justification in the context of Continuum Mechanics seems to be missing in the Literature. In this regard we refer the reader to [41] for a discussion on whether the presence of stresses modifies contact angles or not. In this paper we provide such mathematical validation in *linear elasticity* in the context of thin films starting from the models introduced in [38]. Among our results we in particular find that the classical contact angles determined by the Young-Dupré law are not impacted by the singular elastic fields present at the wedges of the contact corners.

The purpose of this paper is therefore threefold. First, in Theorem 2.3 we analytically derive a variational thin-film model (see Subsection 1.2) with energy $\mathcal{F}$ defined in (1.2), both as Γ-limit of the *transition-layer models* considered in [38], and as relaxation of their associated *sharp-interface* description (see Subsection 2.2). Second, we show that optimal thin-film profiles of $\mathcal{F}$ satisfy the Young-Dupré law for angles $\theta \in [0, \pi/2]$ (see Theorem 2.4). Third, in Theorem 2.5 the regularity of the profile of minimizing configurations of $\mathcal{F}$ is assessed.

1.1. **The Young-Dupré law in Fluid Mechanics.** The first formulation of the law dates back to 1805 and is due to Thomas Young [44] who derived it by computing the mechanical equilibrium of drops resting on planar surfaces under the action of the surface tensions γ_f, γ_s, γ_{fs} of the three involved interfaces, respectively, the drop/gas interface, the substrate/gas interface, and the drop/substrate interface. Notice that we use here the subscript f because in our setting drops coincide with film material. Subsequently in 1869 a zero-angle condition for the case in which $\gamma_f \le \gamma_s - \gamma_{fs}$ (also called *wetting criterion* in [37]) has been included in the relation by Anthanase Dupré and Paul Dupré (see [12]). A formulation of the law that includes both the contributions of [44] and [12] is

$$\cos\theta = \frac{\min\{\gamma_f, \gamma_s - \gamma_{fs}\}}{\gamma_f}, \tag{1.1}$$

where θ is the contact angle between planar surface of the substrate and the film profile (see Figure 1).

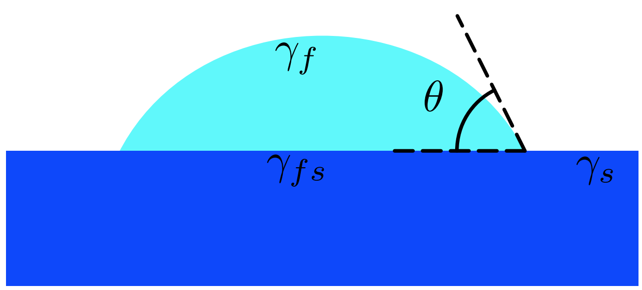

FIGURE 1. Contact angle of a sessile drop.

In 1877 Carl Friedrich Gauss [22] introduced a free energy consisting of four terms: a free surface energy related to the boundary of the drop detached from the substrate, a wetting energy accounting for the adhesion of the drop to the supporting surface and depending of an adhesion coefficient σ, a gravitational energy, and a Lagrange multiplier to include a constraint on the volume of the drop. We recall that a law which includes this adhesion coefficient σ has been formulated by Pierre Simon Laplace in [32] also starting from the ideas in [44]. This law which is often referred to as *Young-Laplace* law in the context of *capillarity problems*, i.e., problems related to fluids in containers, can be stated as $\nu_D \cdot \nu_C = \sigma$, where ν_D and ν_C are the exterior normal to the drop and the container, respectively. We observe that (1.1) is equivalent to the Young-Laplace law when $-\sigma$ corresponds to the right-hand side of (1.1).

However, also the results in the Literature related to the Young-Laplace law seem not to include elasticity. In particular, in [7] the authors prove that, if $\sigma \in (-1, 0)$, than the detached boundary of the minimizing drops of the Gauss free energy is the graph of a function describing the thickness of the drop. This result, although in a different context, is in accordance with our analysis. In fact we assume that the admissible film profiles are *graphs of height functions* and the conditions that we need to impose on γ_f, γ_s, γ_{fs} are such that the right-hand side of (1.1) belongs to $[0, 1]$ (where, though, the boundary values can be included in our analysis). For more general conditions on the adhesion coefficient σ we refer the reader to [2] and [10], where every set of finite perimeter is an admissible drop, and the boundary regularity of optimal drops is studied also in the presence of anisotropy.

1.2. **The thin-film model.** The first rigorous validation of a thin-film model as Γ-limit of the transition-layer model of [38] was performed in the seminal paper [14]. Our analysis moves ahead from [14], as we not only consider the free profile of the film and the surface of the substrate, but also the interface between the film and the substrate, and we take into account the (possible) different elastic properties of the film and substrate materials. This is particularly important to fully treat the often encountered situation of *heteroepitaxy*, i.e., the deposition of a material different from the one of the substrate.

In order to describe our model, we need to introduce some notation. Following [38] we regard the substrate and the film as continua, we work in the framework of the *theory of small deformations* of *linear elasticity*, and, as in [14], we restrict our analysis to two-dimensional profiles (or three-dimensional configurations with planar symmetry). The interface between the film and the substrate is always assumed to be contained in the x-axis and the film thickness is measured by the height function $h : [a, b] \to [0, +\infty)$ with $b > a > 0$. The subgraph

$$\Omega_h := \{(x, y) : \ a < x < b, \, y < h(x)\},$$

is the region occupied by the film and the substrate material, whereas the *graph*

$$\Gamma_h := \partial\Omega_h \cap ((a, b) \times \mathbb{R})$$

of the height function h represents the film profile. The elastic deformations of the film are encoded by the *material displacement* $u : \Omega_h \to \mathbb{R}^2$, and its associated *strain-tensor*, i.e., the symmetric part of the gradient of u, denoted by

$$Eu := \mathrm{sym}\nabla u.$$

In order to account for non-regular profiles, as in [14] the height function is assumed to be lower semicontinuous and with bounded pointwise variation. We denote by

$$\tilde{\Gamma}_h := \partial\overline{\Omega}_h \cap ((a, b) \times \mathbb{R}),$$

and by Γ_h^{cut} the set of *cuts* in the profile of h, namely $\Gamma_h^{cut} := \Gamma_h \setminus \tilde{\Gamma}_h$.

As previously mentioned, elasticity must be included in the model as it plays a major role in heteroepitaxy. Large stresses are in fact induced in the film by the lattice mismatch between the film and the substrate materials [19]. We introduce a parameter $e_0 \geq 0$ to represent such lattice mismatch and, as in [14], we assume that the minimum of the energy is reached at

$$E_0(y) := \begin{cases} e_0\,(\mathbf{e_1} \odot \mathbf{e_1}) & \text{if } y \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $(\mathbf{e_1}, \mathbf{e_2})$ is the standard basis of $\mathbb{R}^2$. In the following we refer to E_0 as the *mismatch strain.*

The model considered in this paper is then characterized by an energy functional $\mathcal{F}$, defined for any film configuration (u, h) as

$$\mathcal{F}(u,h) \,=\, \int_{\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy \\ + \int_{\tilde{\Gamma}_h} \varphi(y)\,d\mathcal{H}^1 \,+\, \gamma_{\rm fs}(b-a) \,+\, 2\gamma_{\rm f}\mathcal{H}^1(\Gamma_h^{cut}), \qquad (1.2)$$

where the surface density φ is given by

$$\varphi(y) := \begin{cases} \gamma_f & \text{if } y > 0, \\ \min\{\gamma_f, \gamma_s - \gamma_{fs}\} & \text{otherwise,} \end{cases}$$

with

$$\gamma_f > 0, \quad \gamma_s > 0, \quad \text{and} \quad \gamma_s - \gamma_{fs} \geq 0. \qquad (1.3)$$

The elastic energy density $W_0 : \mathbb{R} \times \mathbb{M}^{2\times 2}_{\rm sym} \to [0, +\infty)$ satisfies

$$W_0(y, E) := \frac{1}{2} E : \mathbb{C}(y) E$$

for every $(y, E) \in \mathbb{R} \times \mathbb{M}^{2\times 2}_{\rm sym}$. In the expression above $\mathbb{C}(y)$ represents the elasticity tensor,

$$\mathbb{C}(y) := \begin{cases} \mathbb{C}_f & \text{if } y > 0, \\ \mathbb{C}_s & \text{otherwise,} \end{cases}$$

and is assumed to satisfy

$$E : \mathbb{C}(y) E > 0 \qquad (1.4)$$

for every $y \in \mathbb{R}$ and $E \in \mathbb{M}^{2\times 2}_{\rm sym}$. The fourth-order tensors $\mathbb{C}_f$ and $\mathbb{C}_s$ are symmetric and positive-definite, and we allow them to be possibly different to include the case of a different elastic behavior for the film and the substrate. An alternative formulation of heteroepitaxy is to consider $E_0 \equiv 0$ in (1.2), and to impose a *transmission Dirichlet condition* at the interface between film and substrate. We refer to Remark 2.6 to see that the two corresponding minimum problems are equivalent.

Energy functionals of the form (1.2) also appear in the study of *Stress-Driven Rearrangement Instabilities (SDRI)* [24] and well represent the *competition* between the *roughening effect* of the elastic energy and the *regularizing effect* of the surface energy that characterize the formation of such crystal microstructures (see [19, 21, 24] and [13] for the related problem of crystal cavities).

As already mentioned at the beginning of this subsection a similar functional to (1.2) was derived in [14] by Γ-convergence from the transition-layer model introduced in [38] in the case in which $\mathbb{C}_f = \mathbb{C}_s$, and $\gamma_{fs} = 0$. We observe here that in [14] the regularity of the local minimizers of such energy is studied for isotropic film and substrate in the case in which $\gamma_f \leq \gamma_s$, and the local minimizers are shown to be smooth outside of finitely many *cusps* and *cuts* and to form zero contact angles with the substrate (see also [6, 13]). In the same regime in [20] *thresholds* for the film volume (dependent on the lattice mismatch), below which the flat configuration is an absolute minimizer or only a local minimizer, and below which minimizers are smooth, have been identified. We point out that the functional in [14], when restricted to the regime $\gamma_f \leq \gamma_s$ did not present any discontinuity along the film/substrate interface contained in the x-axis. The same applies for the energy in [20]. In our more general setting, instead, (1.2) always presents a sharp discontinuity with respect to the elastic tensors. Additionally the geometrical and regularity results of this paper include the *dewetting regime*, $\gamma_f > \gamma_s - \gamma_{fs}$, for which the surface tension is also discontinuous.

Related SDRI models have been studied in [3, 18, 23]. In [18] the existence and the shape of island profiles, which enforces the presence of nonzero contact angles, has been analyzed in the constraint of faceted profiles. In [23] a mathematical justification of island nucleation was provided by deriving scaling laws for the minimal energy in terms of e_0 and the film volume, and then extended in [3] to the situation of unbounded domains, in the two regimes of small- and large-slope approximations for the profile function h. Finally, the evolutionary problem for thin-film profiles has been studied in dimension two in [15] for the evolution driven by surface diffusion, and in [36] for the growth in the evaporation-condensation case. Recently the analysis of [15] has been extended to three dimensions in [16].

1.3. **Wettability and growth modes.** In applications the importance of determining on which parameters contact angles in epitaxial growth depend, and of precisely characterizing their amplitude, resides on the need to control the film adherance to substrates, which is also referred to as the *film wettability*, that depends on the chemical interactions between the constituents of the two materials. Zero contact angles correspond to complete wetting that occurs when an infinitesimal thin layer of film atoms, the *wetting layer*, spreads freely on the substrate and covers it. Positive angles instead represent the so called situation of nonspreading films, in which the substrate is partially exposed [45]. Since contact angles represent the degree of the wettability of the film, they are also in general referred to as *wetting angles*.

It is exactly because of their various possible morphologies and wettability properties that thin films play nowadays a key role in an ever-growing number of technologies which range from optoelectronics to semiconductor devices, and from solid oxide fuel/hydrolysis cells to photovoltaic devices. In fact, different modes of growth relate to different film wettability: *Volmer-Weber* (VW) mode, in which separated islands form on top of the substrate, or situations in which the substrate is completely covered such as in the *Frank-van der Merwe* (FM) and *Stranski-Krastanov* (SK) modes. FM and SK differ as FM consists in a layer-by-layer growth (next level starting only upon completion of previous layers), while SK presents islands which are nucleated on top of a wetting layer [35].

Therefore, a large effort has been played at the engineering stage to improve the accuracy with which the resulting processed films correspond to the designed geometries. Any advancement in the modeling that improves the engineering of pre-determined profile shapes has therefore a direct economical impact as it contributes to saving computational time needed for simulations, and to reducing the waste of material used in the current work-intensive and expensive trial-and-error production. We notice here that as a byproduct of our analysis, we also deduce that the VW thin-film mode (which corresponds to a positive wetting angle) is exhibited if and only if $\gamma_f > \gamma_s - \gamma_{fs}$.

1.4. Organization of the paper and methodology.

The paper is organized as follows. In Section 2 we introduce the mathematical setting, and we rigorously state our main results (see Theorems 2.3, 2.4, and 2.5).

Section 3 is devoted to the analytical derivation of the energy (1.2) by relaxation and by Γ-convergence, respectively, from the sharp-interface and the transition-layer models (see Theorem 2.3).

In Section 4 we begin studying the regularity properties for the local minimizers of such energy. These results are achieved by performing a volume penalization of the energy to allow more freedom in the allowed variations, and by proving in our setting the *internal-ball condition*, an idea first introduced in [8] and employed also in [13, 14]. In view of these initial regularity results, we develop a novel strategy for deriving contact-angle conditions. The originality of the method consists in implementing in our thin-film setting some ideas used for *transmission problems*. Relying on a *decomposition formula* established in [34], as well as by implementing some regularity results for transmission problems based on properties of the *Mellin transform* and of the *operator pencil* (see [33]), we prove in Proposition 4.6 a *decay estimate* for the displacements corresponding to local minimizers of $\mathcal{F}$.

In Section 5, in view of Proposition 4.6 we are able to perform a blow-up argument at the film/substrate contact points and to pass to the limit in the Euler equation satisfied by local minimizers (by considering variations only with respect to the profile functions). Among the contact points Z_h of minimal profiles h we distinguish the isolated ones from the extrema of non-degenerate intervals in Z_h, and we refer to the first as *valleys* and to the latter as *island borders*. Careful choices of suitable competitors for the minimal profile functions with respect to the different cases of valleys and island borders allow in Proposition 5.1 to identify corresponding contact-angle conditions. In particular the conditions proved in Proposition 5.1 include the Young-Dupré law for the *wetting regime*, $\gamma_f \leq \gamma_s - \gamma_{fs}$. For the *dewetting regime*, $\gamma_f > \gamma_s - \gamma_{fs}$, the Young-Dupré law is obtained in Theorem 2.4 by a further comparison argument, that shows that angles smaller than the one characterized in (1.1) are not energetically convenient in this regime. As a byproduct of our results we also obtain that in the dewetting regime there are no valleys, and hence, that islands are separated.

Finally, in Section 6 an adaptation of the proof strategy of Theorem 2.4, together with improved decay estimates along the lines of [14], allow to reach in Theorem 2.5 the final regularity results for local minimizers.

2. Main results

2.1. Mathematical setting.

In this subsection we introduce the main definitions and the notation used throughout the paper. We begin by characterizing the admissible film profiles. The set AP of admissible film profiles in (a, b) is denoted by

$$AP(a, b) := \{h : [a, b] \to [0, +\infty) \, : \, h \text{ is lower semicontinuous and } \operatorname{Var} h < +\infty\},$$

where $\operatorname{Var} h$ denotes the pointwise variation of h, namely,

$$\operatorname{Var} h := \sup\Big\{ \sum_{i=1}^{n} |h(x_i) - h(x_{i-1})| \, : \\ P := \{x_1, \dots, x_n\} \text{ is a partition of } [a,b] \Big\}.$$

We recall that for every lower semicontinuous function $h : [a,b] \to [0,+\infty)$, to have finite pointwise variation is equivalent to the condition

$$\mathcal{H}^1(\Gamma_h) < +\infty,$$

where

$$\Gamma_h := \partial\Omega_h \cap ((a,b) \times \mathbb{R}).$$

For every $h \in AP(a,b)$, and for every $x \in (a,b)$, consider the left and right limits

$$h(x^{\pm}) := \lim_{z \to x^{\pm}} h(z),$$

we define

$$h^-(x) := \min\{h(x^+), h(x^-)\} = \liminf_{z\to x} h(z),$$

and

$$h^+(x) := \max\{h(x^+), h(x^-)\} = \limsup_{z\to x} h(z).$$

In the following $\operatorname{Int}(A)$ denotes the interior part of a set A. Let us now recall some properties of height functions $h \in AP(a,b)$, regarding their graphs Γ_h, their subgraphs Ω_h, the film and the substrate parts of the subgraph,

$$\Omega_h^+ := \Omega_h \cap \{y > 0\}$$

and

$$\Omega_h^- := \Omega_h \cap \{y \leq 0\}$$

respectively, and the sets

$$\tilde{\Gamma}_h := \partial\bar{\Omega}_h \cap ((a,b) \times \mathbb{R}). \tag{2.1}$$

Any $h \in AP(a,b)$ satisfies the following assertions (see [14, Lemma 2.1]):

1. Ω_h^+ has finite perimeter in $((a,b) \times \mathbb{R})$,
2. $\Gamma_h = \{(x,y) : \, a < x < b, \, h(x) < y < h^+(x)\}$,
3. h^- is lower semicontinuous and $\operatorname{Int}\left(\overline{\Omega}\right) = \{(x,y) : \, a < x < b, \, y < h^-(x)\}$,
4. $\tilde{\Gamma}_h = \{(x,y) : \, a < x < b, \, h^-(x) \leq y \leq h^+(x)\}$,
5. Γ_h and $\tilde{\Gamma}_h$ are connected.

We now characterize various portions of Γ_h. To this aim we denote the *jump* set of a function $h \in AP(a,b)$, i.e., the set of its profile discontinuities, by

$$J(h) := \{x \in (a,b) : \, h^-(x) \neq h^+(x)\}, \tag{2.2}$$

whereas the set identifying *vertical cuts* in the graph of h is given by

$$C(h) := \{x \in (a,b) : \, h(x) < h^-(x)\}. \tag{2.3}$$

The graph Γ_h of a height function h is then characterized by the decomposition

$$\Gamma_h = \Gamma_h^{jump} \sqcup \Gamma_h^{cut} \sqcup \Gamma_h^{graph},$$

where $\sqcup$ denotes the disjoint union, and

$$\Gamma_h^{jump} := \overline{\{(x,y) : \, x \in (a,b) \cap J(h), \, h^-(x) \leq y \leq h^+(x)\}},$$

$$\Gamma_h^{cut} := \{(x,y) : x \in (a,b) \cap C(h),\, h(x) \le y < h^-(x)\}, \tag{2.4}$$
$$\Gamma_h^{graph} := \Gamma_h \setminus (\Gamma_h^{jump} \cup \Gamma_h^{cut}).$$

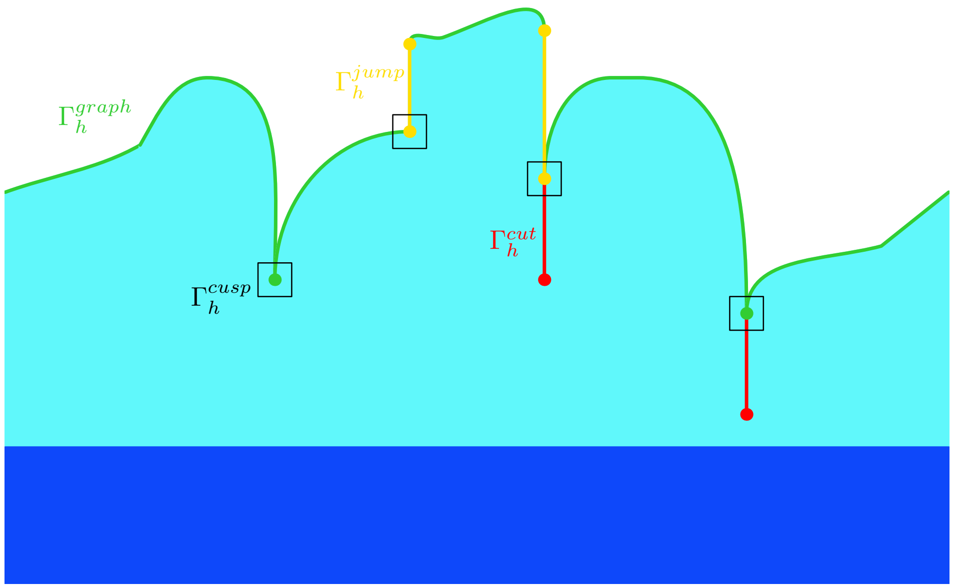

FIGURE 2. In the figure above an admissible profile function h is displayed. The portions of Γ_h corresponding to Γ_h^{graph}, Γ_h^{jump}, and Γ_h^{cut} are represented with the colors green, yellow, and red respectively. The points in Γ_h^{cusp} are marked by enclosing them within squares.

We observe that Γ_h^{graph} represents the regular part of the graph of h, whilst both Γ_h^{jump} and Γ_h^{cut} consist in (at most countable) unions of segments, corresponding to the *jumps* and the *cuts* in the graph of h, respectively (see Figure 2). Notice also that

$$\Gamma_h = \tilde{\Gamma}_h \sqcup \Gamma_h^{cut}.$$

Let us also identify the set of *cusps* in Γ_h by

$$\Gamma_h^{cusp} := \big\{(x, h^-(x)) : \text{either } x \in J(h) \\ \text{or we have that } x \notin J(h) \text{ with } h'_+(x) = +\infty \text{ or } h'_-(x) = -\infty\big\}$$

(see Figure 2).

For every $h \in AP(a,b)$ we indicate its set of of zeros by

$$Z_h := \Gamma_h \cap \{x \in [a,b] : h(x) = 0\}.$$

For every $x \in Z_h$, let $\theta^\pm(x)$ be the internal angles, with amplitude smaller or equal to $\frac{\pi}{2}$, between the x-axis and the tangents to Γ_h in $(x,0)$ from the left and from the right, with slopes $h'_-(x)$ and $h'_+(x)$, respectively. Consider the set

$$I_h := \{(c,d) \subset Z_h : c, d \notin \mathrm{Int}(Z_h)\},$$

and let

$$P_h := Z_h \setminus \bigcup_{(c,d)\in I_h} [c,d].$$

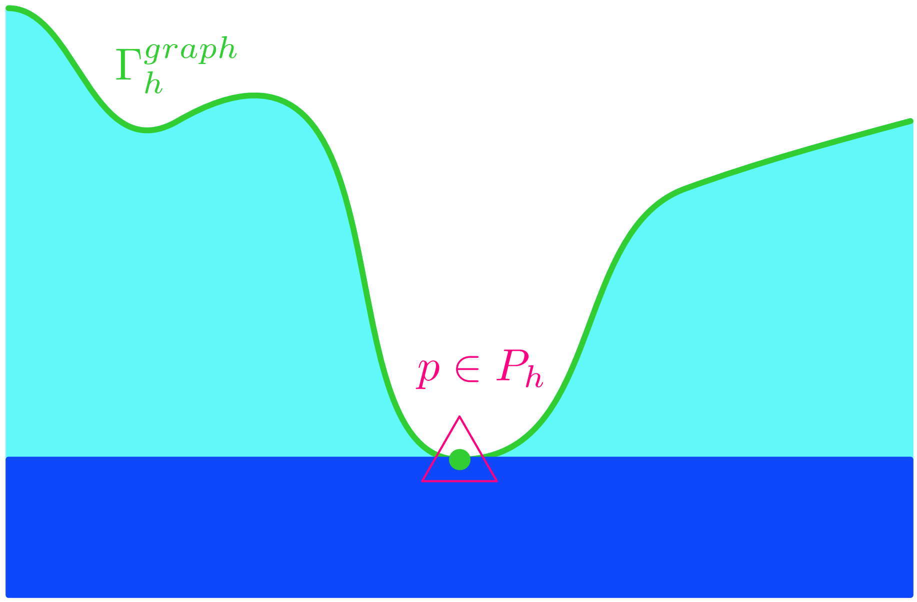

FIGURE 3. A valley at an isolated point $p \in P_h$ is displayed. The point p is indicated by enclosing it in a pink triangle

We will refer to the endpoints c and d of any interval $(c,d) \in I_h$ as *borders of (two different) islands* and to the points in P_h as *valleys*, and we observe that

$$\begin{cases} \theta^-(x) = 0 & \text{for every } x \in (c,d] \\ \theta^+(x) = 0 & \text{for every } x \in [c,d) \end{cases} \tag{2.5}$$

(see Figures 3 and 4).

We now define the family X of admissible film configurations as

$$X := \{(u,h) :\, u \in H^1_{\mathrm{loc}}(\Omega_h;\mathbb{R}^2) \text{ and } h \in AP(a,b)\}$$

and we endow X with the following notion of convergence.

Definition 2.1. We say that a sequence $\{(u_n,h_n)\} \subset X$ converges to $(u,h) \in X$, and we write $(u_n,h_n) \to (u,h)$ in X if

1. $\sup_n \operatorname{Var} h_n < +\infty$,
2. $\mathbb{R}^2 \setminus \Omega_{h_n}$ converges to $\mathbb{R}^2 \setminus \Omega_h$ in the Hausdorff metric,
3. $u_n \rightharpoonup u$ weakly in $H^1(\Omega';\mathbb{R}^2)$ for every $\Omega' \subset\subset \Omega_h$.

Let us also consider the following subfamily X_{Lip} in X of configurations with Lipschitz profiles, namely,

$$X_{\mathrm{Lip}} := \{(u,h) :\, u \in H^1_{\mathrm{loc}}(\Omega_h;\mathbb{R}^2),\ h \text{ is Lipschitz}\}.$$

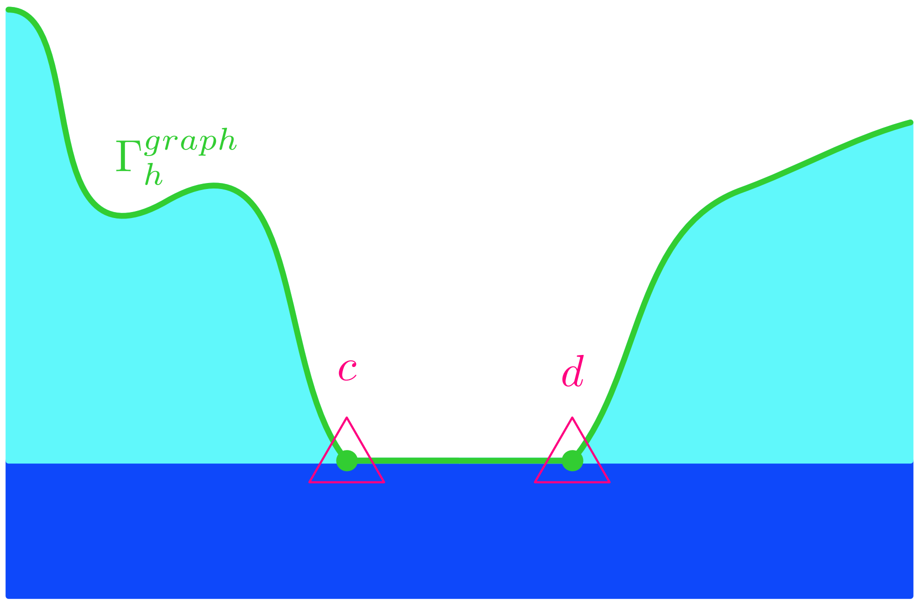

FIGURE 4. An interval $(c, d) \in I_h$ is displayed. The points c, d indicated with pink triangles are the only ones in I_h with non-trivial contact angles.

We recall from Subsection 1.2 that the thin-film model analyzed in this paper is characterized by the energy $\mathcal{F}$ defined by (1.2) on configurations $(u, h) \in X$.

We state here the definition of *μ-local minimizers* of the energy $\mathcal{F}$.

Definition 2.2. We say that a pair $(u, h) \in X$ is a μ-local minimizer of the functional $\mathcal{F}$ if $\mathcal{F}(u, h) < +\infty$ and there exists $\mu > 0$ such that

$$\mathcal{F}(u, h) \leq \mathcal{F}(v, g)$$

for every $(v, g) \in X$ satisfying $|\Omega_g^+| = |\Omega_h^+|$ and $|\Omega_g \Delta \Omega_h| \leq \mu$.

Note that every global minimizer (with or without volume constraint) is a μ-local minimizer.

2.2. **The sharp-interface and the transition-layer models.** We now recall classical thin-film models from the Literature. The sharp-interface model for epitaxy is characterized by a configurational energy $\mathcal{F}_0$ that presents a discontinuous transition both in the elasticity tensors and in the surface tensions, and that encodes the abrupt change in materials across the film/substrate interface at the x-axis. We set

$$\mathcal{F}_0(u, h) := \int_{\Omega_h} W_0(y, Eu(x, y) - E_0(y))\, dx\, dy + \int_{\Gamma_h} \varphi_0(y)\, d\mathcal{H}^1 + \gamma_{fs}\mathcal{H}^1((a, b) \setminus Z_h) \tag{2.6}$$

for every $(u,h) \in X_{\text{Lip}}$, where the energy density $\varphi_0 : \mathbb{R} \to [0,+\infty)$ forces a sharp discontinuity at $\{y=0\}$, namely

$$\varphi_0(y) := \begin{cases} \gamma_f & \text{if } y > 0 \\ \gamma_s & \text{if } y = 0, \end{cases}$$

for positive constants γ_f and γ_s. The same energy functional has been considered in [38], where it appears without the last term since in that framework γ_{fs} is considered to be negligible. We notice that $\mathcal{F}$ and $\mathcal{F}_0$ differ only with respect to the surface energy, and that $\mathcal{F}$ is extended to the set X.

Models presenting regularized discontinuities have been introduced in the Literature because more easy to implement numerically (see, e.g., [31, 38]). They can be considered as an approximation of the sharp-interface functional $\mathcal{F}_0$ where the elastic tensors and/or the surface densities are regularized over a thin transition region of width $\delta > 0$ (see Figure 5).

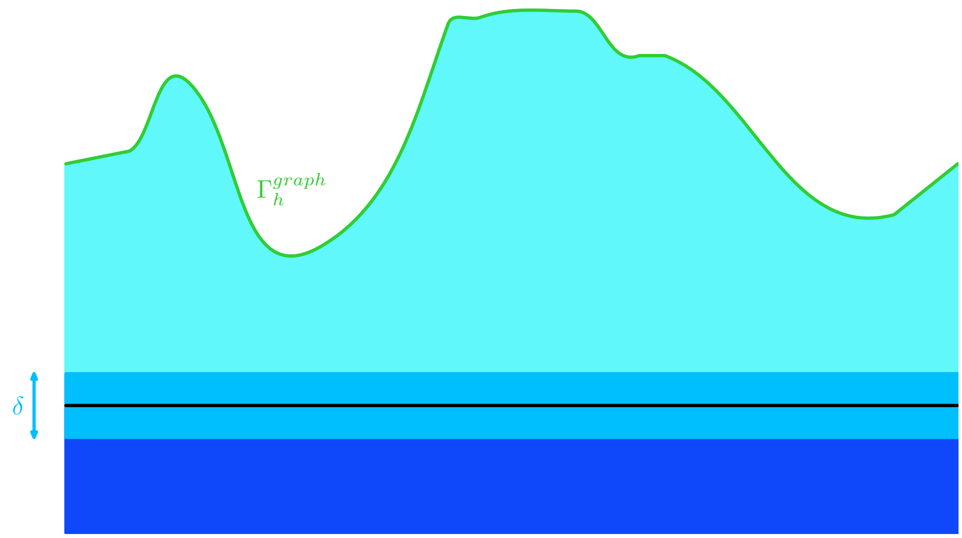

FIGURE 5. In the transition-layer model the elastic tensors and the surface tension are regularized over a (thin) layer with thickness $\delta > 0$.

In order to introduce the energy functional $\mathcal{F}_\delta$ corresponding to the *transition-layer model* with transition layer of width $\delta > 0$, we consider an auxiliary smooth and increasing function f such that $f(0) = 0$, $\lim_{y\to+\infty} f(y) = 1$, $\lim_{y\to-\infty} f(y) = -1$, and

$$\int_{-\infty}^{0} (1 + f(y))^2 \, dy < +\infty. \tag{2.7}$$

We notice that the hypotheses on f are satisfied for example by the *boundary-layer function*

$$r \mapsto \frac{2}{\pi} \arctan(r)$$

proposed in [30, 31] (see also [38]). The regularized mismatch strain is defined as

$$E_\delta(y) := \frac{1}{2}e_0\Big(1 + f\Big(\frac{y}{\delta}\Big)\Big)\mathbf{e_1} \odot \mathbf{e_1} \quad \text{for every } y \in \mathbb{R},$$

whereas the regularized surface energy density takes the form

$$\varphi_\delta(y) := \gamma_f f\left(\frac{y}{\delta}\right) + (\gamma_s - \gamma_{fs})\left(1 - f\left(\frac{y}{\delta}\right)\right),$$

for every $y \in \mathbb{R}$ (see [39]).

The transition-layer energy functional is then given by

$$\mathcal{F}_\delta(u,h) := \int_{\Omega_h} W_\delta(y, Eu(x,y) - E_\delta(y))\,dx\,dy + \int_{\Gamma_h} \varphi_\delta(y)\,d\mathcal{H}^1 + \gamma_{fs}(b-a)$$

for every $(u,h) \in X_{\mathrm{Lip}}$, where $W_\delta(y,E) := \frac{1}{2}E : \mathbb{C}_\delta(y)E$ for every $y \in \mathbb{R}$ and $E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, with

$$\mathbb{C}_\delta(y) := \frac{1}{2}\Big(1 + f\Big(\frac{y}{\delta}\Big)\Big)\mathbb{C}_f + \frac{1}{2}\Big(1 - f\Big(\frac{y}{\delta}\Big)\Big)\mathbb{C}_s + \frac{1}{2}\Big(1 + f\Big(\frac{y}{\delta}\Big)\Big)\Big(1 - f\Big(\frac{y}{\delta}\Big)\Big)(\mathbb{C}_s - \mathbb{C}_f).$$

Notice that $\mathbb{C}_\delta(0) = \mathbb{C}_s$, and that $\mathbb{C}_\delta(y)$ is symmetric and positive-definite for every $y \in \mathbb{R}$. Additionally, there exists a positive constant C such that

$$\mathbb{C}_\delta(y)F : F \leq C|F|^2 \quad \text{for every } F \in \mathbb{M}^{2\times 2}. \tag{2.8}$$

2.3. **Statement of the main results.** The paper contains three main theorems. The first result concerns the derivation of the energy functional $\mathcal{F}$ from the transition-layer functional $\mathcal{F}_\delta$ and the sharp-interface model $\mathcal{F}_0$.

Theorem 2.3 (Model derivation)**.** *The energy $\mathcal{F}$ is both*

1. *The relaxed functional of $\mathcal{F}_0$, i.e.,*

$$\mathcal{F}(u,h) := \inf\Big\{\liminf_{n\to+\infty} \mathcal{F}_0(u_n,h_n) : (u_n,h_n) \in X_{\mathrm{Lip}}, (u_n,h_n) \to (u,h) \text{ in } X, \text{ and } |\Omega^+_{h_n}| = |\Omega^+_h|\Big\}$$

for every $(u,h) \in X$.

2. *The Γ-limit as $\delta \to 0$ of the transition layer energies $\mathcal{F}_\delta$ under the volume constraint.*

We note that Theorem 2.3 holds also for non-isotropic films and substrates. The remaining two main results instead refer to the situation in which $\mathbb{C}_f$ and $\mathbb{C}_s$ are the elasticity tensors of isotropic materials with Lamé coefficients μ_f, λ_f, and μ_s, λ_s, respectively.

Our second result regards the identification of contact angle conditions for the μ-local minimizers $(u,h) \in X$ of $\mathcal{F}$.

Theorem 2.4 (Contact-angle conditions)**.** *Assume that the Lamé coefficients of the film and the substrate satisfy*

$$\mu_s \geq \mu_f > 0 \quad \textit{and} \quad \mu_s + \lambda_s \geq \mu_f + \lambda_f > 0. \tag{2.9}$$

Then, for every μ-local minimizer $(u,h) \in X$ of $\mathcal{F}$ the set $Z_h \setminus (\Gamma_h^{cusp} \cup \Gamma_h^{cut})$ satisfies the following properties:

1. *If $\beta < 1$, then $P_h = \emptyset$, otherwise, if $\beta = 1$, then*
$$\theta^-(x_0) = \theta^+(x_0) = \arccos(\beta)$$
for every $x_0 \in P_h$,
2. *For any $(c,d) \in I_h$, then*
$$\theta^-(c) = \theta^+(d) = \arccos(\beta),$$

where
$$\beta := \frac{\min\{\gamma_f, \gamma_s - \gamma_{fs}\}}{\gamma_f}. \tag{2.10}$$

Additionally, for Γ_h^{jump} the following assert holds true:

3. *if $\beta \neq 0$, then $\Gamma_h^{jump} \cap Z_h = \emptyset$.*

We remark that Theorem 2.4 is the analytical validation of the Young-Dupré law for angles not greater than $\pi/2$. Let us sum up here the possible scenarios for the wetting angles:

Wetting regime: For $\gamma_s - \gamma_{fs} \geq \gamma_f$ all contact angles of $\Gamma_h \setminus (\Gamma_h^{cut} \cup \Gamma_h^{cusp})$ are zero.

Dewetting regime: For $\gamma_s - \gamma_{fs} < \gamma_f$ all nontrivial contact angles θ of points in $Z_h \setminus (\Gamma_h^{cut} \cup \Gamma_h^{cusp})$ are such that
$$\cos\theta = \frac{\gamma_s - \gamma_{fs}}{\gamma_f}.$$

We stress that, in agreement with the Young-Dupré law, jumps at island borders (see Figure 6) are excluded when $\min\{\gamma_f, \gamma_s - \gamma_{fs}\}/\gamma_f \neq 0$. Note also that the contact angles at valleys are always zero (and there are no jumps at valleys), since valleys exist only for the wetting regime when $\beta = 1$.

However, our analysis allows the set $D_h := \left[(\Gamma_h^{cusp} \cup \Gamma_h^{cut}) \setminus \Gamma_h^{jump}\right] \cap Z_h$ to be nonempty. It seems though that this is not a restriction of our method but it is in agreement with the experimental evidence. Points in D_h may represent in fact *dislocations* that are experimentally shown to form as a further mode of strain relief and to migrate at the film/substrate interface. We kindly refer the reader to [17] and the reference therein for more details on dislocations in epitaxy and for a thin-film model accounting for their presence. Some examples of contact angles in D_h are displayed in Figure 7.

Regarding condition (5.2), assuming μ_f, $\mu_s > 0$, $\lambda_f + \mu_f > 0$, and $\lambda_s + \mu_s > 0$ guarantees the ellipticity of the transmission problem associated to the Euler-Lagrange equations of μ-local minimizers of $\mathcal{F}$ (see [26, Lemma 1.3]). The assumption

$$\mu_s \geq \mu_f \quad \text{and} \quad \mu_s + \lambda_s \geq \mu_f + \lambda_f \tag{2.11}$$

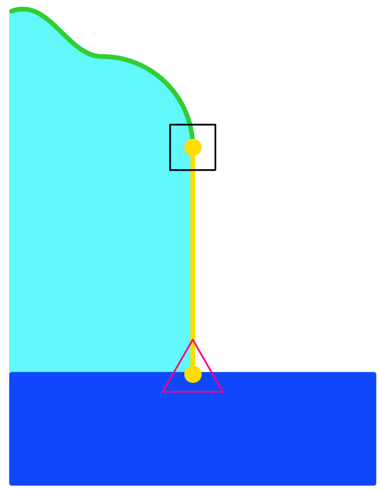

FIGURE 6. Example of a jump at an island border (here indicated with a pink triangle). This is the only type of jump allowed by Theorem 2.4 and only if $\min\{\gamma_f, \gamma_s - \gamma_{fs}\}/\gamma_f = 0$.

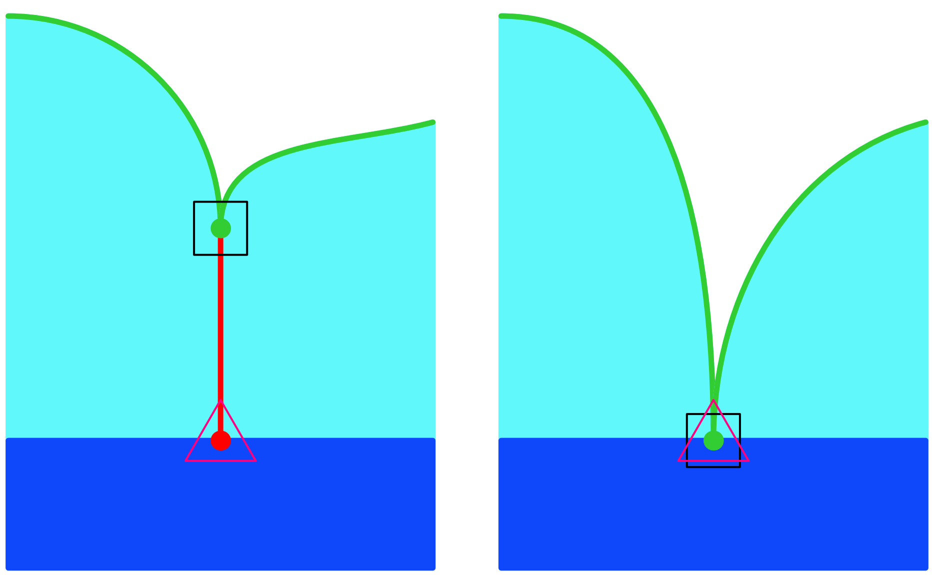

FIGURE 7. Cuts (left) and cusps (right) may represents dislocations at the film/substrate interface.

is a *quasi-monotonicity condition.* This kind of assumptions are classically considered in transmission problems for elliptic systems, we refer the reader to [11] for the first formulation for transmission problems with the Laplace operator (see also [27] and the references therein). As stated in [28] where (2.11) is introduced, "it seems that the quasi-monotonicity condition [...] describes a class of composites which can sustain higher loads before breaking". Furthermore, condition (2.11) implies that the *shear* and the *P-wave moduli* of the substrate are higher than those of the film. As such parameters are *elastic moduli* for the materials, this entails that the substrate is stiffer than the film. Such requirement appears to be natural in the thin-film models here considered from [38], where only the boundary of the film and not the boundary of the substrate is allowed to deform. We recall that in these models the film/substrate interface is forced to coincide with the x-axis. As a matter of fact, quasi-monotonicity conditions are strongly related to the particular geometry in which the transmission problem is considered, and in particular to the position of the transmission interface at boundary corners. Other conditions than (2.11) might be included if the film/substrate interface is not maintained fixed as in [38].

The final main theorem of the paper concerning the regularity of optimal profiles is the following.

Theorem 2.5 (Regularity). *Assume that the Lamé coefficients of the film and the substrate satisfy* (5.2).

Then, every μ-local minimizer $(u,h)\in X$ of $\mathcal{F}$ has the following regularity properties:

1. *Cusps points and vertical cuts are at most finite;*
2. $\Gamma_h^{reg} := \Gamma_h\setminus(\Gamma_h^{cut}\cup\Gamma_h^{cusp})$ *is locally the graph of a Lipschitz function;*
3. Γ_h^{jump} *has no connected components contained in* $\{y>0\}$ *and, if* $\beta\neq 0$, *then* $\Gamma_h^{jump}=\emptyset$*;*
4. $\Gamma_h^{reg}\setminus Y_h$ *is* $C^{1,\alpha}$ *for all* $\alpha\in(0,1/2)$*, where* Y_h *is the subset of* $Z_h\cap\Gamma_h^{reg}$ *containing points with nonzero contact angles for* h*;*
5. *The set*
$$A_h := \begin{cases}\Gamma_h^{reg}\setminus Z_h & \text{if } \mathbb{C}_f\neq\mathbb{C}_s\\ \Gamma_h^{reg}\setminus Y_h & \text{if } \mathbb{C}_f=\mathbb{C}_s\end{cases}$$
is analytic and satisfies the Euler-Lagrange equation
$$\gamma_f k_{A_h} = \tau_{A_h}\left(W_0(\cdot,Eu(\cdot)-E_0)\right)+\lambda_0 \quad \text{on } A_h, \tag{2.12}$$
where the function $k_{A_h}(\cdot)$ denotes the curvature of A_h, $\tau_{A_h}(\cdot)$ is the trace operator on A_h, and λ_0 is a suitable Lagrange multiplier.

We also point out that for $\mathbb{C}_f=\mathbb{C}_s$ in view of Assertion 5. of Theorem 2.5 for every μ-local minimizer (u,h) of $\mathcal{F}$ the set Z_h has either finite cardinality or nonempty interior in the x-axis. Finally, we observe that in the wetting regime for $\mathbb{C}_f=\mathbb{C}_s$ the analytic portion of the graph A_h coincides with Γ_h^{reg} since by the assertions 1. and 2. of Theorem 2.4 we have $Y_h=\emptyset$.

Remark 2.6. The results in Theorems 2.3–2.4 hold also for μ-local minimizers of the energy

$$\begin{aligned}\mathcal{E}(u^+,u^-,h) := &\int_{\Omega_h^+}\mathbb{C}_f Eu^+(x,y):Eu^+(x,y)\,dx\,dy \\ &+\int_{\Omega_h^-}\mathbb{C}_s Eu^-(x,y):Eu^-(x,y)\,dx\,dy+\int_{\Gamma_h}\varphi(y)\,d\mathcal{H}^1 \\ &+\gamma_{fs}(b-a)+2\gamma_f\mathcal{H}^1(\Gamma_h^{cut})\end{aligned} \tag{2.13}$$

for every $(u^+, u^-, h) \in \tilde{X}$, where

$$\begin{aligned}\tilde{X} :=&\{(u^+, u^-, h) :\ u^+ \in H^1_{\rm loc}(\Omega_h^+;\mathbb{R}^2), u^- \in H^1_{\rm loc}(\Omega_h^-;\mathbb{R}^2),\\ &\qquad u^+(\cdot,0) - u^-(\cdot,0) = (e_0\cdot,0), \text{ and } h \in AP(a,b)\}.\end{aligned}$$

In fact, there is a 1-1 correspondence between triples (u^+, u^-, h) that are μ-local minimizers of (2.13), and pairs (u, h) which are μ-local minimizers of (1.2), with

$$u(x,y) := \begin{cases} u^+(x,y) - (e_0 x, 0) & \text{if } y \geq 0 \\ u^-(x,y) & \text{if } y < 0 \end{cases}$$

for $(x, y) \in \Omega_h$. Energy functionals similar to (2.13) are considered for the corresponding evolution problem (see, e.g., [43]).

3. Derivation of the thin-film model

In this section we provide a rigorous justification of the model $\mathcal{F}$ defined in (1.2) by proving Theorem 2.3.

Proof of Theorem 2.3. Assertion 1. and 2. of Theorem 2.3 follow, respectively, from Propositions 3.1 and 3.3 which are proven in the following two subsections. □

3.1. Relaxation from the sharp-interface model.

In this subsection we characterize $\mathcal{F}$ as the lower-semicontinuous envelope of the energy $\mathcal{F}_0$ with respect to the convergence in X, restricted to pairs in $X_{\rm Lip}$.

Proposition 3.1 (Relaxation of the sharp-interface model)**.**

$$\begin{aligned}\mathcal{F}(u,h) = \inf\Big\{&\liminf_{n\to+\infty} \mathcal{F}_0(u_n,h_n) :\ (u_n,h_n) \in X_{\rm Lip},\\ &\qquad (u_n,h_n) \to (u,h) \textit{ in } X, \textit{ and } |\Omega^+_{h_n}| = |\Omega^+_h|\Big\},\end{aligned}$$

for every $(u, h) \in X$*.*

Proof. We preliminary observe that the thesis is equivalent to showing that

$$\begin{aligned}\bar{\mathcal{F}}(u,h) := \inf\Big\{&\liminf_{n\to+\infty} \tilde{\mathcal{F}}_0(u_n,h_n) :\ (u_n,h_n) \in X_{\rm Lip},\\ &\qquad (u_n,h_n) \to (u,h) \text{ in } X, \text{ and } |\Omega^+_{h_n}| = |\Omega^+_h|\Big\} = \tilde{\mathcal{F}}(u,h)\end{aligned} \tag{3.1}$$

for every $(u, h) \in X$, where

$$\tilde{\mathcal{F}}_0(u,h) := \int_{\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy + \int_{\Gamma_h} \tilde{\varphi}_0(y)\,d\mathcal{H}^1,$$

with

$$\tilde{\varphi}_0(y) := \begin{cases} \gamma_f & \text{if } y > 0 \\ \gamma_s - \gamma_{fs} & \text{otherwise} \end{cases},$$

and

$$\tilde{\mathcal{F}}(u,h) := \mathcal{F}(u,h) - \gamma_{fs}(b-a).$$

The proof of the inequality

$$\bar{\mathcal{F}}(u,h) \geq \tilde{\mathcal{F}}(u,h)$$

for every $(u,h) \in X$ follows along the lines of [14, Proof of Theorem 2.8, Step 1], by observing that

$$\liminf_{n\to+\infty} \int_{\Gamma_{h_n}} \tilde{\varphi}_0(y)\, d\mathcal{H}^1 \geq \liminf_{n\to+\infty} \int_{\Gamma_{h_n}} \varphi(y)\, d\mathcal{H}^1,$$

and by applying the argument in [14, (2.22)–(2.26)] directly to the density φ, which is lower-semicontinuous and hence allows to use Reshetnyak's theorem (see [1, Theorem 2.38]).

Fix now $(u,h) \in X$. To prove that

$$\bar{\mathcal{F}}(u,h) \leq \tilde{\mathcal{F}}(u,h)$$

it is enough to construct a sequence $\{(u_n,h_n)\} \subset X_{\rm Lip}$ such that

$$\begin{aligned} &(u_n,h_n) \to (u,h) \quad \text{in } X,\\ &|\Omega^+_{h_n}| = |\Omega^+_h|, \end{aligned}$$

and

$$\limsup_{n\to+\infty} \tilde{\mathcal{F}}_0(u_n,h_n) \leq \tilde{\mathcal{F}}(u,h).$$

We subdivide the argument into two steps.

Step 1. Consider the functional

$$\hat{\mathcal{F}}(v,g) := \int_{\Omega_g} W_0(y, Eu(x,y) - E_0(y))\, dx\, dy + \int_{\Gamma_g} \tilde{\varphi}(y)\, d\mathcal{H}^1$$

for every $(v,g) \in X$. Fix $(u,h) \in X$. By applying the construction in [14, Proof of Theorem 2.8, Steps 2–5], we obtain the existence of a sequence $\{(u_n,h_n)\} \subset X_{\rm Lip}$ such that

$$\begin{aligned} &(u_n,h_n) \to (u,h) \quad \text{in } X,\\ &|\Omega^+_{h_n}| = |\Omega^+_h|, \end{aligned}$$

and

$$\lim_{n\to+\infty} \hat{\mathcal{F}}(u_n,h_n) = \tilde{\mathcal{F}}(u,h). \tag{3.2}$$

In the case in which $\gamma_s - \gamma_{fs} \leq \gamma_f$, there holds

$$\hat{\mathcal{F}}(u_n,h_n) = \tilde{\mathcal{F}}_0(u_n,h_n),$$

thus (3.2) implies the thesis.

Step 2. Consider now the case in which $\gamma_f < \gamma_s - \gamma_{fs}$. In view of Step 1, and by a diagonal argument, the thesis reduces to show that for every $(u,h) \in X_{\rm Lip}$ there exists a sequence $\{(u_n,h_n)\} \subset X_{\rm Lip}$ such that

$$\begin{aligned} &(u_n,h_n) \to (u,h) \quad \text{in } X,\\ &|\Omega^+_{h_n}| = |\Omega^+_h|, \end{aligned}$$

and

$$\lim_{n\to+\infty} \tilde{\mathcal{F}}_0(u_n,h_n) = \hat{\mathcal{F}}(u,h). \tag{3.3}$$

Fix $(u,h) \in X_{\mathrm{Lip}}$. We argue as in [14, Proof of Theorem 2.9, Step 2], and we define the maps

$$h_n(x) := \min\{h(x) + \varepsilon_n, t_n\}$$

for every $x \in [a,b]$, where $\{\varepsilon_n\}$ is a vanishing sequence of positive numbers, and $\{t_n\}$ is chosen so that $t_n > 0$ and $|\Omega^+_{h_n}| = |\Omega^+_h|$ for every $n \in \mathbb{N}$. Choosing $y_0 < 0$ such that $u(\cdot, y_0) \in H^1((a,b);\mathbb{R}^2)$ (the existence of y_0 follows by a slicing argument), we set,

$$u_n(x,y) := \begin{cases} u(x, y-\varepsilon_n) & \text{if } y > y_0 + \varepsilon_n, \\ u(x, y_0) & \text{if } y_0 - \varepsilon_n \leq y \leq y_0 + \varepsilon_n, \\ u(x,y) & \text{if } y < y_0 - \varepsilon_n, \end{cases}$$

for every $(x,y) \in \Omega_{h_n}$. By definition,

$$\lim_{n\to+\infty} \int_{\Omega_{h_n}} W_0(y, Eu_n(x,y) - E_0(y))\,dx\,dy = \int_{\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy,$$

and

$$\tilde{\varphi}_0(y + \varepsilon_n) \to \tilde{\varphi}(y) \tag{3.4}$$

for every $y \geq 0$. Property (3.3) follows then by the observation that

$$\limsup_{n\to+\infty} \int_{\Gamma_{h_n}} \tilde{\varphi}_0(y)\,d\mathcal{H}^1 \leq \lim_{n\to+\infty} \int_{\Gamma_h} \tilde{\varphi}_0(\min\{y+\varepsilon_n, t_n\})\,d\mathcal{H}^1 = \int_{\Gamma_h} \tilde{\varphi}(y)\,d\mathcal{H}^1,$$

where the last equality is a consequence of (3.4), and of the Dominated Convergence Theorem. □

3.2. **Γ-convergence from the transition-layer model.** In this subsection we characterize $\mathcal{F}$ defined in (1.2) as the Γ-limit of the transition-layer functionals $\mathcal{F}_\delta$. The proof of this result is a modification of the arguments in [14, Theorems 2.8 and 2.9] to the situation with possibly $\mathbb{C}_f \neq \mathbb{C}_s$ and $\gamma_{fs} \neq 0$, therefore we here highlight only the main changes for convenience of the reader.

We begin by characterizing the lower-semicontinuous envelope of $\mathcal{F}_\delta$ with respect to the convergence in X, restricted to pairs in X_{Lip}, with the integral formula (3.6).

Proposition 3.2 (Relaxation of the transition-layer functionals)**.** *For every $\delta > 0$, let $\bar{\mathcal{F}}_\delta$ be the relaxed functional of $\mathcal{F}_\delta$, namely*

$$\bar{\mathcal{F}}_\delta(u,h) := \inf\Big\{\liminf_{n\to+\infty} \mathcal{F}_\delta(u_n,h_n) :\ (u_n,h_n) \in X_{\mathrm{Lip}},\ (u_n,h_n) \to (u,h) \text{ in } X, \text{ and } |\Omega^+_{h_n}| = |\Omega^+_h|\Big\} \tag{3.5}$$

for every $(u,h) \in X$. Then

$$\bar{\mathcal{F}}_\delta(u,h) = \int_{\Omega_h} W_\delta(y, Eu(x,y) - E_0(y))\,dx\,dy + \int_{\tilde{\Gamma}_h} \varphi_\delta(y)\,d\mathcal{H}^1 + 2\sum_{x\in S} \int_{h(x)}^{h^-(x)} \varphi_\delta(y)\,dy + \gamma_{fs}(b-a), \tag{3.6}$$

for every $(u,h) \in X$.

Proof. Denote by $\hat{\mathcal{F}}_\delta$ the right-hand side of (3.5). The proof of the inequality

$$\bar{\mathcal{F}}_\delta(u,h) \geq \hat{\mathcal{F}}_\delta(u,h)$$

for every $(u,h)\in X$ is analogous to [14, Proof of Theorem 2.8, Step 1]. To prove the opposite inequality, we argue as in [14, Proof of Theorem 2.8, Steps 3–5], and we construct a sequence $\{h_n\}$ of Lipschitz maps such that

$$\begin{aligned} &0 \leq h_n(x) \leq h(x) \quad \text{for every } x\in[a,b], \\ &(u,h_n) \to (u,h) \quad \text{in } X, \\ &\lim_{n\to+\infty} \mathcal{F}_\delta(u,h_n) = \hat{\mathcal{F}}_\delta(u,h). \end{aligned} \tag{3.7}$$

With a slicing argument we identify $y_0<0$ such that $u(\cdot,y_0)\in H^1((a,b);\mathbb{R}^2)$, and we define the maps

$$u_n(x,y) := \begin{cases} u(x,y-\varepsilon_n) & \text{if } y > y_0+\varepsilon_n, \\ u(x,y_0) & \text{if } y_0 < y \leq y_0+\varepsilon_n, \\ u(x,y) & \text{if } y \leq y_0 \end{cases}$$

for a.e. $(x,y)\in\Omega_{\tilde{h}_n}$, where $\tilde{h}_n(x) := h_n(x)+\varepsilon_n$ for every $x\in[a,b]$, and

$$\varepsilon_n := \frac{1}{b-a}\Big(|\Omega_h^+| - \int_a^b h_n(x)\,dx\Big).$$

It is immediate to see that $|\Omega^+_{\tilde{h}_n}| = |\Omega_h^+|$, and that $(u_n,\tilde{h}_n)\to(u,h)$ in X. Additionally,

$$\lim_{n\to+\infty}\int_{\Gamma_{h_n}} \varphi_\delta(y)\,d\mathcal{H}^1 = \lim_{n\to+\infty}\int_{\Gamma_{\tilde{h}_n}} \varphi_\delta(y)\,d\mathcal{H}^1. \tag{3.8}$$

Regarding the bulk energies, we have

$$\begin{aligned} \int_{\Omega_{\tilde{h}_n}} W_\delta(y, Eu_n(x,y) - E_\delta(y))\,dx\,dy &= \int_a^b\int_{-\infty}^{y_0} W_\delta(y, Eu(x,y)-E_\delta(y))\,dy\,dx \\ &+ \int_a^b\int_{y_0}^{y_0+\varepsilon_n} W_\delta(y, Eu(x,y_0)-E_\delta(y))\,dy\,dx \\ &+ \int_a^b\int_{y_0+\varepsilon_n}^{h_n(x)+\varepsilon_n} W_\delta(y, Eu(x,y-\varepsilon_n)-E_\delta(y))\,dy\,dx. \end{aligned}$$

Thus, by (2.8) there holds

$$\begin{aligned} &\Big|\int_{\Omega_{\tilde{h}_n}} W_\delta(y, Eu_n(x,y)-E_\delta(y))\,dx\,dy - \int_{\Omega_{h_n}} W_\delta(y, Eu(x,y)-E_\delta(y))\,dx\,dy\Big| \\ &\leq C\int_a^b\int_{y_0}^{y_0+\varepsilon_n} |Eu(x,y)-E_\delta(y)|^2\,dy\,dx \\ &+ \int_a^b\int_{y_0}^{h_n(x)} |W_\delta(y+\varepsilon_n, Eu(x,y)-E_\delta(y+\varepsilon_n)) - W_\delta(y, Eu(x,y)-E_\delta(y))|\,dx\,dy \\ &\leq C\int_a^b\int_{y_0}^{y_0+\varepsilon_n} |Eu(x,y)-E_\delta(y)|^2\,dy\,dx + C\int_a^b\int_{y_0}^{h_n(x)} |E_\delta(y+\varepsilon_n)-E_\delta(y)|^2\,dy\,dx \end{aligned} \tag{3.9}$$

$$+ C \int_a^b \int_{y_0}^{h_n(x)} (\mathbb{C}_\delta(y+\varepsilon_n) - \mathbb{C}_\delta(y))(Eu(x,y) - E_\delta(y)) : (Eu(x,y) - E_\delta(y))\, dy\, dx,$$

which converges to zero due to the Dominated Convergence Theorem. By combining (3.7), (3.8), and (3.9) we deduce that

$$\lim_{n\to+\infty} \mathcal{F}_\delta(u_n, \tilde{h}_n) = \lim_{n\to+\infty} \mathcal{F}_\delta(u, h_n) = \hat{\mathcal{F}}_\delta(u,h),$$

which in turn yields

$$\bar{\mathcal{F}}_\delta(u,h) \le \hat{\mathcal{F}}_\delta(u,h)$$

and completes the proof of the proposition. □

Proposition 3.2 is instrumental for the proof of the Γ-convergence result.

Proposition 3.3 (Γ-convergence)**.** *The functional $\mathcal{F}$ is the Γ-limit as $\delta \to 0$ of $\{F_\delta\}_\delta$ under volume constraint. Namely, if $(u_\delta, h_\delta) \to (u,h)$ in X, and $|\Omega^+_{h_\delta}| = |\Omega^+_h|$ for every δ, then*

$$\mathcal{F}(u,h) \le \liminf_{\delta\to 0} \mathcal{F}_\delta(u_\delta, h_\delta).$$

Additionally, for every $(u,h) \in X$, there exists a sequence $\{(u_\delta, h_\delta)\} \subset X$ such that $|\Omega^+_{h_\delta}| = |\Omega^+_h|$ for every δ, and

$$\mathcal{F}(u,h) \ge \limsup_{\delta\to 0} \mathcal{F}_\delta(u_\delta, h_\delta).$$

Proof. We subdivide the proof into two steps.

Step 1. We first show that for all sequences $\{\delta_n\}$, and $\{(u_n, h_n)\} \subset X_{\mathrm{Lip}}$, with $\delta_n \to 0$, $(u_n, h_n) \to (u,h)$ in X, and such that $|\Omega^+_{h_n}| = |\Omega^+_h|$ for every $n \in \mathbb{N}$, there holds

$$\lim_{n\to+\infty} \mathcal{F}_{\delta_n}(u_n, h_n) \ge \mathcal{F}(u,h). \tag{3.10}$$

The liminf inequality for the surface energies follows arguing as in [14, Proof of Theorem 2.9, Step 1]. To study the elastic energies fix $D \subset\subset \Omega_h$ and let $\eta > 0$. Let $\varepsilon > 0$ be small enough so that

$$\int_{D\cap\{|y|\le\varepsilon\}} W_0(y, Eu(x,y) - E_0(y))\, dx\, dy \le \eta. \tag{3.11}$$

We have

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_{\Omega_{h_n}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\, dx\, dy \\
&\quad\ge \liminf_{n\to+\infty} \int_D W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\, dx\, dy \\
&\quad\ge \liminf_{n\to+\infty} \int_{D\cap\{|y|>\varepsilon\}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\, dx\, dy \\
&\qquad + \liminf_{n\to+\infty} \int_{D\cap\{|y|\le\varepsilon\}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\, dx\, dy.
\end{aligned}$$

Now,

$$\int_{D\cap\{|y|>\varepsilon\}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\, dx\, dy \tag{3.12}$$

$$= \int_{D\cap\{|y|>\varepsilon\}} (\mathbb{C}_{\delta_n}(y) - \mathbb{C}(y))(Eu_n(x,y) - E_{\delta_n}(y)) : (Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy$$
$$+ \int_{D\cap\{|y|>\varepsilon\}} W_0(y, Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy.$$

Since $(u_n, h_n) \to (u, h)$ in X, by Definition 2.1 the right-hand side of (3.12) satisfies

$$\liminf_{n\to+\infty} \int_{D\cap\{|y|>\varepsilon\}} W_0(y, Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy \geq \int_{D\cap\{|y|>\varepsilon\}} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy,$$

whereas the first term in the right-hand side of (3.12) can be estimated as

$$\Big| \int_{D\cap\{|y|>\varepsilon\}} (\mathbb{C}_{\delta_n}(y) - \mathbb{C}(y))(Eu_n(x,y) - E_{\delta_n}(y)) : (Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy \Big|$$
$$\leq C \Big\| 1 - f\Big(\frac{y}{\delta_n}\Big) \Big\|_{L^\infty(D\cap\{y>\varepsilon\})} + C \Big\| 1 + f\Big(\frac{y}{\delta_n}\Big) \Big\|_{L^\infty(D\cap\{y<-\varepsilon\})},$$

which converges to zero as $n \to +\infty$ due to the properties of f. Hence, by (3.11),

$$\liminf_{n\to+\infty} \int_{\Omega_{h_n}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy$$
$$\geq \liminf_{n\to+\infty} \int_{D\cap\{|y|>\varepsilon\}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy$$
$$\geq \int_{D\cap\{|y|>\varepsilon\}} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy$$
$$\geq \int_D W_0(y, Eu(x,y) - E_0(y))\,dx\,dy - \eta.$$

By the arbitrariness of η and D we conclude that

$$\liminf_{n\to+\infty} \int_{\Omega_{h_n}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy$$
$$\geq \int_{\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy.$$

Step 2. By Proposition 3.2 to prove the limsup inequality it is enough to show that for all sequences $\{\delta_n\}$ of nonnegative numbers, with $\delta_n \to 0$, and for every $(u, h) \in X$ there exists $\{(u_n, h_n)\} \subset X$ such that $(u_n, h_n) \to (u, h)$ in X, and

$$\limsup_{n\to+\infty} \overline{\mathcal{F}}_{\delta_n}(u_n, h_n) \leq \mathcal{F}(u, h). \tag{3.13}$$

Fix $\{\delta_n\}$. If $\gamma_f \geq \gamma_s - \gamma_{fs}$, take $u_n = u$ and $h_n = h$. Then (3.13) follows by the pointwise convergences

$$\varphi_{\delta_n}(y) \to \varphi_0(y) \quad \text{for every } y \in [0, +\infty),$$

and

$$\mathbb{C}_{\delta_n}(y) \to \mathbb{C}(y) \quad \text{for every } y \in \mathbb{R}. \tag{3.14}$$

If $\gamma_f < \gamma_s - \gamma_{fs}$, construct $\varepsilon_n \to 0$ such that

$$\varphi_{\delta_n}(y + \varepsilon_n) \to \varphi_0(y) = \gamma_f \quad \text{for all } y \in [0, +\infty).$$

Let $y_0 < 0$ be such that $u(\cdot, y_0) \in H^1((a,b);\mathbb{R}^2)$. We define

$$u_n(x) := \begin{cases} u(x, y-\varepsilon_n) & \text{if } y > y_0+\varepsilon_n, \\ u(x, y_0) & \text{if } y_0 < y \le y_0+\varepsilon_n, \\ u(x,y) & \text{if } y \le y_0, \end{cases}$$

and $h_n(x) := \min\{h(x)+\varepsilon_n, t_n\}$, where $t_n > 0$ is such that $|\Omega^+_{h_n}| = d$. The convergence of surface energies follows as in [14, Proof of theorem 2.9, Step 2]. Regarding the bulk energies, we have

$$\int_{\Omega_{h_n}} W_{\delta_n}(y, Eu_n(x,y) - E_{\delta_n}(y))\,dx\,dy = \int_a^b \int_{-\infty}^{y_0} W_{\delta_n}(y, Eu(x,y) - E_{\delta_n}(y))\,dx\,dy \tag{3.15}$$

$$+ \int_a^b \int_{y_0}^{y_0+\varepsilon_n} W_{\delta_n}(y, Eu(x,y_0) - E_{\delta_n}(y))\,dx\,dy$$
$$+ \int_a^b \int_{y_0+\varepsilon_n}^{h_n(x)} W_{\delta_n}(y, Eu(x,y-\varepsilon_n) - E_{\delta_n}(y))\,dx\,dy.$$

The first term in the right-hand side of (3.15) satisfies

$$\lim_{n\to+\infty} \int_a^b \int_{-\infty}^{y_0} W_{\delta_n}(y, Eu(x,y) - E_{\delta_n}(y))\,dx\,dy \tag{3.16}$$
$$= \int_a^b \int_{-\infty}^{y_0} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy,$$

owing to (3.14) and the fact that

$$E_{\delta_n} \to E_0 \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{3.17}$$

By (2.8) the second term in the right-hand side of (3.15) can be bounded from above as

$$\int_a^b \int_{y_0}^{y_0+\varepsilon_n} W_{\delta_n}(y, Eu(x,y_0) - E_{\delta_n}(y))\,dx\,dy \tag{3.18}$$
$$\le C \int_a^b \int_{y_0}^{y_0+\varepsilon_n} |Eu(x,y_0) - E_{\delta_n}(y)|^2\,dy\,dx$$

and hence vanishes, as $n \to +\infty$. Finally, there holds

$$\int_a^b \int_{y_0+\varepsilon_n}^{h_n(x)} W_{\delta_n}(y, Eu(x,y-\varepsilon_n) - E_{\delta_n}(y))\,dx\,dy$$
$$\le \int_{\Omega_h} W_{\delta_n}(y+\varepsilon_n, Eu(x,y) - E_{\delta_n}(y+\varepsilon_n))\,dx\,dy$$
$$= \int_{\Omega_h} (\mathbb{C}_{\delta_n}(y+\varepsilon_n) - \mathbb{C}(y))(Eu(x,y) - E_{\delta_n}(y+\varepsilon_n)) : (Eu(x,y) - E_{\delta_n}(y+\varepsilon_n))\,dx\,dy$$
$$+ \int_{\Omega_h} W_0(y, Eu(x,y) - E_{\delta_n}(y+\varepsilon_n))\,dx\,dy.$$

By the Dominated Convergence Theorem, (3.14), and (3.17), we conclude that

$$\limsup_{n\to+\infty} \int_a^b \int_{y_0+\varepsilon_n}^{h_n(x)} W_{\delta_n}(y, Eu(x,y-\varepsilon_n) - E_{\delta_n}(y))\,dx\,dy \tag{3.19}$$

$$\leq \int_{\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy.$$

Inequalities (3.16)–(3.19) imply the convergence of the elastic energies and complete the proof of (3.13). ☐

4. Properties of local minimizers

In this section we start analyzing the regularity of μ-local minimizers (u,h) of (1.2). In the first subsection we employ an argument first introduced in [8] to prove that optimal profiles h satisfy the *internal-ball condition*. The second subsection is devoted to establish a decay estimate for the minimizing displacements u, and relies on some techniques introduced in the setting of transmission problems for elliptic systems (see [26, 27, 33, 34]).

4.1. Internal-ball condition.

In order to prove the internal-ball condition we need to perform local variations. Therefore, we first show that the area constraint in the minimization problem of Definition 2.2 can be replaced with a suitable penalization in the energy functional.

Proposition 4.1. *Let $(u,h) \in X$ be a μ-local minimizer for the functional $\mathcal{F}$. Then there exists $\lambda_0 > 0$ such that*

$$\mathcal{F}(u,h) = \min\left\{\mathcal{F}(v,g) + \lambda\big||\Omega_h^+| - |\Omega_g^+|\big| : (v,g) \in X,\ |\Omega_g \Delta \Omega_h| \leq \frac{\mu}{2}\right\} \tag{4.1}$$

for all $\lambda \geq \lambda_0$.

Proof. The proof strategy is analogous to [14, Proof of Proposition 3.1], and consists in first establishing the existence of a solution (u_λ, h_λ) of the minimum problem (M_λ) in the right side of (4.1) for every fixed $\lambda > 0$, then in observing that

$$\mathcal{F}(u,h) \geq \mathcal{F}(u_\lambda, h_\lambda) \tag{4.2}$$

since (u,h) is admissible for (M_λ), and finally in showing that there exists $\lambda_0 > 0$ such that the reverse inequality

$$\mathcal{F}(u,h) \leq \mathcal{F}(u_\lambda, h_\lambda) \tag{4.3}$$

holds true for every $\lambda \geq \lambda_0$. Since (u,h) is a μ-local minimizer of the functional $\mathcal{F}$, to prove (4.3) it is enough to show that $|\Omega_{g_\lambda}^+| = |\Omega_h^+|$ for all $\lambda \geq \lambda_0$.

The key modification in our setting consists in observing that any sequence $\{(u_k, h_k)\} \subset X$, for which there exists a constant C such that

$$\sup_{k\in\mathbb{N}} \mathcal{F}(u_k, h_k) < C,$$

satisfies the uniform bound

$$\mathcal{H}^1(\Gamma_{h_k}) \leq M(C), \tag{4.4}$$

where the constant $M(C) > 0$ is given by

$$M(C) := \begin{cases} \frac{C}{\beta\gamma_f} & \text{if } \beta \neq 0, \\ \frac{C}{\min\{\gamma_f, \gamma_{fs}\}} & \text{if } \beta = 0, \end{cases}$$

and where β is the quantity defined in (2.10). Note that, by (1.3), when $\beta = 0$ then $\gamma_{fs} = \gamma_s > 0$. The bound (4.4) is used a first time to prove the sequential compactness of any minimizing sequence for the problem (M_λ), and to deduce the existence of a minimizer (u_λ, h_λ). In view of (4.2), an application of (4.4) to the sequence $\{(u_\lambda, h_\lambda)\}$ with $C = \mathcal{F}(u, h)$ allows to check that $\{(u_\lambda, h_\lambda)\}$ satisfies the assumptions of [14, Lemma 3.2], and to complete the proof of (4.3).

□

We are now ready to establish the internal-ball condition for optimal profiles.

Proposition 4.2 (Internal-ball condition)**.** *Let $(u, h) \in X$ be a μ-local minimizer for the functional $\mathcal{F}$. Then, there exists $\rho_0 > 0$ such that for every $z \in \overline{\Gamma}_h$ we can choose a point P_z for which $B(P_z, \rho_0) \cap ((a, b) \times \mathbb{R}) \subset \Omega_h$, and*

$$\partial B(P_z, \rho_0) \cap \overline{\Gamma}_h = \{z\}.$$

Proof. Let λ_0 be as in Proposition 4.1 and let β be the quantity defined in (2.10). The case in which $\beta = 1$ can be treated as in [14, Proposition 3.3], despite the fact that in our setting the two elasticity tensors $\mathbb{C}_f$ and $\mathbb{C}_s$ are allowed to be different. Also in the case $\beta < 1$ the argument of [14, Proposition 3.3] can be implemented. We highlight the main differences with respect to the case $\beta = 1$ for convenience of the reader.

We begin by proving the following claim: there exists $\rho_0 > 0$ such that, for any $P \in \mathbb{R}^2$ for which $B(P, \rho_0) \cap ((a, b) \times \mathbb{R}) \subset \Omega_h$, the intersection between $\partial B(P, \rho_0)$ and $\overline{\Gamma}_h$ contains at most one point. Once this claim is proved, the uniform internal-ball condition of the assert follows then by the argument of [8, Lemma 2].

By contradiction, assume that for every $r > 0$ there exists $\rho_r < \frac{r}{2}$ for which three points P_1^r, P_2^r, and P_r can be chosen so that

$$B(P_r, \rho_r) \cap ((a, b) \times \mathbb{R}) \subset \Omega_h$$

and

$$\partial B(P_r, \rho_r) \cap \overline{\Gamma}_h \supset \{P_1^r, P_2^r\}.$$

Denote by $[P_1^r, P_2^r]$ the segment

$$[P_1^r, P_2^r] := \{P_1^r + t(P_2^r - P_1^r) \,:\, 0 \leq t \leq 1\}$$

and by $\Gamma_{P_1^r, P_2^r}$ the set

$$\Gamma_{P_1^r, P_2^r} := \left(\tilde{\Gamma}_h \cap ([x_1^r, x_2^r] \times \mathbb{R})\right) \cup \bigcup_{i=1}^{2} \left\{(x_i^r, y) \,:\, y_i^r \leq y \leq h^+(x_i^r)\right\}$$

where $P_1^r =: (x_1^r, y_1^r)$ and $P_2^r =: (x_2^r, y_2^r)$. The case in which either $y_1^r \neq 0$ or $y_2^r \neq 0$ follows exactly as in [14, Proposition 3.3], thus we assume that $y_1^r = y_2^r = 0$. Consider the pair $(u, h^r) \in X$ with h^r defined by

$$h^r := \begin{cases} 0 & \text{if } x_1^r < x < x_2^r, \\ h(x) & \text{otherwise.} \end{cases}$$

Note that $\Omega_h \setminus \Omega_{h^r} \subset \overline{D}^r$ where D^r is the portion of $\mathbb{R}^+$ enclosed by the curve $\Gamma_{P_1^r, P_2^r} \cup [P_1^r, P_2^r]$.

Fix

$$0 < \varepsilon_0 < \frac{\mu}{4(b-a)}, \tag{4.5}$$

and

$$0 < \varepsilon < \frac{\varepsilon_0}{2}, \tag{4.6}$$

and consider the finite set $A \subset (a,b)$ such that

$$\sum_{x\in J(h)\setminus A} (h^+(x) - h^-(x)) + \sum_{x\in C(h)\setminus A} (h^-(x) - h(x)) < \frac{\varepsilon}{2}$$

(see [14, (3.33) and (3.34)]). Let $r_0 > 0$ be such that

$$r_0 < \min\{|x - x'| \,:\, x \neq x' \text{ for any } x,\, x' \in A\}$$

and

$$\sup\{\mathscr{M}(I \setminus A) \,:\, I \in \mathcal{I}\} < \frac{\varepsilon}{2}$$

where $\mathcal{I}$ is the family of intervals $I \subset (a,b)$ with $|I| \leq r_0$, and $\mathscr{M}$ is the measure obtained by projecting $\mathcal{H}^1_{|\Gamma_h}$ on the x-axis. By choosing

$$r := \min\left\{\frac{\varepsilon_0}{4}, \frac{r_0}{2}\right\}, \tag{4.7}$$

it follows that the set $[P_1^r, P_2^r] \cap A$ contains at most one point. Arguing as in [14, Proof of (3.37)] we deduce the estimate

$$\mathcal{H}^1(\Gamma_{P_1^r,P_2^r}) \leq 2\varepsilon + r. \tag{4.8}$$

In view of (4.6), (4.7), and (4.8), there holds

$$\begin{aligned} h^+(x) - h^r(x) &\leq \mathcal{H}^1(\Gamma_{P_1^r,P_2^r}) + \mathcal{H}^1([P_1^r, P_2^r]) \\ &\leq 2\varepsilon + r + 2\rho \leq 2\varepsilon + 2r \leq 2\varepsilon_0 \end{aligned}$$

for every $x \in (x_1^r, x_2^r)$, and hence

$$|D^r| = \int_{x_1^r}^{x_2^r} (h^+(x) - h^r(x))\,dx \leq 2\varepsilon_0(b-a), \tag{4.9}$$

and by (4.5),

$$|\Omega_h^+ \Delta \Omega_{h^r}^+| \leq \frac{\mu}{2},$$

namely (u, h^r) is an admissible competitor for the minimum problem (4.1) with $\lambda = \lambda_0$. The minimality of (u,h) yields the estimate

$$\mathcal{F}(u,h) \leq \mathcal{F}(u,h^r) + \lambda_0 \big||\Omega_{h^r}^+| - |\Omega_h^+|\big|. \tag{4.10}$$

On the other hand,

$$\begin{aligned} \mathcal{F}(u,h^r) &= \int_{\Omega_{h^r}} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy + \int_{\tilde{\Gamma}_{h^r}} \varphi(y)\,d\mathcal{H}^1 \\ &\quad + 2\gamma_f \mathcal{H}^1(\Gamma_{h^r}^{cut}) + \gamma_{fs}(b-a) \\ &\leq \int_{\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy + \gamma_f\big(\mathcal{H}^1(\tilde{\Gamma}_h \cap \{y>0\}) - \mathcal{H}^1(\Gamma_{P_1^r,P_2^r})\big), \\ &\quad + \min\{\gamma_f, \gamma_s - \gamma_{fs}\}\big(\mathcal{H}^1(\tilde{\Gamma}_h \cap \{y=0\}) + \mathcal{H}^1([P_1^r, P_2^r])\big) \\ &\quad + 2\gamma_f \mathcal{H}^1(\Gamma_h^{cut}) + \gamma_{fs}(b-a) \\ &= \mathcal{F}(u,h) - \gamma_f\big(\mathcal{H}^1(\Gamma_{P_1^r,P_2^r}) - \beta\mathcal{H}^1([P_1^r, P_2^r])\big), \end{aligned} \tag{4.11}$$

where β is the quantity defined in (2.10). By combining (4.10) and (4.11) we deduce that

$$\mathcal{H}^1(\Gamma_{P_1^r,P_2^r}) - \beta\mathcal{H}^1([P_1^r, P_2^r]) \leq \frac{\lambda_0}{\gamma_f}|D^r|. \tag{4.12}$$

Arguing as in the proof of [8, Lemma 1], the isoperimetric inequality in the plane (see [1]) yields

$$\sqrt{|D^r|} \leq \frac{\mathcal{H}^1(\partial D^r)}{2\sqrt{\pi}} = \frac{(\theta^r+1)\mathcal{H}^1([P_1^r,P_2^r])}{2\sqrt{\pi}} \tag{4.13}$$

where

$$\theta^r := \frac{\mathcal{H}^1(\Gamma_{P_1^r,P_2^r})}{\mathcal{H}^1([P_1^r,P_2^r])} > 1. \tag{4.14}$$

Substituting (4.12) in (4.13) we obtain the estimate

$$|D^r| \leq \frac{\lambda_0^2}{4\pi\gamma_f^2}\frac{(\theta^r+1)^2}{(\theta^r-\beta)^2}|D^r|^2.$$

In view of (4.9),

$$|D^r| \leq \frac{2\lambda_0^2\varepsilon_0(b-a)}{4\pi\gamma_f^2}\frac{(\theta^r+1)^2}{(\theta^r-\beta)^2}|D^r|,$$

which in turn implies

$$\frac{2\lambda_0^2\varepsilon_0(b-a)}{4\pi\gamma_f^2}\frac{(\theta^r+1)^2}{(\theta^r-\beta)^2} \geq 1.$$

By the previous inequality, as ε_0 vanishes, then θ^r must approach β. Since $\beta < 1$, we have a contradiction with (4.14). This completes the proof of the claim and of the proposition.

□

We notice that in view of Proposition 4.2 the upper-end point of each *cut* is a cusp point (see Figure 2).

The following proposition is a consequence of the internal-ball condition.

Proposition 4.3. *Let $(u,h) \in X$ be a μ-local minimizer for the functional $\mathcal{F}$. Then for any $z_0 \in \overline{\Gamma}_h$ there exist an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, and a rectangle*

$$Q := \{z_0 + s\mathbf{v}_1 + t\mathbf{v}_2 :\ -a' < s < a',\ -b' < t < b'\},$$

$a', b' > 0$, such that $\Omega_h \cap Q$ has one of the following two representations:

1. *There exists a Lipschitz function $g : (-a', a') \to (-b', b')$ such that $g(0) = 0$ and*

   $$\Omega_h \cap Q := \{z_0 + s\mathbf{v}_1 + t\mathbf{v}_2 :\ -a' < s < a',\ -b' < t < g(s)\} \cap ((a,b)\times\mathbb{R}).$$

   In addition, the function g admits left and right derivatives at all points that are, respectively, left and right continuous.
2. *There exist two Lipschitz functions $g_1, g_2 : [0, a') \to (-b', b')$ such that $g_i(0) = (g_i)'_+(0) = 0$ for $i = 1, 2$, $g_1 \leq g_2$, and*

   $$\Omega_h \cap Q := \{z_0 + s\mathbf{v}_1 + t\mathbf{v}_2 :\ 0 < s < a',\ -b' < t < g_1(s) \text{ or } g_2(s) < t < b'\}.$$

   In addition, the functions g_1, g_2 admit left and right derivatives at all points that are, respectively, left and right continuous.

For the proof of Proposition 4.3 we refer the reader to [8, Lemma 3] and [14, Proposition 3.5]. In particular Proposition 4.3 entails that the set

$$\Gamma_h^{reg} = \Gamma_h \setminus (\Gamma_h^{cusp} \cup \Gamma_h^{cut})$$

is locally Lipschitz (see the proof of Theorem 2.5 in Section 6 for more details).

4.2. **Decay estimate.** From now on we work under the assumption that both the film and the substrate are made of linearly elastic isotropic materials, and we denote by μ_f, λ_f, μ_s, λ_s their Lamé coefficients. Note that

$$\mathbb{C}_\sigma Eu = 2\mu_\sigma Eu + \lambda_\sigma(\operatorname{div} u)Id, \quad \sigma = f, s,$$

for every $u \in H^1(\Omega_h; \mathbb{R}^2)$.

In order to prove the decay estimate of Proposition 4.6 for minimizing configurations (u, h) at the points of Γ_h^{reg} a *blow-up* around such points is needed. As the graph is allowed to touch the film/substrate interface, we are lead to consider transmission problems for Lamé systems in conical sets. We first state a preliminary lemma, relying on [25, Theorem 1.5.2.8], and whose proof is contained in [14, Lemma 3.12]).

Lemma 4.4. *Let $\mathcal{C}$ be a circular sector of amplitude $\theta \in (0, 2\pi)$ and radius $R > 0$. Assume that $\mathcal{C}$ is the reference configuration of a linearly elastic isotropic material whose Lamé coefficients are denoted by μ and λ. Let $g \in H^{1/2}(\partial\mathcal{C}; \mathbb{R}^2)$ be a function vanishing in a neighborhood of the origin. Then there exists a function $v \in H^2(\mathcal{C}; \mathbb{R}^2)$ such that*

$$[2\mu Ev + \lambda(\operatorname{div} v)Id]\, \nu_{\mathcal{C}} = g \quad \text{on } \partial\mathcal{C},$$

where $\nu_{\mathcal{C}}$ is the outer unit normal to $\mathcal{C}$ (where it exists), and

$$v = 0 \quad \text{on } \partial\mathcal{C}.$$

In the following proposition we assess the regularity of weak solutions to transmission problems for Lamé systems in conical sets.

Proposition 4.5. *Let $\mathcal{C}$ be the set given by*

$$\mathcal{C} := \operatorname{Int}\left(\bigcup_{i=1}^{3} \overline{\mathcal{C}_i}\right)$$

where $\mathcal{C}_i$, $i = 1, 2, 3$, are the circular sectors defined by

$$\mathcal{C}_i := \{(x, y) : x = \rho\cos(\theta),\, y = \rho\sin(\theta), \text{ with } 0 < \rho < R, \text{ and } \theta_{i-1} < \theta < \theta_i\}$$

with $R > 0$, and $0 =: \theta_0 \le \theta_1 < \theta_2 \le \theta_3 < 2\pi$ (see Figure 8). Denote by

$$\Gamma_{1,0} := (0, R),$$

and

$$\Gamma_{3,0} := \{(\rho\cos(\theta_3), \rho\sin(\theta_3)) \in \mathbb{R}^2 \text{ with } 0 < \rho < R\},$$

the two external sides of $\mathcal{C}$, and by

$$\Gamma_i := \{(x, y) : x = R\cos\theta,\, y = R\sin\theta, \text{ with } \theta_{i-1} \le \theta < \theta_i\},$$

for $i = 1, 2, 3$ the curvilinear portions of its boundary. Finally, consider the transmission interfaces

$$\Gamma_{i,i+1} := \partial\mathcal{C}_i \cap \partial\mathcal{C}_{i+1} \quad \text{for } i = 1, 2.$$

We assume that each set $\mathcal{C}_i$ is the reference configuration of a linearly elastic, isotropic material whose Lamé coefficients are denoted by μ_i and λ_i, with $\mu_3 := \mu_1$ and $\lambda_3 := \lambda_1$, and satisfy the quasi-monotonicity condition:

$$\mu_2 \ge \mu_1 > 0 \quad \text{and} \quad \mu_2 + \lambda_2 \ge \mu_1 + \lambda_1 > 0.$$

Let $(u_1, u_2, u_3) \in \prod_{i=1}^{3} H^1(\mathcal{C}_i; \mathbb{R}^2)$ *be a weak solution of the transmission problem:*

$$\begin{cases} \mu_i \Delta u_i + (\lambda_i + \mu_i)\nabla(\operatorname{div} u_i) = f_i & \text{in } \mathcal{C}_i,\ i = 1,2,3, \\ [2\mu_i E u_i + \lambda_i(\operatorname{div} u_i) Id]\, \nu_{i,0} = 0 & \text{on } \Gamma_{i,0},\ i = 1,3, \\ [2\mu_i E u_i + \lambda_i(\operatorname{div} u_i) Id]\, \nu_i = g_i & \text{on } \Gamma_i,\ i = 1,2,3, \\ u_i - u_{i+1} = 0 & \text{on } \Gamma_{i,i+1},\ i = 1,2, \\ \Big[2\mu_i E u_i - 2\mu_{i+1} E u_{i+1} & \\ \qquad + \lambda_i(\operatorname{div} u_i) Id - \lambda_{i+1}(\operatorname{div} u_{i+1}) Id \Big]\, \nu_{i,i+1} = 0 & \text{on } \Gamma_{i,i+1},\ i = 1,2, \end{cases} \tag{4.15}$$

where the data f_i *and* g_i *satisfy* $f_i \in L^2(\mathcal{C}_i)$, $g_i \in H^{1/2}(\Gamma_i, \mathbb{R}^2)$, $i = 1,2,3$, *the vectors* $\nu_{i,i+1}$ *are the normal to* $\Gamma_{i,i+1}$ *external to* $\mathcal{C}_i$, $i = 1,2$, *and the vectors* $\nu_{1,0}$, $\nu_{3,0}$, *and* ν_i *are the outer unit normals to* $\Gamma_{1,0}$, $\Gamma_{3,0}$, *and* Γ_i, $i = 1,2,3$, *respectively.*

If there exists a vector $\tau \in \mathbb{R}^2 \setminus \{0\}$ *such that* $\tau \in \mathcal{C}_2$ *and* $-\tau \notin \overline{\mathcal{C}}$, *then there exists a neighbourhood* U *of the origin such that*

$$u \in H^{3/2+\varepsilon}(U \cap \mathcal{C}_i)$$

for some $\varepsilon > 0$ *and for* $i = 1,2,3$.

Proof. Let $\varphi \in C_c^\infty(\mathcal{C})$ be a cut-off function such that $0 \le \varphi \le 1$ in $\mathcal{C}$, with $\operatorname{supp}\varphi \subset\subset B_{R/2}$, and $\varphi \equiv 1$ in $B_{R/3}$, where here $B_{R/2}$ and $B_{R/3}$ are the balls centered in the origin and with radii $R/2$ and $R/3$, respectively. Consider the maps $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \in \prod_{i=1}^{3} H^1(\mathcal{C}_i; \mathbb{R}^2)$, defined as $\tilde{u}_i := \varphi u_i$, $i = 1,2,3$. By straightforward computation, and in view of (4.32), the triple $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ solves the transmission problem

$$\begin{cases} \mu_i \Delta \tilde{u}_i + (\lambda_i + \mu_i)\nabla(\operatorname{div} \tilde{u}_i) = \tilde{f}_i & \text{in } \mathcal{C}_i,\ i = 1,2,3, \\ [2\mu_i E \tilde{u}_i + \lambda_i(\operatorname{div} \tilde{u}_i) Id]\, \nu_{i,0} = \tilde{g}_i & \text{on } \Gamma_{i,0},\ i = 1,3, \\ [2\mu_i E \tilde{u}_i + \lambda_i(\operatorname{div} \tilde{u}_i) Id]\, \nu_i = 0 & \text{on } \Gamma_i,\ i = 1,2,3, \\ \tilde{u}_i - \tilde{u}_{i+1} = 0 & \text{on } \Gamma_{i,i+1},\ i = 1,2, \\ \Big[2\mu_i E \tilde{u}_i - 2\mu_{i+1} E \tilde{u}_{i+1} & \\ \qquad + \lambda_i(\operatorname{div} \tilde{u}_i) Id - \lambda_{i+1}(\operatorname{div} \tilde{u}_{i+1}) Id \Big]\, \nu_{i,i+1} = \tilde{h}_i & \text{on } \Gamma_{i,i+1},\ i = 1,2, \end{cases} \tag{4.16}$$

where $\tilde{f}_i \in L^2(\mathcal{C}_i)$, $\tilde{g}_i \in H^{1/2}(\Gamma_{i,0}; \mathbb{R}^2)$, and $\tilde{h}_i \in H^{1/2}(\Gamma_{i,i+1}; \mathbb{R}^2)$ for every i, and the maps $\tilde{g}_i$ and $\tilde{h}_i$ vanish in the intersection of their domains with $B_{R/3}$. By applying Lemma 4.4 to both sets $\mathcal{C}_i$, $i = 1,3$, with

$$g = \begin{cases} \tilde{g}_i & \text{on } \Gamma_{i,0}, \\ 0 & \text{on } \Gamma_i, \\ \tilde{h}_i & \text{on } \Gamma_{i,i+1}, \end{cases} \tag{4.17}$$

we obtain functions $v_i \in H^2(\mathcal{C}_i; \mathbb{R}^2)$ such that

$$[2\mu E v_i + \lambda(\operatorname{div} v_i) Id]\, \nu_{\mathcal{C}_i} = g \quad \text{on } \partial\mathcal{C}_i, \tag{4.18}$$

where $\nu_{\mathcal{C}_i}$ is the outer unit normal to $\mathcal{C}_i$ (where it exists), and

$$v_i = 0 \quad \text{on } \partial\mathcal{C}_i. \tag{4.19}$$

Setting $(w_1, w_2, w_3) := (\tilde{u}_1 - v_1, \tilde{u}_2, \tilde{u}_3 - v_3) \in \prod_{i=1}^{3} H^1(\mathcal{C}_i; \mathbb{R}^2)$, by (4.16), and (4.17)–(4.19) there holds

$$\begin{cases} \mu_i \Delta w_i + (\lambda_i + \mu_i)\nabla(\operatorname{div} w_i) = \hat{f}_i & \text{in } \mathcal{C}_i,\ i = 1,2,3, \\ [2\mu_i E w_i + \lambda_i(\operatorname{div} w_i) Id]\, \nu_{i,0} = 0 & \text{on } \Gamma_{i,0},\ i = 1,3, \\ [2\mu_i E w_i + \lambda_i(\operatorname{div} w_i) Id]\, \nu_i = 0 & \text{on } \Gamma_i,\ i = 1,2,3, \\ w_i - w_{i+1} = 0 & \text{on } \Gamma_{i,i+1},\ i = 1,2, \\ \Big[2\mu_i E w_i - 2\mu_{i+1} E w_{i+1} & \\ \qquad + \lambda_i(\operatorname{div} w_i) Id - \lambda_{i+1}(\operatorname{div} w_{i+1}) Id\Big]\, \nu_{i,i+1} = 0 & \text{on } \Gamma_{i,i+1},\ i = 1,2, \end{cases} \tag{4.20}$$

where $\hat{f}_i \in L^2(\mathcal{C}_i)$ for $i = 1,2,3$. By [28, Theorem 2] we obtain that there exists a neighborhood $\tilde{U}$ of the origin such that $w_i \in H^{3/2+\varepsilon}(\tilde{U} \cap \mathcal{C}_i)$ for $i = 1,2,3$. The thesis follows by observing that on $U := \tilde{U} \cap B_{R/3}$, the triple (u_1, u_2, u_3) satisfies

$$(u_1, u_2, u_3) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (w_1 + v_1, w_2, w_3 + v_3),$$

and by the regularity of the maps v_i, $i = 1,3$. □

We are now ready to provide a decay estimate for the gradient of minimizing displacements at the points in which the graph of the corresponding minimizing profile is locally Lipschitz.

Proposition 4.6 (Decay estimate). *Let $(u,h) \in X$ be a μ-local minimizer for the functional $\mathcal{F}$ and assume that the Lamé coefficients of film and substrate satisfy the monotonicity condition (5.2). Let $z_0 := (x_0, h(x_0)) \in \Gamma_h \setminus (\Gamma_h^{cut} \cup \Gamma_h^{cusp})$. Then there exists a constant $C > 0$, a radius r_0, and an exponent $\frac{1}{2} < \alpha < 1$, such that*

$$\int_{B(z_0,r) \cup \Omega_h} |\nabla u|^2 \, dx\, dy \le C r^{2\alpha}$$

for all $0 < r < r_0$.

Proof. We begin by considering the case in which $h(x_0) = 0$. If there exists a constant C such that

$$\limsup_{r \to 0} \frac{1}{r^2} \int_{B(z_0,r) \cap \Omega_h} |\nabla u|^2 \, dx\, dy \le C,$$

then there is nothing to prove. Thus, we assume that this does not hold and that there exists a sequence $\{r_n\} \subset \mathbb{R}$ such that $r_n \to 0$ and

$$\limsup_{n \to +\infty} \frac{1}{r_n^2} \int_{B(z_0,r_n) \cap \Omega_h} |\nabla u|^2 \, dx\, dy = +\infty. \tag{4.21}$$

We subdivide the proof into three steps.

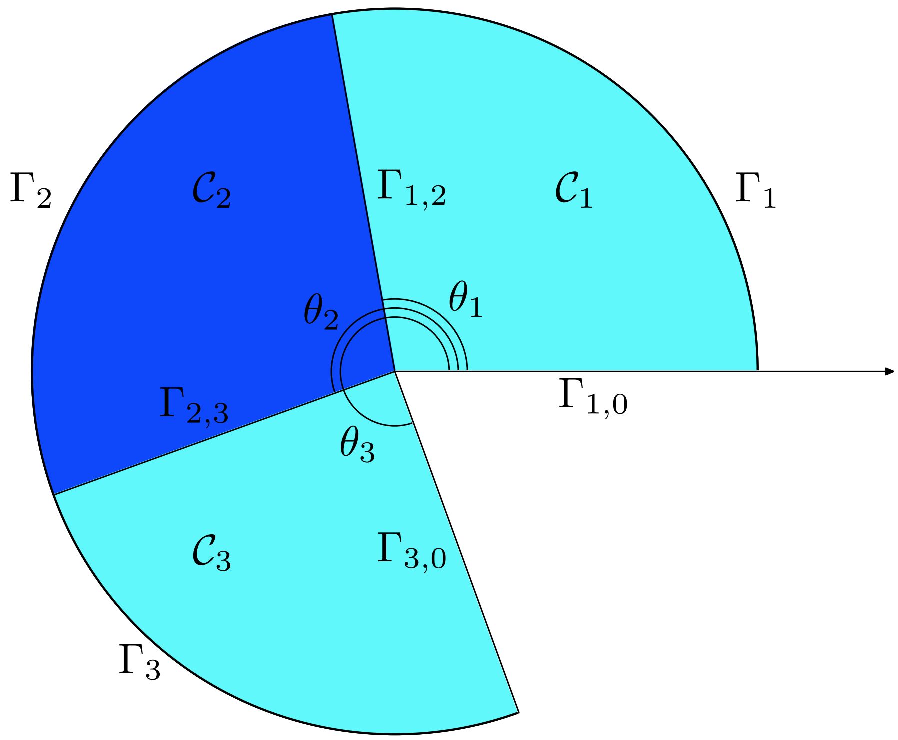

FIGURE 8. The geometry of the set $\mathcal{C}$ on which we consider the transmission problem of Proposition 4.5 is depicted. The lines $\Gamma_{1,2}$ and $\Gamma_{2,3}$ are transmission interfaces for such problem.

Step 1: We claim that there exist an orthonormal basis $\{v_1, v_2\}$ of $\mathbb{R}^2$, three constants $C_1, L > 0$, $\tau_0 \in (0,1)$, and an exponent $\frac{1}{2} < \beta < 1$ such that for all $\tau \in (0, \tau_0)$ there exists a radius $0 < r_\tau < 1$ satisfying

$$\int_{C(z_0,\tau r_n)} |\nabla u|^2 \, dx \, dy \leq C_1 \tau^{2\beta} \int_{C(z_0,r_n)} (1 + |\nabla u|^2) \, dx \, dy, \tag{4.22}$$

for all $0 < r_n < r_\tau$, where

$$C(z_0, r_n) := \Omega_h \cap \{z_0 + sv_1 + tv_2 : \, -r_n < s < r_n, \, -4Lr_n < t < 4Lr_n\}.$$

We point out that, once claim (4.22) is proved, the assert of the theorem follows arguing as in [14, Theorem 3.13, Step 6].

To prove (4.22) we first observe that, since $z_0 \in \Gamma_h^{reg}$, we can apply Proposition 4.3 to obtain a Lipschitz function $g : (-a', a') \to (-b', b')$ with $\operatorname{Lip} g \leq L$ for some $L > 1$ such that $g(0) = 0$, and

$$\Omega_h \cap Q = \{z_0 + sv_1 + tv_2 : \, -a' < s < a', \, -b' < t < g(s)\},$$

where

$$Q = \{z_0 + sv_1 + tv_2 : \, -a' < s < a', \, -b' < t < b'\}.$$

Note that g has left (right) derivative in every point that is left (right) continuous. By Korn inequality in Lipschitz domains we deduce that $u \in H^1(\Omega_h \cap Q; \mathbb{R}^2)$. If

$r_n \leq \min\{a', \frac{b'}{4L}\}$, then $C(z_0, r_n) \subset Q \cap \Omega_h$. Therefore,

$$C(z_0, r_n) = \{z_0 + sv_1 + tv_2 : \, -r_n < s < r_n, \, -4Lr_n < t < g(s)\}.$$

Fix $C_1 > 0$, $\tau_0 \in (0,1)$ and $\beta > \frac{1}{2}$ to be determined later, and assume by contradiction that (4.22) is false for some $\tau \in (0, \tau_0)$. Up to the extraction of a (non-relabeled) subsequence there holds

$$\int_{C(z_0, \tau r_n)} |\nabla u|^2 \, dx \, dy > C_1 \tau^{2\beta} \int_{C(z_0, r_n)} (1 + |\nabla u|^2) \, dx \, dy, \tag{4.23}$$

for a sequence $r_n \to 0$. Define the sets

$$C_n := \frac{1}{r_n}(-z_0 + C(z_0, r_n)) = \left\{ sv_1 + tv_2 : \, -1 < s < 1, \, -4L < t < \frac{g(r_n s)}{r_n} \right\}.$$

We have

$$\chi_{C_n} \to \chi_{C_\infty} \quad \text{a.e. in } \mathbb{R}^2, \tag{4.24}$$

where

$$C_\infty := \{sv_1 + tv_2 : \, -1 < s < 1, \, -4L < t < g_\infty(s)\},$$

the function g_∞ is defined as

$$g_\infty(s) := \begin{cases} g'_-(0)s & \text{for } s < 0, \\ g'_+(0)s & \text{for } s > 0, \end{cases}$$

and χ_{C_n} and χ_{C_∞} are the characteristic functions of the sets C_n and C_∞, respectively.

Define the maps

$$u_n(z) := \frac{u(z_0 + r_n z) - a_n}{\lambda_n r_n}, \quad \text{for every } z \in C_n,$$

where

$$a_n := \frac{1}{|C(z_0, r_n)|} \int_{C(z_0, r_n)} u(x,y) \, dx \, dy, \quad \lambda_n^2 := \frac{1}{|C(z_0, r_n)|} \int_{C(z_0, r_n)} |\nabla u|^2 \, dx \, dy. \tag{4.25}$$

We point out that

$$\frac{1}{|C_n|} \int_{C_n} |\nabla u_n|^2 \, dz = \frac{1}{\lambda_n^2 |C(z_0, r_n)|} \int_{C(z_0, r_n)} |\nabla u|^2 \, dx \, dy = 1 \tag{4.26}$$

and

$$\begin{aligned} \int_{C_n} u_n \, dz &= \frac{1}{\lambda_n r_n} \int_{C_n} u(z_0 + r_n z) \, dx - \frac{a_n |C_n|}{\lambda_n r_n} \\ &= \frac{1}{\lambda_n r_n^3} \Big(\int_{C(z_0, r_n)} u \, dx \, dy - a_n |C(z_0, r_n)| \Big) = 0. \end{aligned}$$

Extend the maps u_n to the rectangle

$$R := \{sv_1 + tv_2 : \, -1 < s < 1, \, -4L < t < 4L\}$$

so that $u_n \in W^{1,2}(R; \mathbb{R}^2)$. By (4.26) we obtain the uniform bound

$$\|u_n\|_{W^{1,2}(R;\mathbb{R}^2)} \leq C \|\nabla u_n\|_{L^2(C_n; \mathbb{M}^{2\times 2})} \leq C.$$

Thus, there exist $u_\infty \in W^{1,2}(R; \mathbb{R}^2)$, and $\lambda_\infty \in [0, +\infty]$ such that, up to the extraction of a (non-relabelled) subsequence, there holds

$$u_n \rightharpoonup u_\infty \quad \text{weakly in } W^{1,2}(R; \mathbb{R}^2), \tag{4.27}$$

and

$$\lambda_n \to \lambda_\infty. \tag{4.28}$$

In addition,

$$\frac{1}{r_n^2}\int_{B(x_0,r_n)\cap\Omega_h} |\nabla u|^2\,dx\,dy \le \frac{1}{r_n^2}\int_{C(x_0,r_n)} |\nabla u|^2\,dx\,dy = \lambda_n^2\frac{|C(z_0,r_n)|}{r_n^2}$$
$$= \lambda_n^2|C_n| \le 12L\lambda_n^2.$$

Hence, by (4.21) we conclude that

$$\lambda_\infty = +\infty. \tag{4.29}$$

In view of a change of variable, the maps u_n satisfy the Euler-Lagrange equations

$$\int_{C_n} E\varphi(z) : \mathbb{C}(r_n z_2)Eu_n(z)\,dz = \frac{1}{\lambda_n}\int_{C_n} E\varphi(z) : \mathbb{C}(r_n z_2)E_0(r_n z_2)\,dz$$

for every $\varphi \in C_0^1(R;\mathbb{R}^2)$. Thus, by (4.24), (4.27), (4.28), and (4.29) we deduce that

$$\int_{C_\infty} E\varphi(z) : \mathbb{C}(z_2)Eu_\infty(z)\,dz = 0 \quad \text{for every } \varphi \in C_0^1(R;\mathbb{R}^2). \tag{4.30}$$

Step 2: Fix a ball B such that

$$B \subset\subset \{sv_1 + tv_2 :\ -1 < s < 1,\ -4L < t < -3L\}.$$

We claim that

$$\lim_{n\to+\infty}\int_{C_n} \psi^2|\nabla u_n - \nabla u_\infty|^2\,dz = 0 \tag{4.31}$$

for every $\psi \in C_0^1(R)$ vanishing in B. Arguing as in [14, Theorem 3.13, Step 2] we obtain that

$$\lim_{n\to+\infty}\int_{C_n} \psi^2(z)(Eu_n(z) : \mathbb{C}(r_n z_2)Eu_n(z) - Eu_\infty(z) : \mathbb{C}(z_2)Eu_\infty(z))\,dz = 0,$$

hence

$$\lim_{n\to+\infty}\int_{C_n} E(\psi(z)(u_n(z) - u_\infty(z))) : \mathbb{C}_\infty(z)E(\psi(z)(u_n(z) - u_\infty(z)))\,dz = 0.$$

Claim (4.31) follows then from Korn's inequality (see [14, Theorem 4.2]).

Step 3: by Step 1, we deduce that u_∞ is a weak solution of the transmission problem

$$\begin{aligned}
&\mu_f\Delta u_\infty^+ + (\lambda_f+\mu_f)\nabla(\operatorname{div} u_\infty^+) = 0 && \text{in } C_\infty^+,\\
&\mu_s\Delta u_\infty^- + (\lambda_s+\mu_s)\nabla(\operatorname{div} u_\infty^-) = 0 && \text{in } C_\infty^-,\\
&(2\mu_f Eu_\infty^+ + \lambda_f(\operatorname{div} u_\infty^+)Id)\nu_\infty = 0 && \text{on } \Gamma_{g_\infty},\\
&u_\infty^+ - u_\infty^- = 0 && \text{on } \{z_2 = 0\},\\
&(2\mu_f Eu_\infty^+ - 2\mu_s Eu_\infty^- + \lambda_f(\operatorname{div} u_\infty^+)Id - \lambda_f(\operatorname{div} u_\infty^-)Id)e_2 = 0 && \text{on } \{z_2 = 0\},
\end{aligned}$$

where $C_\infty^+ := C_\infty \cap \{z_2 > 0\}$, $C_\infty^- := C_\infty \cap \{z_2 < 0\}$, $\Gamma_{g_\infty} := \{(s, g_\infty(s)) : -1 < s < 1\}$, $u_\infty^+ := u_\infty|_{C_\infty^+}$, $u_\infty^- := u_\infty|_{C_\infty^-}$, and ν_∞ is the outer unit normal to Γ_{g_∞}, wherever it exists. Note that the fourth condition in (4.32) holds because Sobolev maps are absolutely continuous on almost every line, whereas the other equations in (4.32) are a consequence of (4.30).

In view of (5.2) and the geometry of the problem we can apply Proposition 4.5 with $R \leq 1$ to u_∞, with $\theta_2 = \pi$, $\mu_1 = \mu_f$, $\lambda_1 = \lambda_f$, $\mu_2 = \mu_s$, $\lambda_2 = \lambda_s$, and with data $f_i = 0$, and

$$g_i := [2\mu_i E u_\infty + \lambda_i (\operatorname{div} u_\infty) Id]\, \nu_i \quad \text{on } \Gamma_i,\ i = 1,2,3,$$

where $\Gamma_1 = \partial B(O,R) \cap C_\infty^+ \cap \{z_1 < 0\}$, $\Gamma_2 = \partial B(O,R) \cap C_\infty^-$, and $\Gamma_3 = \partial B(O,R) \cap C_\infty^+ \cap \{z_1 > 0\}$ where O denotes the origin $(0,0)$. Therefore, we conclude that there exists a ball $B \subset B(O,R)$ centered in the origin, and such that

$$u_\infty \in H^{3/2+\varepsilon}\big(B \cap \big(C_\infty^+ \cup C_\infty^-\big); \mathbb{R}^2\big).$$

Thus, by Hölder inequality we obtain

$$\begin{aligned}\int_{B(O,r)\cap C_\infty} |\nabla u_\infty|^2\, dz &= \int_{B(O,r)\cap C_\infty^+} |\nabla u_\infty|^2\, dz + \int_{B(O,r)\cap C_\infty^-} |\nabla u_\infty|^2\, dz \\ &\leq r^{2-\frac{4}{s}} \big(\|\nabla u\|^2_{L^s(B\cap C_\infty^+;\mathbb{M}^{2\times 2})} + \|\nabla u\|^2_{L^s(B\cap C_\infty^-;\mathbb{M}^{2\times 2})}\big) \leq C r^{2\beta}\end{aligned}$$

for every $r > 0$ small enough, where $4 < s < \frac{4}{1-2\varepsilon}$, and $\beta = 1 - \frac{2}{s} > \frac{1}{2}$, where we used the fact that

$$H^{3/2+\varepsilon}\big(B \cap \big(C_\infty^+ \cup C_\infty^-\big); \mathbb{R}^2\big) \subset L^s\big(B \cap \big(C_\infty^+ \cup C_\infty^-\big); \mathbb{R}^2\big)$$

for every $s \in \left[1, \frac{4}{1-2\varepsilon}\right]$. Choosing τ_0 such that

$$\tau_0 C_\infty \subset (B(O,1) \cap C_\infty) \setminus \{se_1 + te_2 :\ -1 < s < 1,\ -4L < t < -3L\},$$

by Step 2 we deduce that for $0 < \tau \leq \tau_0$ there holds

$$\begin{aligned}\lim_{n\to+\infty} \frac{\int_{C(z_0,\tau r_n)} |\nabla u|^2\, dx\, dy}{\int_{C(z_0,r_n)} |\nabla u|^2\, dx\, dy} &= \frac{1}{|C_\infty|} \lim_{n\to+\infty} \int_{\tau C_n} |\nabla u_n|^2\, dz \\ = \frac{1}{|C_\infty|} \int_{\tau C_\infty} |\nabla u_\infty|^2\, dz &\leq \frac{1}{|C_\infty|} \int_{B\left(O,\frac{\tau}{\tau_0}\right)\cap C_\infty} |\nabla u_\infty|^2\, dz \leq \frac{C_2 \tau^{2\beta}}{|C_\infty|}.\end{aligned}$$

This leads to a contradiction to (4.23) provided that $C_1 \geq \frac{C_2\tau^{2\beta}}{|C_\infty|}$, and thus completes the proof of (4.22) in the case $h(x_0) = 0$. The same argument works for $h(x_0) > 0$ by noticing that in this latter scenario after the blow-up $\mathbb{C}_\infty(h(x_0) + r_n \cdot) \equiv \mathbb{C}_f$ (see also [14, Theorem 3.13]). □

5. Contact-Angle conditions

This section is devoted to the proof of Theorem 2.4. For every profile function h we denote by $h'_-(x)$ and $h'_+(x)$, respectively, the left and right derivative of h in $x \in [a,b]$, whenever they exist. In the following we denote by θ^* the angle

$$\theta^* := \arccos \beta, \tag{5.1}$$

where β is the quantity defined in (2.10). We first provide a preliminary characterization of contact-angle conditions.

Proposition 5.1. *Assume that the Lamé coefficients of the film and the substrate satisfy*

$$\mu_s \geq \mu_f > 0 \quad \textit{and} \quad \mu_s + \lambda_s \geq \mu_f + \lambda_f > 0. \tag{5.2}$$

Then, for every μ-local minimizer $(u,h) \in X$ of $\mathcal{F}$ and for $z_0 := (x_0, 0) \in Z_h \cap \Gamma_h^{reg}$ the following asserts hold true:

1. *For every* $x_0 \in P_h$ *we have that* $\theta^-(x_0), \theta^+(x_0) \in [0, \theta^*]$ *and, if* $\theta^-(x_0) = \theta^+(x_0)$ *then* $\theta^-(x_0) = \theta^+(x_0) = 0$,
2. *For any* $(c, d) \in I_h$*, there holds* $\theta^-(c), \theta^+(d) \in [0, \theta^*]$.

Additionally, Γ_h^{jump} *satisfies the following property*

3. *If* $\beta \neq 0$*, then* $\Gamma_h^{jump} \cap Z_h = \emptyset$.

Proof. Let (u, h) be a μ-local minimizer of $\mathcal{F}$, and let $z_0 = (x_0, h(x_0)) \in Z_h \cap (\Gamma_h^{reg} \cup \Gamma_h^{jump})$. As a consequence of Assertion 1. of Proposition 4.3 there exist $a' > 0$ and $b' > 0$ such that the function $g : (-a', a') \to (-b', b')$ defined as

$$g(x) := h(x) - h(x_0) \quad \text{for every } x \in (-a', a')$$

satisfies one of the following conditions:

(c_1) g is a Lipschitz function in $(-a', a')$ with Lipschitz constant $\operatorname{Lip} g \leq L$ for some $L > 1$;
(c_2) g is a Lipschitz function in $(-a', 0)$ with Lipschitz constant $\operatorname{Lip} g \leq L$ for some $L > 1$, and $g'_+(0) = \infty$;
(c_3) g is a Lipschitz function in $(0, a')$ with Lipschitz constant $\operatorname{Lip} g \leq L$ for some $L > 1$, and $g'_-(0) = -\infty$.

We also point out that in view of the internal-ball condition (see Proposition 4.2), under condition (c_1), the angle between $g'_-(0)$ and $g'_+(0)$ intersecting Ω_h^- is always in the interval $[\pi, 2\pi)$.

In the following we denote the intersection of a given a set U with the half-planes $\{x < 0\}$ and $\{x > 0\}$ by $U^2 := U \cap \{x < 0\}$ and $U^3 := U \cap \{x > 0\}$, respectively. We also set $U^1 := U$.

Choose an infinitesimal sequence $r_n \to 0$, and consider the sets

$$C(z_0, r_n) := \{z_0 + (x, y) \in \mathbb{R}^2 : \, -r_n < x < r_n, \, -4Lr_n < y < g(x)\},$$

and

$$C_n := \frac{1}{r_n}(C(z_0, r_n) - z_0) = \left\{ z \in \mathbb{R}^2 : -1 < z_1 < 1, -4L < z_2 < \frac{g(r_n z_1)}{r_n} \right\}.$$

We observe that for $k = 1, 2, 3$ we have that $\chi_{C_n^k} \to \chi_{C_\infty^k}$ a.e., where

$$C_\infty := \left\{ z \in \mathbb{R}^2 : -1 < z_1 < 1, -4L < z_2 < g_\infty(z_1) \right\},$$

the function $g_\infty : (-1, 1) \to \mathbb{R}$ is defined as

$$g_\infty(z_1) := \begin{cases} g'_-(0) z_1 & \text{for } z_1 < 0, \\ g'_+(0) z_1 & \text{for } z_1 > 0. \end{cases},$$

and $\chi_{C_n^k}$, $\chi_{C_\infty^k}$ denote the characteristic functions of the sets C_n^k and C_∞^k, respectively. In particular, $C_\infty^k \subset R^k$ for $k = 1, 2, 3$, where

$$R := \{z \in \mathbb{R}^2 : -1 < z_1 < 1, \, -4L < z_2 < 4L\}.$$

With a slight abuse of notation under each condition (c_k) we identify the map u with its H^1-extension to the set $\Omega_h \cup R^k(z_0, r_n)$, where

$$R(z_0, r_n) := z_0 + r_n R.$$

Note that this extension is well-defined owing to Assertion 1. of Proposition 4.3, which guarantees that the graph of h, aside from cusps and cuts, is locally Lipschitz.

Let I^k be defined as

$$I^k := \begin{cases} (-1,1) & \text{if } k=1, \\ (-1,0) & \text{if } k=2, \\ (0,1) & \text{if } k=3. \end{cases}$$

For every $0<\delta<1$ under each condition (c_k) we consider a function $\psi_\delta \in W^{1,\infty}(I^k)$ to be specified later, satisfying the following properties

$$\sup_\delta \|\psi_\delta\|_{L^\infty(I^k)} \leq C, \tag{5.3}$$

$$0 \leq g_\infty(z_1) + \delta\psi_\delta(z_1) < 4L \quad \text{for every } z_1 \in I^k, \tag{5.4}$$

$$\operatorname{supp}\psi_\delta = [x_\delta^-, x_\delta^+], \tag{5.5}$$

with

$$\begin{cases} x_\delta^- < 0 < x_\delta^+ & \text{for condition } (c_1), \\ x_\delta^- < 0 \text{ and } x_\delta^+ = 0 & \text{for condition } (c_2), \\ x_\delta^- = 0 \text{ and } x_\delta^+ > 0 & \text{for condition } (c_3). \end{cases} \tag{5.6}$$

Define the maps

$$\psi_\delta^n(x) := \begin{cases} r_n\psi_\delta\left(\frac{x-x_0}{r_n}\right) & \text{for every } x \in [x_0 + r_n x_\delta^-, x_0 + r_n x_\delta^+], \\ 0 & \text{otherwise in } (a,b). \end{cases}$$

Note that for r_n small enough,

$$\Omega_{h+\delta\psi_\delta^n} \subset \Omega_h \cup R^k(z_0, r_n),$$

and

$$|\Omega_{h+\delta\psi_\delta^n}\Delta\Omega_h| \leq \frac{\mu}{2}. \tag{5.7}$$

By Proposition 4.1 there exists $\lambda_0 > 0$ such that

$$\mathcal{F}(u,h) = \min\left\{\mathcal{F}(v,\tilde h) + \lambda||\Omega_h^+| - |\Omega_{\tilde h}^+|| : (v,\tilde h) \in X,\, |\Omega_{\tilde h}\Delta\Omega_h| \leq \frac{\mu}{2}\right\}$$

for all $\lambda \geq \lambda_0$. In the following we denote by $\mathcal{G}$ the volume-penalized functional defined as

$$\mathcal{G}(v,\tilde h) := \mathcal{F}(v,\tilde h) + \lambda_0||\Omega_h^+| - |\Omega_{\tilde h}^+|| \tag{5.8}$$

for every $(v,\tilde h) \in X$. By the minimality of (u,h), and by (1.2), (2.4), (2.10), (5.7), and (5.8), there holds

$$0 \leq \frac{\mathcal{G}(u, h+\delta\psi_\delta^n) - \mathcal{G}(u,h)}{\delta r_n} := A_n^1 - A_n^2 + B_n + D_n + E_n, \tag{5.9}$$

where

$$A_n^1 := \frac{1}{\delta r_n}\int_{\Omega_{h+\delta\psi_n}\setminus\Omega_h} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy, \tag{5.10}$$

$$A_n^2 := \frac{1}{\delta r_n}\int_{\Omega_h\setminus\Omega_{h+\delta\psi_\delta^n}} W_0(y, Eu(x,y) - E_0(y))\,dx\,dy, \tag{5.11}$$

$$\begin{aligned} B_n := {} & \frac{\gamma_f\beta}{\delta r_n}(\mathcal{H}^1(\tilde\Gamma_{h+\delta\psi_\delta^n} \cap \{y=0\}) - \mathcal{H}^1(\tilde\Gamma_h \cap \{y=0\})) \\ & + \frac{\gamma_f}{\delta r_n}(\mathcal{H}^1(\tilde\Gamma_{h+\delta\psi_\delta^n} \cap \{y>0\}) - \mathcal{H}^1(\tilde\Gamma_h \cap \{y>0\})), \end{aligned} \tag{5.12}$$

$$D_n := \frac{\lambda_0}{\delta r_n}\left|\int_a^b h-(h+\delta\psi^n_\delta)\,dx\right|, \tag{5.13}$$

and

$$E_n := \frac{2\gamma_f}{\delta r_n}\sum_{x\in C(h)}((h+\delta\psi^n_\delta)^-(x)-(h+\delta\psi^n_\delta)(x))-\frac{2\gamma_f}{\delta r_n}\sum_{x\in C(h)}(h^-(x)-h(x)). \tag{5.14}$$

We begin by noticing that

$$E_n = 0 \tag{5.15}$$

by the regularity of ψ^n_δ, and that

$$D_n \leq \frac{\lambda_0}{r_n}\int_{r_n x^-_\delta+x_0}^{r_n x^+_\delta+x_0}|\psi^n_\delta(x)|\,dx = \lambda_0 r_n\int_{x^-_\delta}^{x^+_\delta}|\psi_\delta(z_1)|\,dz_1 \to 0 \tag{5.16}$$

as $n\to+\infty$, by the change of variable

$$x = x_0 + r_n z_1. \tag{5.17}$$

Step 1 (Convergence of the elastic-energy terms). We show that $A^1_n \to 0$. The proof that $A^2_n \to 0$ is analogous. We begin by assuming that the quantities

$$\lambda_n := \frac{1}{r_n}\left(\int_{C(z_0,r_n)}|\nabla u|^2\,dx\,dy\right)^{\frac{1}{2}}$$

satisfy

$$\limsup_{n\to+\infty}\lambda_n < +\infty. \tag{5.18}$$

In this situation we define the maps $v_n : C^k_n \to \mathbb{R}^2$, as

$$v_n(z) := \frac{u(z_0+r_n z) - \fint_{C^k(z_0,r_n)}u(x,y)\,dx\,dy}{r_n}.$$

Notice that by construction we have $\int_{C^k_n} v_n(x,y)\,dx\,dy = 0$. Since $u \in H^1(\Omega_h \cup R(z_0,r_n)^k;\mathbb{R}^2)$, in each case (c_k), $k=1,2,3$, the map v_n satisfies $v_n \in H^1(R^k;\mathbb{R}^2)$, $k=1,2,3$, and

$$\|v_n\|_{W^{1,2}(R^k;\mathbb{R}^2)} \leq C\|\nabla v_n\|_{L^2(C^k_n;\mathbb{M}^{2\times 2})} \leq C\lambda^2_n \leq C \tag{5.19}$$

for n big enough, where the last inequality follows from (5.18). Therefore for each case (c_k), $k=1,2,3$ we conclude that

$$\begin{aligned}
A^1_n &= \frac{1}{\delta r_n}\int_{I^\delta_n}\int_{h(x)}^{h(x)+\delta\psi^n_\delta(x)}\mathbb{C}_f(Eu(x,y)-E_0(y)):(Eu(x,y)-E_0(y))\,dy\,dx\\
&= \frac{r_n}{\delta}\int_{I^\delta}\int_{\frac{h(x_0+r_n z_1)}{r_n}}^{\frac{h(x_0+r_n z_1)}{r_n}+\delta\psi_\delta(z_1)}\mathbb{C}_f(Ev_n(z)-E_0(z_2)):(Ev_n(z)-E_0(z_2))\,dz_2\,dz_1\\
&\leq \frac{Cr_n}{\delta}\|Ev_n-E_0\|_{L^2(R^k;\mathbb{M}^{2\times 2}_{\rm sym})} \leq \frac{Cr_n}{\delta},
\end{aligned}$$

with $I^\delta_n := (x_0+r_n x^-_\delta, x_0+r_n x^+_\delta)\cap\{\psi^n_\delta \geq 0\}$, $I^\delta := (x^-_\delta, x^+_\delta)\cap\{\psi_\delta\geq 0\}$, where in the second equality we performed the change of variable

$$(x,y) = (x_0 + r_n z_1, r_n z_2), \tag{5.20}$$

and where the last inequality follows from (5.19), and (5.3)–(5.6).

When (5.18) does not hold, we have

$$\limsup_{n\to+\infty}\lambda_n=+\infty.$$

Then, in view of Proposition 4.6 there holds

$$r_n\lambda_n^2=\frac{1}{r_n}\int_{C(z_0,r_n)}|\nabla u|^2\,dx\,dy\leq Cr_n^{2\alpha-1}\to 0. \tag{5.21}$$

We define the maps $w_n:C_n^k\to\mathbb{R}^2$, as

$$w_n(z):=\frac{u(z_0+r_nz)-\fint_{C^k(z_0,r_n)}u(x,y)\,dx\,dy}{\lambda_nr_n},$$

for every $z\in C_n^k$. Note that $\int_{C_n^k}w_n(x)\,dx=0$ by construction. Again, the fact that $u\in H^1(\Omega_h\cup R^k(z_0,r_n);\mathbb{R}^2)$ implies that $w_n\in H^1(R^k;\mathbb{R}^2)$, with

$$\|w_n\|_{W^{1,2}(R^k;\mathbb{R}^2)}\leq C\|\nabla w_n\|_{L^2(C_n^k;\mathbb{M}^{2\times 2})}\leq C. \tag{5.22}$$

By employing the same change of variable (5.20) of the first case we observe that

$$\begin{aligned}A_n^1&=\frac{1}{\delta r_n}\int_{I_n^\delta}\int_{h(x)}^{h(x)+\delta\psi_\delta^n(x)}\mathbb{C}_f(Eu(x,y)-E_0(y)):(Eu(x,y)-E_0(y))\,dy\,dx\\&=\frac{\lambda_n^2r_n}{\delta}\int_{I^\delta}\int_{\frac{h(x_0+r_nz_1)}{r_n}}^{\frac{h(x_0+r_nz_1)}{r_n}+\delta\psi_\delta(z_1)}\mathbb{C}_f(Ew_n(z)-E_0(z_2)):(Ew_n(z)-E_0(z_2))\,dz_2\,dz_1\\&\leq\frac{C\lambda_n^2r_n}{\delta}\|Ew_n-E_0\|_{L^2(R^k;\mathbb{M}^{2\times 2}_{\rm sym})}\leq\frac{C}{\delta}\lambda_n^2r_n\end{aligned}$$

where now in the last inequality we used (5.22) and (5.3)–(5.6). The claim follows from (5.21).

Step 2 (Surface-energy convergence under condition (c_1))**.** In this step we study the convergence of the terms B_n under condition (c_1). To this aim, we treat in three different subsections the cases of island borders, of valleys with no vanishing contact angles, and of valleys with one vanishing contact angle. In particular, the first subsection yields Assertion 2. of the proposition, whereas Assertion 1. is proved in the second and third subsections.

Island borders. In this subsection we prove Assertion 2. of the proposition, namely we consider $x_0=c$ for some $(c,d)\in I_h$, and we prove that $\theta^-(c)\leq\theta^*$ (see Figure 9). The case of $x_0=d$ and $\theta^+(d)$ is analogous by symmetry. For simplicity we denote in the following $\theta^-(c)$ by θ.

We begin by considering $\theta^*>0$. Note that

$$\tan(\theta)=-g'_-(0)\quad\text{and}\quad\tan(\theta^*)=\frac{\sqrt{1-\beta^2}}{\beta}. \tag{5.23}$$

Assume by contradiction that

$$\theta>\theta^*. \tag{5.24}$$

Then by (5.23), we have $0 < \tan(\theta^*) < -g'_-(0)$. We define ψ_δ by

$$\psi_\delta(s) = \begin{cases} -\left(\frac{g'_-(0)+\tan(\theta^*)}{\delta}\right) s + \frac{\tan(\theta^*)}{g'_-(0)} + 1 & \text{for } \frac{\delta}{g'_-(0)} < s \leq 0, \\ \frac{-\tan(\theta^*)}{\delta} s + \frac{\tan(\theta^*)}{g'_-(0)} + 1 & \text{for } 0 < s < \delta\left(\frac{1}{\tan(\theta^*)} + \frac{1}{g'_-(0)}\right), \\ 0 & \text{otherwise.} \end{cases}$$

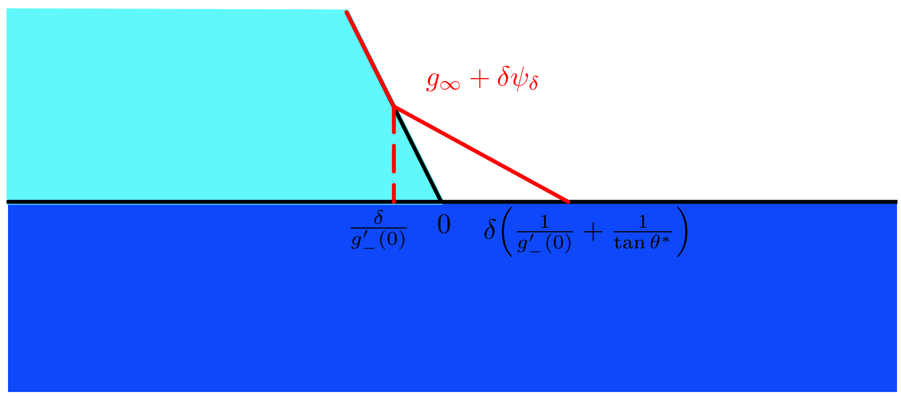

FIGURE 9. The blow-up at island borders (and at valleys with one vanishing contact angle) is displayed. The profile of g_∞ and the perturbation $g_\infty + \delta\psi_\delta$ in the case $\theta \geq \theta^* = \arccos\beta$ for island borders are highlighted in black and in red, respectively

We observe that

$$\mathcal{H}^1(\tilde{\Gamma}_{h+\delta\psi_\delta^n} \cap \{y=0\}) - \mathcal{H}^1(\tilde{\Gamma}_h \cap \{y=0\}) = -\delta r_n \left(\frac{1}{\tan(\theta^*)} + \frac{1}{g'_-(0)}\right)$$

as shown in Figure 9. We now observe that by condition (c_1) the map h is Lipschitz in $(-a', a')$ and hence, its derivative h' exists a.e. in $(-a', a')$, and h'_- and h'_+ are, respectively, left and right continuous. These properties together with the definition of ψ_δ imply that

$$\begin{aligned} B_n &= \frac{\gamma_f}{\delta r_n} \int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{\delta r_n\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right)+x_0} \sqrt{1+\left(h'(x)+\delta\psi'_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}\,dx \\ &\quad - \frac{\gamma_f}{\delta r_n} \int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{x_0} \sqrt{1+(h'(x))^2}\,dx - \gamma_f\beta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right) \\ &= \frac{\gamma_f}{\delta} \int_{\frac{\delta}{g'_-(0)}}^{\delta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right)} \sqrt{1+(h'(x_0+r_n s)+\delta\psi'_\delta(s))^2}\,ds \\ &\quad - \frac{\gamma_f}{\delta} \int_{\frac{\delta}{g'_-(0)}}^{0} \sqrt{1+(h'(x_0+r_n s))^2}\,ds - \gamma_f\beta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right), \end{aligned}$$

where in the last equality we used the change of variable (5.17). Furthermore, in view of the fact that $h'_-(x_0+r_nz)\to g'_-(0)$ and $h'_+(x_0+r_nz)\to g'_+(0)$ as $n\to+\infty$, the Lebesgue Dominated Convergence Theorem yields that

$$B_n\to -\gamma_f\beta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right)+\frac{\gamma_f}{\beta\tan(\theta^*)}+\gamma_f\frac{\sqrt{1+(g'_-(0))^2}}{g'_-(0)}. \tag{5.25}$$

By (5.9), (5.16), and Step 1, there holds

$$-\beta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right)+\frac{1}{\beta\tan(\theta^*)}+\frac{\sqrt{1+(g'_-(0))^2}}{g'_-(0)}\geq 0, \tag{5.26}$$

which in turn implies

$$\beta\tan(\theta^*)\sqrt{1+(\tan(\theta))^2}\leq(1-\beta^2)\tan(\theta)+\beta^2\tan(\theta^*). \tag{5.27}$$

Substituting (5.23) in (5.27), dividing by $\sqrt{1-\beta^2}$, and taking the squares of both sides of the resulting inequality, we obtain

$$\left(\beta\tan(\theta)-\sqrt{1-\beta^2}\right)^2\leq 0, \tag{5.28}$$

and hence, again by (5.23), $\theta=\theta^*$ which is in contradiction with (5.24).

Consider now the case in which $\theta^*=0$, i.e., $\beta=1$. Assume by contradiction that

$$\theta>\theta^*=0. \tag{5.29}$$

Then, for δ small enough, by (5.23), we have $0=\tan(\theta^*)<\delta<-g'_-(0)$. We define ψ_δ by

$$\psi_\delta(s)=\begin{cases}-\left(\frac{g'_-(0)+\varepsilon_\delta}{\delta}\right)s+\frac{\varepsilon_\delta}{g'_-(0)}+1 & \text{for } \frac{\delta}{g'_-(0)}<s\leq 0,\\ \frac{-\varepsilon_\delta}{\delta}s+\frac{\varepsilon_\delta}{g'_-(0)}+1 & \text{for } 0<s<\delta\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right),\\ 0 & \text{otherwise,}\end{cases}$$

where $\varepsilon_\delta<<\delta$ is such that $\delta\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right)<1$.

The same computations as in the case $\theta^*>0$ yield

$$\mathcal{H}^1(\tilde\Gamma_{h+\delta\psi^n_\delta}\cap\{y=0\})-\mathcal{H}^1(\tilde\Gamma_h\cap\{y=0\})=-\delta r_n\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right),$$

and hence, since here $\beta=1$,

$$\begin{aligned}B_n&=\frac{\gamma_f}{\delta r_n}\int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{\delta r_n\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right)+x_0}\sqrt{1+\left(h'(x)+\delta\psi'_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}\,dx\\&\quad-\frac{\gamma_f}{\delta r_n}\int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{x_0}\sqrt{1+(h'(x))^2}\,dx-\gamma_f\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right)\\&=\frac{\gamma_f}{\delta}\int_{\frac{\delta}{g'_-(0)}}^{\delta\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right)}\sqrt{1+(h'(x_0+r_ns)+\delta\psi'_\delta(s))^2}\,ds\\&\quad-\frac{\gamma_f}{\delta}\int_{\frac{\delta}{g'_-(0)}}^{0}\sqrt{1+(h'(x_0+r_ns))^2}\,ds-\gamma_f\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right),\end{aligned}$$

which in turn, by the Dominated Convergence Theorem, implies

$$B_n \to -\gamma_f \left(\frac{1}{\varepsilon_\delta} + \frac{1}{g'_-(0)}\right) + \gamma_f \frac{\sqrt{1+\varepsilon_\delta^2}}{\varepsilon_\delta} + \gamma_f \frac{\sqrt{1+(g'_-(0))^2}}{g'_-(0)}. \tag{5.30}$$

Since the function $x \to \frac{\sqrt{1+x^2}}{x} - \frac{1}{x}$ is strictly increasing in $(-\infty, 0)$, inequality (5.30) gives

$$0 > g'_-(0) \geq -\varepsilon_\delta.$$

By the arbitrary smallness of ε_δ we conclude that $g'_-(0) = 0$. This contradicts (5.29), and completes the proof of Assertion 2. of the proposition.

Valleys with no vanishing contact angles. In this subsection we begin the proof of Assertion 1. of the proposition, namely we consider a point $x_0 \in P_h$ and we prove that at least one between $g'_-(0)$ and $g'_+(0)$ is zero. Notice that since the profile of the film is a graph we have

$$g'_-(0) \leq 0 \leq g'_+(0).$$

Assume by contradiction that

$$g'_-(0) < 0 < g'_+(0), \tag{5.31}$$

and define ψ_δ by

$$\psi_\delta(s) := \begin{cases} 0 & \text{for } s < \frac{\delta}{g'_-(0)} \text{ and } s > \frac{\delta}{g'_+(0)}. \\ \frac{\delta - g_\infty(s)}{\delta} & \text{for } s \in \left[\frac{\delta}{g'_-(0)}, \frac{\delta}{g'_+(0)}\right] \end{cases}$$

for every $s \in (-1, 1)$ (see Figure 10). Since $\psi_\delta \geq 0$, by (5.12) we obtain

$$B_n = \frac{\gamma_f}{\delta r_n}(\mathcal{H}(\tilde{\Gamma}_{h+\delta\psi_\delta^n} \cap \{y > 0\}) - \mathcal{H}(\tilde{\Gamma}_h \cap \{y > 0\})).$$

As before, since h is Lipschitz in $(-a', a')$, the definition of ψ_δ implies that

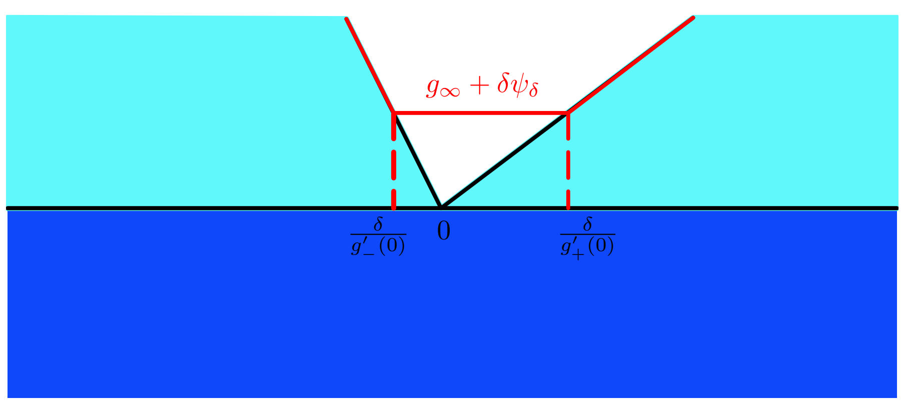

FIGURE 10. The blow-up at a valley with no vanishing contact angle is displayed. The profile of g_∞ and the perturbation $g_\infty + \delta\psi_\delta$ are highlighted in black and in red, respectively

$$
\begin{aligned}
B_n &= \frac{\gamma_f}{\delta r_n}\int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{\frac{\delta r_n}{g'_+(0)}+x_0} \left(\sqrt{1+\left(h'(x)+\delta\psi'_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}-\sqrt{1+(h'(x))^2}\right)dx\\
&= \frac{\gamma_f}{\delta r_n}\int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{x_0} \left(\sqrt{1+\left(h'(x)-g'_-(0)\right)^2}-\sqrt{1+(h'(x))^2}\right)dx\\
&+ \frac{\gamma_f}{\delta r_n}\int_{x_0}^{\frac{\delta r_n}{g'_+(0)}+x_0} \left(\sqrt{1+\left(h'(x)-g'_+(0)\right)^2}-\sqrt{1+(h'(x))^2}\right)dx\\
&= \frac{\gamma_f}{\delta}\int_{\frac{\delta}{g'_-(0)}}^{0} \left(\sqrt{1+\left(h'_-(x_0+r_nz)-g'_-(0)\right)^2}-\sqrt{1+(h'_-(x_0+r_nz))^2}\right)dz\\
&+ \frac{\gamma_f}{\delta}\int_{0}^{\frac{\delta}{g'_+(0)}} \left(\sqrt{1+\left(h'_+(x_0+r_nz)-g'_+(0)\right)^2}-\sqrt{1+(h'_+(x_0+r_nz))^2}\right)dx
\end{aligned}
$$

where in the last equality we used the change of variable (5.17). Furthermore, in view of the fact that $h'_-(x_0+r_nz)\to g'_-(0)$ and $h'_+(x_0+r_nz)\to g'_+(0)$ as $n\to+\infty$, the Lebesgue Dominated Convergence Theorem yields that

$$
B_n \to \gamma_f\left(\frac{g'_-(0)}{1+\sqrt{1+(g'_-(0))^2}}-\frac{g'_+(0)}{1+\sqrt{1+(g'_+(0))^2}}\right).
$$

as $n\to+\infty$. By (5.9), (5.15), (5.16), and Step 1, there holds

$$
\frac{g'_-(0)}{1+\sqrt{1+(g'_-(0))^2}} \geq \frac{g'_+(0)}{1+\sqrt{1+(g'_+(0))^2}}. \tag{5.32}
$$

We observe that, setting $f(x):=\frac{x}{1+\sqrt{1+x^2}}$ for every $x\in\mathbb{R}$, there holds $f'(x)>0$ for every $x\in\mathbb{R}$. Thus (5.32) yields that $g'_-(0)\geq g'_+(0)$ which is in contradiction with (5.31).

Valleys with one vanishing contact angle. In this subsection we conclude the proof of Assertion 1. of the proposition. From the previous subsection it remains to prove that if $x_0\in P_h$ is such that $\theta^+(x_0)=0$, then $\theta^-(x_0)\leq\theta^*$ (see Figure 9). In the symmetric case, in which $x_0\in P_h$ is such that $\theta^-(x_0)=0$, analogous arguments imply that $\theta^+(x_0)\leq\theta^*$.

Let $x_0\in P_h$ with $\theta^+(x_0)=0$. We first consider the case $\theta^*:=\arccos(\beta)>0$. Assume by contradiction that

$$
\theta := \theta^-(x_0) > \theta^*. \tag{5.33}
$$

We define ψ_δ as in the case of island borders, by

$$
\psi_\delta(s) = \begin{cases} -\left(\frac{g'_-(0)+\tan(\theta^*)}{\delta}\right)s+\frac{\tan(\theta^*)}{g'_-(0)}+1 & \text{for } \frac{\delta}{g'_-(0)}<s\leq 0,\\ \frac{-\tan(\theta^*)}{\delta}s+\frac{\tan(\theta^*)}{g'_-(0)}+1 & \text{for } 0<s<\delta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right),\\ 0 & \text{otherwise.}\end{cases}
$$

Differently from the case of island borders, we have

$$
\mathcal{H}^1(\tilde{\Gamma}_{h+\delta\psi^n_\delta}\cap\{y=0\})-\mathcal{H}^1(\tilde{\Gamma}_h\cap\{y=0\})=0,
$$

and

$$
\begin{aligned}
B_n = {} & \frac{\gamma_f}{\delta r_n} \int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{\delta r_n\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right)+x_0} \sqrt{1+\left(h'(x)+\delta\psi'_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}\,dx \\
& - \frac{\gamma_f}{\delta r_n}\int_{\frac{\delta r_n}{g'_-(0)}+x_0}^{x_0} \sqrt{1+(h'(x))^2}\,dx - \gamma_f\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right) \\
= {} & \frac{\gamma_f}{\delta}\int_{\frac{\delta}{g'_-(0)}}^{\delta\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right)} \sqrt{1+(h'(x_0+r_n s)+\delta\psi'_\delta(s))^2}\,ds \\
& - \frac{\gamma_f}{\delta}\int_{\frac{\delta}{g'_-(0)}}^{0}\sqrt{1+(h'(x_0+r_n s))^2}\,ds - \gamma_f\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right).
\end{aligned}
$$

Arguing as in Step 2 in the case of island borders, by the Dominated Convergence Theorem, we obtain

$$
B_n \to -\gamma_f\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right) + \frac{\gamma_f}{\beta\tan(\theta^*)} + \gamma_f\frac{\sqrt{1+(g'_-(0))^2}}{g'_-(0)} \geq 0.
$$

Since $\beta \leq 1$, the previous inequality implies (5.26), which in turn, arguing as in the case of island borders, yields $\theta = \theta^*$. This contradicts (5.33).

Consider now the case in which $\theta^* = 0$, and assume by contradiction that

$$
\theta := \theta^-(x_0) > \theta^*, \tag{5.34}
$$

namely $\beta = 1$. Then, for δ small enough, by (5.23), we have $0 = \tan(\theta^*) < \delta < -g'_-(0)$. We define ψ_δ as in the case of island borders, by

$$
\psi_\delta(s) = \begin{cases} -\left(\frac{g'_-(0)+\varepsilon_\delta}{\delta}\right)s + \frac{\varepsilon_\delta}{g'_-(0)} + 1 & \text{for } \frac{\delta}{g'_-(0)} < s \leq 0, \\ \frac{-\varepsilon_\delta}{\delta}s + \frac{\varepsilon_\delta}{g'_-(0)} + 1 & \text{for } 0 < s < \delta\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right), \\ 0 & \text{otherwise.} \end{cases}
$$

where $\varepsilon_\delta << \delta$ is such that $\delta\left(\frac{1}{\varepsilon_\delta}+\frac{1}{g'_-(0)}\right) < 1$. Analogous computations to the case $\theta^* > 0$, as well as the fact that $\beta = 1$, yield the inequality

$$
-\gamma_f\left(\frac{1}{\tan(\theta^*)}+\frac{1}{g'_-(0)}\right) + \gamma_f\frac{\sqrt{1+\varepsilon_\delta^2}}{\varepsilon_\delta} + \gamma_f\frac{\sqrt{1+(g'_-(0))^2}}{g'_-(0)} \geq 0, \tag{5.35}
$$

which is the same relation that we obtained in (5.30). As in Step 2, in the case of island borders with $\theta^* = 0$, we deduce that $0 > g'_-(0) \geq -\varepsilon_\delta$, and, by the arbitrary smallness of ε_δ, that $g'_-(0) = 0$. This contradicts (5.34) and completes the proof of Assertion 1.

Step 3 (Surface-energy convergence under conditions (c_2) or (c_3))**.** We point out that conditions (c_2) or (c_3) correspond to z_0 being a lower-endpoint of a connected component of Γ_h^{jump}. In this step we prove Assertion 3. of the proposition, namely we show that conditions (c_2) and (c_3) are never satisfied except when $\beta = 0$. As in the previous step we distinguish the case of island borders, of valleys with no vanishing contact angles, and of valleys with one vanishing contact angle. We only consider condition (c_3). The same arguments work under condition (c_2).

Jumps: Island borders. Here we prove that if $\theta^* \neq \frac{\pi}{2}$ then there are no jumps at island borders. Assume by contradiction that there exists $x_0 = c$ for some $(c,d) \in I_h$ such that $\theta^-(c) = \pi/2$, and that $\theta^* \neq \frac{\pi}{2}$.

We first consider the case in which $\theta^* > 0$. We choose ψ_δ such that

$$\psi_\delta(s) := \begin{cases} -\frac{\tan(\theta^*)}{\delta} s & \text{for } 0 < s < \frac{\delta}{\tan(\theta^*)}, \\ 0 & \text{otherwise .} \end{cases}$$

We observe that

$$\mathcal{H}(\tilde{\Gamma}_{h+\delta\psi^n_\delta} \cap \{y=0\}) - \mathcal{H}(\tilde{\Gamma}_h \cap \{y=0\}) = -\frac{\delta r_n}{\tan(\theta^*)}.$$

Therefore

$$\begin{aligned} B_n &= \frac{\gamma_f}{\delta r_n} \int_{x_0}^{\frac{\delta r_n}{\tan(\theta^*)}+x_0} \sqrt{1 + \left(h'(x) + \delta\psi'_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}\, dx - \frac{\gamma_f \beta}{\tan(\theta^*)} - \gamma_f \\ &= \frac{\gamma_f}{\delta} \int_0^{\frac{\delta}{\tan(\theta^*)}} \sqrt{1 + (h'(x_0 + r_n s) + \delta\psi'_\delta(s))^2}\, ds - \frac{\gamma_f \beta}{\tan(\theta^*)} - \gamma_f. \end{aligned}$$

By the Dominated Convergence Theorem, we obtain

$$B_n \to -\frac{\gamma_f \beta}{\tan(\theta^*)} + \frac{\gamma_f}{\beta \tan(\theta^*)} - \gamma_f.$$

By (5.9), (5.16), and Step 1, there holds

$$-1 - \frac{\beta}{\tan(\theta^*)} + \frac{1}{\beta\tan(\theta^*)} \geq 0.$$

Thus, in view of (5.23) we conclude that

$$\sin(\theta^*) = \sqrt{1-\beta^2} \geq 1, \tag{5.36}$$

namely, a contradiction.

Consider now the case in which $\theta^* = 0$, and choose

$$\psi_\delta(s) := -s$$

for every $s \in (0,1)$. Then,

$$\mathcal{H}(\tilde{\Gamma}_{h+\delta\psi^n_\delta} \cap \{y=0\}) - \mathcal{H}(\tilde{\Gamma}_h \cap \{y=0\}) = -r_n,$$

and, since $\beta = 1$,

$$\begin{aligned} B_n &= \frac{\gamma_f}{\delta r_n} \int_{x_0}^{r_n+x_0} \sqrt{1 + \left(h'(x) + \delta\psi'_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}\, dx - \frac{\gamma_f}{\delta} - \gamma_f \\ &= \frac{\gamma_f}{\delta} \int_0^1 \sqrt{1 + (h'(x_0 + r_n s) + \delta\psi'_\delta(s))^2}\, ds - \frac{\gamma_f}{\delta} - \gamma_f. \end{aligned}$$

By the Dominated Convergence Theorem, we have

$$B_n \to -\frac{\gamma_f}{\delta} + \gamma_f \frac{\sqrt{1+\delta^2}}{\delta} - \gamma_f.$$

Thus, properties (5.9), (5.16), and Step 1 imply that

$$-1 - \frac{1}{\delta} + \frac{\sqrt{1+\delta^2}}{\delta} \geq 0.$$

Since $-1-\frac{1}{\delta}+\frac{\sqrt{1+\delta^2}}{\delta}<0$ for every $\delta>0$, we reach also in this case a contradiction.

Jumps: Valleys with no vanishing contact angles. In this subsection we prove that for every θ^* there are no jumps at valleys with no vanishing contact angles. Consider $x_0\in P_h$ and such that $\theta^-(x_0)=\frac{\pi}{2}$. We want to prove that $g'_+(0)=0$. Assume by contradiction that $g'_+(0)>0$. Let

$$\psi_\delta(s):=\begin{cases}0 & \text{for } s>\frac{\delta}{g'_+(0)}.\\ \frac{\delta-g_\infty(s)}{\delta} & \text{for } s\in\left(0,\frac{\delta}{g'_+(0)}\right],\end{cases}$$

for every $s\in(0,1)$. By the definition of ψ_δ there holds

$$B_n=\frac{\gamma_f}{\delta r_n}(\mathcal{H}(\tilde\Gamma_{h+\delta\psi^n_\delta}\cap\{y>0\})-\mathcal{H}(\tilde\Gamma_h\cap\{y>0\})).$$

In particular, we obtain

$$\begin{aligned}B_n&=\frac{\gamma_f}{\delta r_n}\int_{x_0}^{\frac{\delta r_n}{g'_+(0)}+x_0}\left(\sqrt{1+\left(h'(x)+\delta\psi'^n_\delta\left(\frac{x-x_0}{r_n}\right)\right)^2}-\sqrt{1+(h'(x))^2}\right)dx-\gamma_f\\ &=\frac{\gamma_f}{\delta r_n}\int_{x_0}^{\frac{\delta r_n}{g'_+(0)}+x_0}\left(\sqrt{1+\left(h'(x)-g'_+(0)\right)^2}-\sqrt{1+(h'(x))^2}\right)dx-\gamma_f\\ &=\frac{\gamma_f}{\delta}\int_0^{\frac{\delta}{g'_+(0)}}\left(\sqrt{1+\left(h'(x_0+r_nz)-g'_+(0)\right)^2}-\sqrt{1+(h'(x_0+r_nz))^2}\right)dx-\gamma_f.\end{aligned}$$

By applying the Dominated Convergence Theorem we conclude that

$$B_n\to-\gamma_f\frac{g'_+(0)}{1+\sqrt{1+(g'_+(0))^2}}-\gamma_f$$

as $n\to+\infty$. Therefore, properties (5.9), (5.16), and Step 1 yield

$$g'_+(0)\le-1-\sqrt{1+(g'_+(0))^2},$$

which contradicts the non negativity of $g'_+(0)$.

Jumps: Valleys with one vanishing contact angle. Here we prove that if $\theta^*\neq\frac{\pi}{2}$ then there are no jumps at valleys with one vanishing contact angle (and hence, by the previous subsection, at every valley). Assume by contradiction that $\theta^*\neq\frac{\pi}{2}$, and that there exists $x_0\in P_h$ with $\theta^-(x_0)=\frac{\pi}{2}$ and $g'_+(0)=0$.

In the situation in which $\theta^*>0$ we argue choosing ψ_δ as in the corresponding situation in Step 3, in the case of island borders. The same computations as in that subsection yield

$$B_n\to-\frac{\gamma_f}{\tan(\theta^*)}+\frac{\gamma_f}{\beta\tan(\theta^*)}-\gamma_f\ge0.$$

Since $\beta<1$, this implies

$$-\frac{\gamma_f\beta}{\tan(\theta^*)}+\frac{\gamma_f}{\beta\tan(\theta^*)}-\gamma_f\ge0,$$

which in turn yields to (5.36) and to a contradiction.

The situation in which $\theta^*=0$ can be dealt with exactly in the same way as in the corresponding setting in Step 3 for island borders.

□

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. We observe that Assertion 3. of Theorem 2.4 coincides with Assertion 3. of Proposition 5.1. In the wetting regime $\beta = 1$ also Assertions 1. and 2. of Theorem 2.4 follow directly from Assertions 1. and 2. of Proposition 5.1. Furthermore, in the dewetting regime $\beta < 1$ from Proposition 5.1 for any $(c, d) \in I_h$ and $p \in P_h$ the angles $\theta^-(p)$, $\theta^+(p)$, $\theta^-(c)$, and $\theta^+(d)$ are smaller or equal to θ^* (and at least one between $\theta^-(p)$ and $\theta^+(p)$ is zero). It remains therefore to assume that $\beta < 1$, and in turn

$$\theta^* > 0, \tag{5.37}$$

and to show that for any $(c, d) \in I_h$ the angles $\theta^-(c)$ and $\theta^+(d)$ are not strictly smaller than θ^*, and that $P_h = \emptyset$.

To this aim we observe that it is enough to show the following claim: for every $z_0 = (x_0, h(x_0)) \in Z_h$ which is a valley or an island border, there holds

$$\theta^-(x_0) \geq \theta^*.$$

In fact, we already know that in the dewetting regime any $p \in P_h$ has at least a zero contact angle from Proposition 5.1.

To show the claim, we argue by contradiction and we assume that there exists a point $z_0 = (x_0, h(x_0)) \in Z_h$ such that

$$\theta^-(x_0) < \theta^*. \tag{5.38}$$

The case with $\theta^+(x_0) < \theta^*$ follows by symmetry. We start by defining a competitor profile function $h_\varepsilon \in AP(a, b)$ by

$$h_\varepsilon(x) := \begin{cases} h(x) & \text{if } x \notin [x_0 - \varepsilon, x_0], \\ -\tan(\theta^*)(x - x_0 + \varepsilon) + h(x_0 - \varepsilon) & \text{if } x \in [x_0 - \varepsilon, x_0 - \varepsilon + \ell_\varepsilon], \\ 0 & \text{if } x \in [x_0 - \varepsilon + \ell_\varepsilon, x_0], \end{cases} \tag{5.39}$$

for every $x \in (a, b)$ and $\varepsilon > 0$ small enough, where the quantity

$$\ell_\varepsilon := \frac{h(x_0 - \varepsilon)}{\tan(\theta^*)} \tag{5.40}$$

is well defined owing to (5.37). We observe that $h \geq h_\varepsilon$ and that

$$|\Omega_h| - |\Omega_{h_\varepsilon}| \leq \int_{x_0-\varepsilon}^{x_0} h(x)dx = \varepsilon \int_{-1}^{0} h(x_0 - \varepsilon y)dy \tag{5.41}$$

by the change of variable $x = x_0 + \varepsilon y$. Furthermore, we notice that the integral on the right-hand side of (5.41) converges to zero by the Lebesgue Dominated Convergence Theorem because h is null and continuous at x_0. Therefore, $(u, h_\varepsilon) \in X$ is admissible for the penalized minimum problem (4.1) for every $\varepsilon > 0$ small enough.

From the minimality of (u, h) and Proposition 4.1 it follows that

$$\begin{aligned} \mathcal{F}(u, h) &\leq \mathcal{F}(u, h_\varepsilon) + \lambda_0 ||\Omega_h| - |\Omega_{h_\varepsilon}|| \\ &\leq \mathcal{F}(u, h_\varepsilon) + \lambda_0 \varepsilon \int_{-1}^{0} h(x_0 - \varepsilon y)dy, \end{aligned} \tag{5.42}$$

where in the last inequality we again used (5.41). By (1.2), (2.10), (5.39), and (5.40) we obtain

$$\begin{aligned}
\mathcal{F}(u,h_\varepsilon) &= \int_{\Omega_{h_\varepsilon}} W_0(y, Eu(x,y)-E_0(y))\,dx\,dy + \int_{\tilde\Gamma_{h_\varepsilon}} \varphi(y)\,d\mathcal{H}^1 \\
&\quad + 2\gamma_f \mathcal{H}^1(\Gamma^{cut}_{h_\varepsilon}) + \gamma_{fs}(b-a) \\
&\le \int_{\Omega_h} W_0(y, Eu(x,y)-E_0(y))\,dx\,dy + \int_{\tilde\Gamma_h} \varphi(y)\,d\mathcal{H}^1 \\
&\quad - \gamma_f \int_{x_0-\varepsilon}^{x_0} \sqrt{1+(h'(x))^2}dx + \gamma_f\sqrt{h^2(x_0-\varepsilon)+\ell_\varepsilon^2} + \beta\gamma_f(\varepsilon-\ell_\varepsilon) \\
&\quad + 2\gamma_f \mathcal{H}^1(\Gamma^{cut}_{h}) + \gamma_{fs}(b-a) \\
&= \mathcal{F}(u,h) - \gamma_f\varepsilon \int_{-1}^{0} \sqrt{1+(h'_-(x_0+\varepsilon y))^2}dy \\
&\quad + \gamma_f\sqrt{h^2(x_0-\varepsilon)+\ell_\varepsilon^2} + \beta\gamma_f(\varepsilon-\ell_\varepsilon).
\end{aligned} \tag{5.43}$$

Inequalities (5.42) and (5.43) yield

$$\begin{aligned}
0 \le &\frac{\lambda_0}{\gamma_f}\int_{-1}^{0} h(x_0-\varepsilon y)dy - \int_{-1}^{0}\sqrt{1+(h'_-(x_0+\varepsilon y))^2}dy \\
&\qquad + \frac{h(x_0-\varepsilon)}{\varepsilon}\frac{\sqrt{1+\tan^2\theta^*}}{\tan\theta^*} + \beta\left(1-\frac{\ell_\varepsilon}{\varepsilon}\right).
\end{aligned} \tag{5.44}$$

By applying again the Lebesgue Dominated Convergence Theorem together with the observation that both h and h'_- are left continuous at x_0, $h(x_0)=0$, and $h'_-(x_0) = -\tan(\theta^-(x_0))$, we obtain that

$$\begin{aligned}
0 \le -\sqrt{1+\tan^2(\theta^-(x_0))} + \tan\left(\theta^-(x_0)\right)\frac{\sqrt{1+\tan^2\theta^*}}{\tan\theta^*} \\
+ \beta\left(1-\frac{\tan(\theta^-(x_0))}{\tan\theta^*}\right).
\end{aligned} \tag{5.45}$$

If $\tan(\theta^-(x_0)) = 0$, inequality (5.45) implies that $\beta \ge 1$, which contradicts the fact that $\beta < 1$.

Assume now that $\tan(\theta^-(x_0)) \ne 0$. By dividing (5.45) by $\tan(\theta^-(x_0))$, we have

$$0 \le -\frac{\sqrt{1+\tan^2(\theta^-(x_0))}}{\tan(\theta^-(x_0))} + \frac{\sqrt{1+\tan^2\theta^*}}{\tan\theta^*} + \beta\left(\frac{1}{\tan(\theta^-(x_0))} - \frac{1}{\tan\theta^*}\right)$$

from which we conclude that

$$(\beta\tan\left(\theta^-(x_0)\right) - \sqrt{1-\beta^2})^2 \le 0, \tag{5.46}$$

in the same way as done for passing from (5.25) to (5.28). From (5.46) it follows that

$$\tan\left(\theta^-(x_0)\right) = \frac{\sqrt{1-\beta^2}}{\beta} = \tan\theta^*.$$

This contradicts (5.38), and therefore the claim and the theorem follow.

□

6. Regularity of local minimizers

In this section we prove Theorem 2.5 by improving the regularity results already contained in Section 4. In particular the results follow from Proposition 4.3, the decay estimate of Proposition 4.6, from implementing some arguments used for Proposition 5.1, and from proving a second decay estimate which is independent from the specific point on the graph $\Gamma_h \setminus (\Gamma_h^{cut} \cup \Gamma_h^{cusp})$ (see (6.1)).

Proof of Theorem 2.5. We begin by observing that Assertions 1. and 2. are direct consequences of Proposition 4.3. In fact, as pointed out in [14, Remark 3.6], the only situation in which case (ii) of Proposition 4.3 arises is when z_0 is either a cusp point or the lower-end point of a vertical cut. Then, by combining Proposition 4.3 with a compactness argument it follows that the set $\Gamma_h^{cusp} \cup \{(x, h(x)) : x \in C(h)\}$ where $C(h)$ is the set defined in (2.3) has finite cardinality.

Assertion 3. directly follows from Theorem 2.4 and the fact that the same argument employed in Step 3 of Proposition 5.1 in the case of valleys can be applied also to points $z_0 = (x_0, h(x_0)) \in \Gamma_h \setminus (\Gamma_{\rm cut} \cup \Gamma_{\rm cusp})$ with $h(x_0) > 0$ (by using Step 1 of Proposition 5.1 for $\mathbb{C}_f = \mathbb{C}_s$). This in particular yields that

$$(x, h^-(x)) \in \Gamma_h^{jump} \qquad \text{iff} \qquad (h^-(x) = 0 \text{ and } \beta = 0).$$

To obtain Assertion 4. we note that, by employing a similar argument to the one of Step 2 (valleys with one vanishing contact angle) of the Proof of Proposition 5.1 in the case of valleys and for the situation of $z_0 = (x_0, h(x_0)) \in \Gamma_h \setminus (\Gamma_h^{cut} \cup \Gamma_h^{cusp})$ with $h(x_0) > 0$ (by using Step 1 of Proposition 5.1 for $\mathbb{C}_f = \mathbb{C}_s$) we also prove that $\Gamma_h^{reg} \cap \{y > 0\}$ is C^1 and hence, $\Gamma_h^{reg} \setminus Y_h \in C^1$. In view of this regularity we can implement the argument used in [14], which is based on the following decay estimate: For every parameter $0 < \sigma < 1$ there exist a constant $C > 0$ and a radius r_0 such that

$$\int_{B(z_0,r)\cup\Omega_h} |\nabla u|^2 \, dx \, dy \leq C r^{2\sigma}, \tag{6.1}$$

for all $z_0 \in \Gamma_h \setminus (\Gamma_h^{cut} \cup \Gamma_h^{cusp})$ and $0 < r < r_0$. In view of (6.1) it is possible to prove as in [14, Theorem 3.17] that

$$\mathcal{H}^1(\Gamma_h \cap B(z_0, r)) \leq C r^{2\sigma_0} \tag{6.2}$$

for $\sigma_0 \in (1/2, 1)$ and r small enough. We note that (6.2) follows by a perturbation argument which we can reproduce also in the dewetting regime. In fact by Theorem 2.4 the set $Z_h \setminus Y_h$ does not include island borders, and so the profile h is only perturbed in $\{y > 0\}$. The conclusion then follows from (6.2) by arguing as in the proof of Theorem 6.1 of [5] (see [5, Proposition 6.4]).

Assertion 5. follows as in [14, Theorem 3.19] by taking special care for the case $\mathbb{C}_f \neq \mathbb{C}_s$. In this case infact, when showing that h is a classical solution of the Euler Lagrange equation (2.12) in $\Gamma_h^{reg} \setminus Z_h$ it is not possible to extend the argument to $\Gamma_h^{reg} \setminus Y_h$ arguing by approximation. This difficulty is due to the presence of the transmission problem. □

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(Elisa Davoli) Department of Mathematics, University of Vienna, Oskar-Morgenstern Platz 1, 1090 Vienna (Austria)

E-mail address, E. Davoli: `elisa.davoli@univie.ac.at`

(Paolo Piovano) Department of Mathematics, University of Vienna, Oskar-Morgenstern Platz 1, 1090 Vienna (Austria)

E-mail address, P.Piovano: `paolo.piovano@univie.ac.at`