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To be discussed on March 11, 2019

Convex Optimization

Exercise session 1

1. Let U be a convex subset of a real normed space and $\alpha, \beta \in [0, +\infty)$. Show that
$$(\alpha + \beta)U = \alpha U + \beta U.$$

2. Let U be a convex subset of a real normed space. Show that the following statements are true:
   (i) $\operatorname{cl} U$ is convex;
   (ii) for every $x \in \operatorname{int} U$ and every $y \in \operatorname{cl} U$ it holds $\lambda x + (1 - \lambda)y \in \operatorname{int} U \ \forall \lambda \in (0, 1]$;
   (iii) $\operatorname{int} U$ is convex;
   (iv) if $\operatorname{int} U \neq \emptyset$, then $\operatorname{cl}(\operatorname{int} U) = \operatorname{cl} U$ and $\operatorname{int}(\operatorname{cl} U) = \operatorname{int} U$.

3. Let U be a convex subset of the real normed space X. The *algebraic interior* of the set U is defined as
$$\operatorname{core} U := \{x \in U : X = \cup_{\lambda > 0} \lambda(U - x)\}.$$
Show that: if $\operatorname{int}(U) \neq \emptyset$, then $\operatorname{int}(U) = \operatorname{core}(U)$.

4. Give an example of a convex subset of a real normed space with empty interior and non-empty algebraic interior.

5. Let X be a real normed space, $f : X \to \overline{\mathbb{R}}$ a convex function and $x_0 \in X$ such that $f(x_0) = -\infty$. Show that $f(x) = -\infty$ for every $x \in \operatorname{core}(\operatorname{dom} f)$.

6. Let X be a real normed space, $f : X \to \overline{\mathbb{R}}$ a given function and $x \in X$. Show that f is lower semicontinuous at x if and only if for every sequence $(x^k)_{k \geq 0}$ such that $x^k \to x$ as $k \to +\infty$ it holds $\liminf_{k \to +\infty} f(x^k) \geq f(x)$.