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Vienna, March 18, 2019
To be discussed on March 26, 2019

Convex Optimization

Exercise session 2

7. (a) Let X be a real normed space and $f : X \to \overline{\mathbb{R}}$ be a given function. Prove that f is convex if and only if for every $k \geq 1$, $x^1, ..., x^k \in X$, $\lambda_i \geq 0, i = 1, ..., k$, fulfilling $\sum_{i=1}^k \lambda_i = 1$, it holds
$$f\left(\sum_{i=1}^k \lambda_i x^i\right) \leq \sum_{i=1}^k \lambda_i f(x^i).$$

   (b) Prove that for every $k \geq 1$, $x_1, ..., x_k \in [0, +\infty)$, $\alpha_1, ..., \alpha_k \in (0, 1)$ fulfilling $\alpha_1 + ... + \alpha_k = 1$, it holds
$$\alpha_1 x_1 + ... + \alpha_k x_k \geq x_1^{\alpha_1} \cdots x_k^{\alpha_k}.$$

8. Let X be a real normed space, $f : X \to \mathbb{R}$ a convex function and $\lambda > \inf_{x \in X} f(x)$. Prove that
$$\operatorname{core}(\{x \in X : f(x) \leq \lambda\}) = \{x \in X : f(x) < \lambda\}.$$
Moreover, if f is continuous, then
$$\operatorname{int}(\{x \in X : f(x) \leq \lambda\}) = \{x \in X : f(x) < \lambda\}.$$

9. Let X be a real normed space and $U \subseteq X$ be a convex neighborhood of 0. The *Minkowski gauge* of U is defined by
$$\gamma_U : X \to \overline{\mathbb{R}}, \gamma_U(x) := \inf\{\lambda \geq 0 : x \in \lambda U\}.$$
Prove that γ_U is finite, sublinear and continuous and it holds
$$\operatorname{int} U = \{x \in X : \gamma_U(x) < 1\} \text{ and } \operatorname{cl} U = \{x \in X : \gamma_U(x) \leq 1\}.$$

10. Let X, Y be real normed spaces and $T : X \to Y$ be a linear continuous operator. Prove that the function
$$f : Y \to \overline{\mathbb{R}},\ f(y) := \inf\{\|x\| : Tx = y\},$$
is sublinear. Moreover, if T is an *open* operator (maps open sets of X to open sets of Y), then $\operatorname{dom} f = X$ and f is continuous.

11. Let U be a convex subset of a real normed space. Prove that U is weakly closed if and only if it is closed.

12. Let X be a real normed space and $f : X \to \overline{\mathbb{R}}$ be a proper and convex function. Prove that f is continuous on $\operatorname{int}(\operatorname{dom} f)$ if and only if $\operatorname{int}(\operatorname{epi} f)$ is nonempty.