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Convex Optimization

Exercise session 3

13. Let $(X, \|\cdot\|)$ be a real normed space.

    (a) Let $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$ be a proper and even function and $f : X \to \overline{\mathbb{R}}$, $f(x) = \varphi(\|x\|)$. Prove that $f^*(x^*) = \varphi^*(\|x^*\|_*)$ $\forall x^* \in X^*$.

    (b) Let be $f : X \to \mathbb{R}, f(x) = \frac{1}{p}\|x\|^p$ for $p \in (1, +\infty)$. Calculate f^*.

14. Let $(X, \|\cdot\|)$ be a real normed space. Calculate the convex subdifferential of the following functions:

    (a) $f : X \to \mathbb{R}, f(x) = \|x\|$;

    (b) $f : X \to \mathbb{R}, f(x) = \frac{1}{2}\|x\|^2$.

15. Let X be a real normed space and $f : X \to \overline{\mathbb{R}}$ be a given function. Prove that the following statements are true:

    (a) $\partial(\lambda f)(x) = \lambda \partial f(x)$ $\forall \lambda > 0$ $\forall x \in X$;

    (b) $\partial(f + x^*)(x) = \partial f(x) + x^*$ $\forall x^* \in X^*$ $\forall x \in X$;

    (c) if $g : X \to \overline{\mathbb{R}}, g(x) = f(x + z)$, for $z \in X$, then $\partial g(x) = \partial f(x + z)$.

16. Let U be a convex subset of a real normed space X and $x \in U$. The set

$$N_U(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \ \forall y \in U\}$$

    is called *normal cone* of U in x. Prove that the following statements are true:

    (a) $N_U(x)$ is a convex cone;

    (b) $N_U(x) = \partial \delta_U(x)$;

    (c) if $x \in \operatorname{core}(U)$, then $N_U(x) = \{0\}$.

    Hint. A set $K \subseteq X$ is called *cone*, if for every $\lambda \geq 0$ it holds $\lambda K \subseteq K$.

17. (a) Find a proper and convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and an element $x \in \operatorname{dom} f$ such that $f'(x; d) \in \mathbb{R}$ for every $d \in \mathbb{R}^n$, while f is not Gâteaux-differentiable at x.

    (b) Let be

$$f : \mathbb{R} \to \overline{\mathbb{R}}, f(x) = \begin{cases} -\sqrt{1 - x^2}, & \text{if } |x| \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

    Calculate $f'(-1; d)$ for every $d > 0$.

18. Let $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \max\{x_1, ..., x_n\}$.

   (a) Prove that f is a convex function.

   (b) Calculate $f'(x; d)$ for every $x, d \in \mathbb{R}^n$.

   (c) Calculate $\partial f(x)$ for every $x \in \mathbb{R}^n$.