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Convex Optimization

Exercise session 4

19. Let X and Y be real normed spaces.

    (a) Let $f : Y \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $A : X \to Y$ a linear and continuous operator such that $A^{-1}(\operatorname{dom} f) \neq \emptyset$. Prove that $(f \circ A)^* = \overline{A^* f^*}$.

    (b) Let $f_i : X \to \overline{\mathbb{R}}, i = 1, ..., m$, be proper, convex and lower semicontinuous functions such that $\cap_{i=1}^m \operatorname{dom} f_i \neq \emptyset$. Prove that $(f_1 + ... + f_m)^* = \overline{f_1^* \Box ... \Box f_m^*}$.

    Hint. Use the Fenchel-Moreau Theorem.

20. Let $(X, \|\cdot\|)$ be a real normed space, $U \subseteq X$ a nonempty, open and convex set and $f : U \to \mathbb{R}$ a Gâteaux differentiable function on U. Prove that the following statements are equivalent:

    (i) f is convex on U;

    (ii) $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) \ \forall x, y \in U$;

    (iii) $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0 \ \forall x, y \in U$;

    (iv) when f is twice Gâteaux differentiable function on U, $\nabla^2 f(x)(d, d) \geq 0 \ \forall x \in U \ \forall d \in X$.

21. Let X be a real normed space and, for $m \geq 2$, let $f_i : X \to \overline{\mathbb{R}}, i = 1, ..., m$, be proper and convex functions fulfilling the qualification condition:

    $$\exists x' \in \bigcap_{i=1}^m \operatorname{dom} f_i \text{ such that the functions } f_i, i = 1, .., m-1, \text{ are continuous at } x'.$$

    Prove that the following statements are true:

    (a) $(f_1 + ... + f_m)^*(x^*) = \min \left\{ \sum_{i=1}^m f_i^*(x^{i*}) : \sum_{i=1}^m x^{i*} = x^* \right\} \ \forall x^* \in X^*$;

    (b) $\partial(f_1 + ... + f_m)(x) = \partial f_1(x) + ... + \partial f_m(x) \ \forall x \in X$.

    Hint. Use Theorem 7.3 and Theorem 7.4 from the lecture.

22. Let be the proper and convex functions

    $$f : \mathbb{R} \to \overline{\mathbb{R}}, f(x) = \begin{cases} x(\ln x - 1), & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$g : \mathbb{R} \to \overline{\mathbb{R}}, g(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } x \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Investigate the existence of strong duality for

$$\inf_{x \in \mathbb{R}} \{f(x) + g(x)\}$$

and its Fenchel dual problem.

23. For $f : \mathbb{R}^2 \to \mathbb{R}, f(x_1, x_2) = x_1^2 + x_2^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2, g(x_1, x_2) = (x_1 + x_2 - 1, 1 - x_1 - x_2)^T$, consider the convex optimization problem:

$$\begin{array}{ll} \inf & f(x_1, x_2). \\ \text{s.t.} & (x_1, x_2) \in \mathbb{R}^2 \\ & g(x_1, x_2) \leqq_{\mathbb{R}^2_+} 0 \end{array}$$

   (a) Show that the weak Slater qualification condition (condition (QC_5^L) in the lecture) is fulfilled.

   (b) Find the optimal objective value of the primal optimization problem and an optimal solution of its Lagrange dual.

24. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Formulate the Lagrange duals of the following linear optimization problems:

   (a)
$$\begin{array}{ll} \inf & c^T x; \\ \text{s.t.} & x \in \mathbb{R}^n \\ & Ax \leqq_{\mathbb{R}^m_+} b \end{array}$$

   (b)
$$\begin{array}{ll} \inf & c^T x. \\ \text{s.t.} & x \in \mathbb{R}^n_+ \\ & Ax = b \end{array}$$

Formulate the weakest possible qualification conditions which guarantee strong duality for the resulting primal-dual pairs of optimization problems.