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To be discussed on May 21, 2019

Convex Optimization

Exercise session 5

25. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Prove that for every $x, y \in H$ and every $\alpha \in \mathbb{R}$

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

26. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ and $D \subseteq H$ a nonempty set. For $i = 1, ..., k$, let $T_i : D \to H$ be α_i-averaged operators, $\alpha_i \in (0,1)$ and $\omega_i \geq 0$ such that $\sum_{i=1}^k \omega_i = 1$. Show that $\sum_{i=1}^k \omega_i T_i$ is α-averaged, where $\alpha = \max_{i=1,\dots,k} \alpha_i$.

27. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ and $D \subseteq H$ a nonempty set. For $i = 1, ..., k$, let $T_i : D \to D$ be α_i-averaged operators, $\alpha_i \in (0,1)$. Show that $T = T_1 \dots T_k$ is α-averaged, where

$$\alpha = \frac{m}{m - 1 + \frac{1}{\max_{i=1,\dots,k} \alpha_i}}.$$

28. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$, $f : H \to \overline{\mathbb{R}}$ a proper function and $\beta > 0$. Show that the following statements are equivalent:

   (i) f is strongly convex with constant β;

   (ii) $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.

29. Let $(X, \|\cdot\|)$ be a real normed space, $U \subseteq X$ a nonempty, open and convex set, $f : U \to \mathbb{R}$ a Gâteaux differentiable function on U and consider the following statements:

   (i) f is strictly convex on U;

   (ii) $\langle \nabla f(x), y - x\rangle < f(y) - f(x)$ $\forall x, y \in U, x \neq y$;

   (iii) $\langle \nabla f(y) - \nabla f(x), y - x\rangle > 0$ $\forall x, y \in U, x \neq y$;

   (iv) when f is twice Gâteaux differentiable on U, $\nabla^2 f(x)(d,d) > 0$ $\forall x \in U$ $\forall d \in X, d \neq 0$.

   Prove that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftarrow (iv)$.

30. Prove that the function $x \mapsto \exp(x)$ defined on $\mathbb{R}$ is strictly convex and it is not strongly convex.