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Vienna, May 28, 2019
To be discussed on June 4, 2019

Convex Optimization

Exercise session 6

31. (a) For $\underline{\alpha}, \overline{\alpha}$ real numbers, such that $\underline{\alpha} < \overline{\alpha}$, calculate the proximal operator of

$$\sigma_{[\underline{\alpha},\overline{\alpha}]} : \mathbb{R} \to \mathbb{R},\ \sigma_{[\underline{\alpha},\overline{\alpha}]}(x) = \begin{cases} \underline{\alpha}x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \overline{\alpha}x, & \text{if } x \geq 0. \end{cases}$$

(b) Calculate the proximal operator of the function $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$.

32. For $\alpha > 0$, calculate the proximal operator of the following real-valued functions defined on $\mathbb{R}$:

(a) $x \mapsto \max\{|x| - \alpha, 0\}$;

(b)

$$x \mapsto \begin{cases} \frac{1}{2}x^2, & \text{if } |x| \leq \alpha \\ \alpha|x| - \frac{\alpha^2}{2}, & \text{otherwise.} \end{cases}$$

33. Let H be a real Hilbert spaces.

(a) For $f : H \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function and $\gamma > 0$, prove that

$$\operatorname{prox}_{\gamma({}^1 f)}(x) = \frac{x + \gamma \operatorname{prox}_{(\gamma+1)f}(x)}{\gamma + 1},$$

where

$${}^1 f : H \to \mathbb{R},\ {}^1 f(x) = \inf_{y \in H} \left\{ f(y) + \frac{1}{2}\|y - x\|^2 \right\},$$

denotes the Moreau envelope of f of parameter 1.

(b) For $C \subseteq H$ a nonempty, convex and closed set, calculate the proximal operator of the squared distance function $\frac{1}{2}d_C^2$.

34. Let H be a real Hilbert space.

(a) If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function and $u \in H$, determine the proximal operator of the function $x \mapsto f(\langle u, x \rangle)$.

(b) If H is a finite-dimensional space, $e_1, ..., e_n$ an orthonormal basis of H and $f : H \to \overline{\mathbb{R}}$, $f(x) = \sum_{i=1}^n f_i(\langle x, e_i \rangle)$, where $f_i : \mathbb{R} \to \overline{\mathbb{R}}, i = 1, ..., n$ are proper, convex and lower semicontinuous functions, determine the proximal operator of f.

35. Let H be a real Hilbert space, $f : H \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function and $\gamma > 0$.

   (a) Prove that $\operatorname{prox}_{(\gamma f)}(x) = x + \frac{1}{\gamma+1}\left(\operatorname{prox}_{(\gamma+1)f}(x) - x\right) \ \forall x \in H$.

   (b) For $g : H \to \mathbb{R}, g(x) = \frac{1}{2\gamma}\|x\|^2 - ({}^{\gamma}f)(x)$, prove that $g(x) = (f + \frac{1}{2\gamma}\|\cdot\|^2)^*\left(\frac{1}{\gamma}x\right) \ \forall x \in H$ and deduce from here that g is convex.

   (c) Show that $\operatorname{prox}_g(x) = x - \frac{1}{\gamma}\operatorname{prox}_{\frac{\gamma^2}{\gamma+1}f}\left(\frac{\gamma}{\gamma+1}x\right) \ \forall x \in H$.

   Hint. For proving (a) use Exercise 33.

36. Implement the proximal point algorithm with variable stepsizes in MATLAB. Apply the algorithm to minimize the convex function

$$f : \mathbb{R}^n \to \mathbb{R}, \ f(x) = \frac{\lambda}{2}\|x\|^2 + \|x\|_1,$$

   (i) by considering different values for the dimension n (for instance, $n = 1, 10, 100, 1000$) and for the starting point x^0;

   (ii) by considering different values for the parameter λ ($\lambda = 0, 1, 10, 100, 1000, 10000$);

   (iii) by using as stopping criterion $\|x^k - x^*\| \leq 10^{-3}$, where x^* denotes the unique minimizer of f;

   (iv) by using both constant stepsizes $\gamma_k = 1, \forall k \geq 0$, and variable stepsizes $\gamma_k = \frac{1}{k+1}, \forall k \geq 0$.

   Display the RMSE ($\frac{1}{\sqrt{n}}\|x^k - x^*\|, k = 0, 1, 2, ...$) and the objective function values ($f(x^k) - f(x^*), k = 0, 1, 2, ...$) as functions of the number of iterations k (in two separate plots).

   Send the generated files by June 11, 2019 at `radu.bot@univie.ac.at`.