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To be discussed on October 11, 2019

Nonlinear Optimization

Exercise session 1

1. Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $x_0 \in X$. Show that the following statements are true:

   (i) $T_X(x_0)$ is a nonempty closed cone;

   (ii) if X is a convex set, then $T_X(x_0) = \operatorname{cl}\left(\bigcup_{\lambda \geq 0} \lambda(X - x_0)\right)$ is a convex set;

   (iii) if X is a convex set, then $(T_X(x_0))^* = -N_X(x_0)$.

   (4 points)

2. Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $\operatorname{dist}_X : \mathbb{R}^n \to \mathbb{R}$, $\operatorname{dist}_X(y) = \inf\{\|y - x\| : x \in X\}$, the *distance function associated to* X. For a given function $f : \mathbb{R}^n \to \mathbb{R}$, let

   $$f'(x_0; d) = \lim_{t \downarrow 0} \frac{f(x_0 + td) - f(x_0)}{t}$$

   denote its *directional derivative* at $x_0 \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$. Prove that: if $X \subseteq \mathbb{R}^n$ is a nonempty and convex set and $x_0 \in X$, then

   $$T_X(x_0) = \{d \in \mathbb{R}^n : (\operatorname{dist}_X)'(x_0; d) = 0\}.$$

   (3 points)

3. Consider the general constrained optimization problem

   $$\begin{array}{ll} \min & f(x), \\ \text{s. t.} & g_i(x) \leq 0, i = 1, ..., m \\ & h_j(x) = 0, j = 1, ..., p \\ & x \in \mathbb{R}^n \end{array}$$

   where $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, $j = 1, ..., p$, are continuously differentiable functions. Let x_0 be a feasible element of this problem and

   $$\begin{array}{ll} X_{\text{lin}} := \{x \in \mathbb{R}^n : & g_i(x_0) + \nabla g_i(x_0)^T(x - x_0) \leq 0, i = 1, ..., m \\ & h_j(x_0) + \nabla h_j(x_0)^T(x - x_0) = 0, j = 1, ..., p\}. \end{array}$$

   Prove that $T_{\text{lin}}(x_0) = T_{X_{\text{lin}}}(x_0)$.

   Hint. Show that $T_{\text{lin}}(x_0) = \bigcup_{\lambda \geq 0} \lambda(X_{\text{lin}} - x_0)$ and use Exercise 1(ii). (3 points)

4. Let be

$$g : \mathbb{R}^2 \to \mathbb{R}^4, g(x,y) = (\pi - 2x, -y - 1, 2x - 3\pi, y - \sin(x))^T$$

and

$$X = \left\{(x,y) \in \mathbb{R}^2 : g(x,y) \leqq 0\right\}.$$

   (i) Represent the set X graphically.
   (ii) Find the tangent cones $T_X(x_i)$, $i = 1,2,3$, where $x_1 = (\frac{\pi}{2}, 1)^T$, $x_2 = (\pi, 0)^T$ and $x_3 = (\frac{3\pi}{2}, -1)^T$, and represent these sets graphically.
   (iii) Find the linearized tangent cones $T_{\text{lin}}(x_i)$ for $i \in \{1,2,3\}$.
   (iv) Find $i \in \{1,2,3\}$ for which $T_X(x_i) = T_{\text{lin}}(x_i)$.

(4 points)

5. Let be $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Prove by using the strong duality theorem of linear optimization that the following statements are equivalent:

   (i) The system $Ax = b$ has a solution $x \geqq 0$.
   (ii) It holds $b^T d \geq 0$ for every $d \in \mathbb{R}^m$ with $A^T d \geqq 0$.

(3 points)

6. For the functions $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $x_0 \in \mathbb{R}^2$ given below, find $X = \{(x,y) \in \mathbb{R}^2 : g(x,y) \leqq 0\}$, the tangent cone and the linearized tangent cone to X at x_0 and find out if x_0 fulfills (ABADIE-CQ):

   (i) $g(x,y) = (y - x^3, -y)^T, x_0 = (0,0)^T$;
   (ii) $g(x,y) = (y^2 - x + 1, 1 - x - y)^T, x_0 = (1,0)^T$.

(3 points)