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To be discussed on October 25, 2019

Nonlinear Optimization

Exercise session 2

7. Let (x^*, λ^*, μ^*) be a KKT point of the optimization problem

$$(P) \qquad \begin{array}{ll} \min & f(x), \\ \text{s.t.} & g_i(x) \leq 0, i = 1, ..., m \\ & h_j(x) = 0, j = 1, ..., p, \\ & x \in \mathbb{R}^n \end{array}$$

with $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, $j = 1, ..., p$, continuously differentiable functions. Prove that x^* is a *critical point* of (P), namely, it holds

$$\nabla f(x^*)^T d \geq 0 \ \forall d \in T_X(x^*),$$

where $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, ..., m, h_j(x) = 0, j = 1, ..., p\}$. Given a critical point x^* of (P), when do Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ exist, such that (x^*, λ^*, μ^*) is a KKT point of (P)? (3 points)

8. Consider the optimization problem

$$\begin{array}{ll} \min & x_1^2 + (x_2 + 1)^2. \\ \text{s.t.} & x_2 - x_1^2 \leq 0 \\ & -x_2 \leq 0 \end{array}$$

Show that $x^* = (0, 0)^T$ fulfills (ABADIE-CQ) and that it does not fulfill (MFCQ). (2 points)

9. Consider the optimization problem

$$\begin{array}{ll} \min & x_1^2 + (x_2 + 1)^2. \\ \text{s.t.} & -x_1^3 - x_2 \leq 0 \\ & -x_2 \leq 0 \end{array}$$

Show that $x^* = (0, 0)^T$ fulfills (MFCQ) and that it does not fulfill (LICQ). (2 points)

10. Let $X \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : \mathbb{R}^n \to \mathbb{R}$ a convex function. Show that every local minimum of f with respect to X is a global minimum of f with respect to X. (2 points)

11. Let $U \subseteq \mathbb{R}^n$ be a nonempty, open and convex set and $f : U \to \mathbb{R}$ a differentiable function on U. Prove that the following statements are equivalent:

    (i) f is convex on U;

(ii) $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) \; \forall x, y \in U$;

(iii) $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0 \; \forall x, y \in U$;

(iv) if f is twice differentiable on U, then $\nabla^2 f(x)$ is positively semidefinite for every $x \in U$.

(4 points)

12. Let be the functions $c : \mathbb{R} \to \mathbb{R}$,

$$c(y) = \begin{cases} (y+1)^2, & y < -1, \\ 0, & -1 \leq y \leq 1, \\ (y-1)^2, & y > 1, \end{cases}$$

and $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$, $g_1(x_1, x_2) = c(x_1) - x_2$, $g_2(x_1, x_2) = c(x_1) + x_2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex and continuously differentiable function. Show that for the convex optimization problem

$$\begin{array}{rl} \min & f(x) \\ \text{s.t.} & x \in \mathbb{R}^2 \\ & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \end{array}$$

(ABADIE-CQ) holds at $x^* = (0,0)^T$, while (SLATER-CQ) is not satisfied. (2 points)