Lecture Notes for

Mathematical Finance in discrete time

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Contents

Preface

The present lecture notes are based on the following literature.

- F. Delbaen and W. Schachermayer. *The mathematics of arbitrage*. Springer Finance. Springer-Verlag, Berlin, 2006.
- H. Föllmer and A. Schied. *Stochastic finance*. Walter de Gruyter & Co., Berlin, extended edition, 2011. An introduction in discrete time.
- S. E. Shreve. *Stochastic calculus for finance. I*. Springer Finance. Springer-Verlag, New York, 2004. The binomial asset pricing model.

Throughout we consider models of financial markets in *discrete time*, i.e., trading is only allowed at discrete time points $0 = t_0 < t_1 < \cdots < t_N = T$. Here, $T > 0$ denotes a finite time horizon. This is in contrast to models in continuous time, where continuous trading during the interval $[0, T]$ is possible.

The following topics of mathematical finance will be covered:

- arbitrage theory;
- completeness of financial markets;
- superhedging;
- pricing of derivatives (European and American options);
- concrete modeling of financial markets via the Binomial asset price model and (its convergence to) the Black Scholes model.

From a mathematical point of view, probability theory and stochastic analysis play a key role in mathematical finance.

Chapter 1

Basic notions from probability theory

We recall here basic notions from probability theory which we will need for modeling financial markets.

1.1 Filtered probability spaces, random variables and stochastic processes

Let us start by recalling the ingredients of a probability space. A probability space consists of three parts:

- a non-empty set Ω (Ergebnismenge), which is the set of possible outcomes;
- a σ-algebra $\mathcal{F}$, i.e., a set consisting of sets of Ω to model all possible events (Ereignisse) (where an event is a set containing zero or more outcomes);
- a probability measure P assigning probabilities to each event.

The precise mathematical definition of these notions are as follows:

Definition 1.1.1. *A set $\mathcal{F} \subset \mathcal{P}(\Omega)$ is called* σ-algebra *if it satisfies*

- $\Omega \in \mathcal{F}$*;*
- $A \in \mathcal{F} \Rightarrow A^c = \Omega \setminus A \in \mathcal{F}$*;*
- $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} \in \mathcal{F}$*.*

The above definition implies that a σ-algebra is closed under countable intersections.

Definition 1.1.2. *Let $(\Omega, \mathcal{F})$ be a measurable space, i.e. $\mathcal{F}$ is σ-algebra on Ω. Then a probability measure is a function $P : \mathcal{F} \to [0, 1]$ such that*

- *$P[\Omega] = 1$;*
- *it is σ-additive, i.e. for any sequence of pairwise disjoint sets in $\mathcal{F}$ (i.e., $A_n \cup A_m = \emptyset$ for $n \neq m$), we have $P[\bigcup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} P[A_n]$.*

Definition 1.1.3.
- *Two probability measures P, Q are called equivalent, which is denoted by $P \sim Q$ if*
$$P[A] = 0 \Leftrightarrow Q[A] = 0, \quad A \in \mathcal{F}.$$
- *Q is absolutely continuous with respect to P, which is denoted by $Q \ll P$ if*
$$P[A] = 0 \Rightarrow Q[A] = 0, \quad A \in \mathcal{F}.$$

Remark 1.1.4.
- From the above definition, we immediately get
$$Q \sim P \Leftrightarrow P \ll Q, Q \ll P.$$
and
$$Q \ll P \Leftrightarrow Q[A] > 0 \Rightarrow P[A] > 0.$$
- In the case when Ω consists of finitely many elements and $P[\{\omega\}] > 0$ for every ω, then for every probability measure Q we have $Q \ll P$. Equivalence means $Q[\{\omega\}] > 0$ for every ω.

Let us recall the notion of an atom:

Definition 1.1.5. *Given a probability space $(\Omega, \mathcal{F}, P)$, then a set A is called atom if $P[A] > 0$ and for any measurable subset $B \subset A$ with $P[B] < P[A]$ we have $P[B] = 0$. In the case of a finite probability space where only the empty set has probability zero, we have the following equivalent definition a set A is called atom if $P[A] > 0$ and for any measurable subset $B \subset A$ with $P[B] < P[A]$ we have $B = \emptyset$.*

Example 1.1.6. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$. Consider a probability measure P which satisfies $P[\omega_i] > 0$. Then the atoms are $\{\omega_i\}$, $i \in \{1, \ldots, 4\}$. If the σ-Algebra is given by $\mathcal{F} = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_3\}\}$, then the atoms are $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$.

Definition 1.1.7. *A family of σ-algebras with $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_T$ is called* filtration *and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{\{t \in [0,\ldots,T]\}}, P)$ filtered probability space.*

Remark 1.1.8. $\mathcal{F}_t$ is interpreted as the set of all events which can happen up to time t or equivalently as the information which is available up to time t.

Assumption. *Unless explicitly mentioned, we shall assume that $\mathcal{F}_T = \mathcal{F}$. We do not assume $\mathcal{F}_0$ to be necessarily the trivial σ-algebra $(\emptyset, \Omega)$, although in many applications this is the case.*

For modeling asset prices we consider *stochastic processes* which are families of *random variables*, whose definition we recall subsequently.

Definition 1.1.9. *Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be two measurable spaces. A* random variable *X with values in E is a $(\mathcal{F}$-$\mathcal{E})$-measurable function $X : \Omega \to E$, i.e. the preimage of any measurable set $B \in \mathcal{E}$ is in $\mathcal{F}$: $\forall B \in \mathcal{E}$, we have $X^{-1}(B) \in \mathcal{F}$.*

In our setting $(E, \mathcal{E})$ is typically $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra, defined as the smallest σ-algebra containing the open sets of $\mathbb{R}^n$.

Remark 1.1.10. In the case $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\mathcal{F}$-$\mathcal{B}(\mathbb{R}))$-measurability (or simply $\mathcal{F}$-measurability) is equivalent to

$$\forall a \in \mathbb{R} : \{\omega \in \Omega : X(\omega) \in (-\infty, a]\} \in \mathcal{F}.$$

Definition 1.1.11. *Let Ω be some set and $(E, \mathcal{E})$ be a measurable spaces. Consider a function $X : \Omega \to E$. Then the σ-algebra generated by X, denoted by $\sigma(X)$, is the collection of all inverse images $X^{-1}(B)$ of the sets B in $\mathcal{E}$, i.e.,*

$$\sigma(X) = \{X^{-1}(B) \,|\, B \in \mathcal{E}\}.$$

Definition 1.1.12. *Let $\mathcal{T}$ be an index set, either $\{0, 1, \ldots, T\}$ or $\{1, \ldots, T\}$, and $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ two measurable spaces. A stochastic process with values in $(E, \mathcal{E})$ is a family of random variables $X = (X_t)_{t \in \mathcal{T}} = \{X_t \,|\, t \in \mathcal{T}\}$ (i.e. $\mathcal{F}$-measurable).*

Definition 1.1.13. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \{0,1\ldots,T\}}, P)$ a filtered probability space.*

1. *A stochastic process X is called* adapted *with respect to the filtration $(\mathcal{F}_t)$ if for every $t \in \{0, 1, \ldots, T\}$, X_t is $\mathcal{F}_t$-measurable.*

2. *A stochastic process Y is called* predictable *with respect to the filtration $(\mathcal{F}_t)$ if for every $t \in \{1, \ldots, T\}$, Y_t is $\mathcal{F}_{t-1}$-measurable.*

Example 1.1.14. Let $T = 2$, $\Omega = \{1, 2, 3, 4\}$ and $E = \mathbb{R}$. Consider the following filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}\}$ and $\mathcal{F}_2 = \mathcal{P}(\Omega)$. Question: How do adapted stochastic processes look like? Answer: For $t = 0$, a ($\mathcal{F}_0$-measurable) random variable is constant, for $t = 1$ a ($\mathcal{F}_1$-measurable) random variable is piece-wise constant (constant on $\{1, 2\}$ and $\{3, 4\}$) and for $t = 2$ all functions are ($\mathcal{F}_2$-measurable) random variables.

1.2 L^p spaces

Let us now pass to $\mathcal{L}^p$ spaces which are spaces of random variables whose p^{th} power is integrable.

Definition 1.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. For random variables $X : \Omega \to \mathbb{R}$ we define*

$$\|X\|_p := \left(\int_\Omega |X(\omega)|^p dP(\omega)\right)^{\frac{1}{p}} = E\left[|X|^p\right]^{\frac{1}{p}}, \quad \text{if } p \in [1, \infty)$$

and for $p = \infty$

$$\|X\|_\infty := \inf\{K \geq 0 : P[\{|X| > K\}] = 0\}.$$

For every $p \in [1, \infty]$, $\mathcal{L}^p(\Omega, \mathcal{F}, P)$ is the vector space for which the above expressions are finite, i.e.,

$$\mathcal{L}^p(\Omega, \mathcal{F}, P) := \{X : \Omega \to \mathbb{R} \text{ is } \mathcal{F}\text{-measurable and } \|X\|_p < \infty\}.$$

This definition implies that $\|\cdot\|$ is a semi-norm, i.e., for all $X, Y \in \mathcal{L}^p(\Omega, \mathcal{F}, P)$ und $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} \|X\|_p \geq 0 \text{ for all } X \text{ and } \|X\|_p = 0, &\text{ if } X = 0 \ P\text{-a.s.}, \\ \|\alpha X\|_p = \alpha \|X\|_p&, \\ \|X + Y\|_p \leq \|X\|_p + \|Y\|_p&. \end{aligned}$$

In other words all properties of a norm are satisfied except that $\|X\|_p = 0 \Rightarrow X = 0$. Indeed we have

$$\|X\|_p = 0 \Leftrightarrow X = 0 \ P\text{-a.s.}$$

In order to make $\|\cdot\|_p$ to be a true norm, we define

$$\mathcal{N} = \{X \text{ is } \mathcal{F}\text{-measurable and } X = 0 \ P\text{-a.s.}\}.$$

For every $p \in [1, \infty]$, $\mathcal{N}$ is a subvector space of $\mathcal{L}^p(\Omega, \mathcal{F}, P)$. We can thus build the quotient space via the equivalence relation $X \sim Y$, if $X = Y$ P-a.s.

Definition 1.2.2. *For $p \in [1, \infty]$, the vector space $L^p(\Omega, \mathcal{F}, P)$ is defined as the quotient space*

$$L^p(\Omega, \mathcal{F}, P) = \mathcal{L}^p(\Omega, \mathcal{F}, P)/\mathcal{N} = \{[X] := X + \mathcal{N} \mid X \in \mathcal{L}^p(\Omega, \mathcal{F}, P)\}.$$

For $[X] \in L^p(\Omega, \mathcal{F}, P)$ we set $\|[X]\|_p = \|X\|_p$ and $\int [X] dP = \int X dP$ with $X \in [X]$.

On $L^p(\Omega, \mathcal{F}, P)$, $\|\cdot\|_p$ is a true norm. Moreover it is complete with respect to this norm, i.e. every Cauchy-sequence converges. Such a space is called *Banach space*.

For $p = 0$, $L^0(\Omega, \mathcal{F}, P)$ denotes the vector space of equivalence classes of random variables, i.e.,

$$L^0(\Omega, \mathcal{F}, P) = \{[X] := X + \mathcal{N} \,|\, X : \Omega \to \mathbb{R}, X \text{ is } \mathcal{F} - \text{measurable}\}.$$

For notational convenience we usually omit the brackets $[\cdot]$ when we talk about elements in $L^p(\Omega, \mathcal{F}, P)$.

1.2.1 The case of finite Ω

In the case where Ω consists only of finitely many elements, i.e.

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$

for some $N \in \mathbb{N}$ and a probability measure P such that

$$P[\omega_n] = p_n \geq 0, \text{ for } n = \{1, \ldots, N\},$$

the above notions simplify as follows. A general random variable $X : \Omega \to \mathbb{R}$ corresponds to a vector in $\mathbb{R}^N$

$$X = (X(\omega_1), \ldots, X(\omega_N))^\top =: (x_1, \ldots, x_N)^\top,$$

where x_n is the evaluation of X at ω_n. Two random variables X and Y are equivalent, if $x_n = y_n$ for all n for which $p_n > 0$. This means we identify random variables whose j^{th} component is different, if $p_j = 0$ (in the case of finite Ω it is also possible to remove those elements ω_j which have probability 0.) For $p \in [1, \infty)$, $L^p(\Omega, \mathcal{F}, P)$ are now equivalence classes of vectors with the following norm

$$\|X\|_p = \left(\sum_{n=1}^N |X(\omega_n)|^p P[\omega_n] \right)^{\frac{1}{p}} = \left(\sum_{n=1}^N |x_n|^p p_n \right)^{\frac{1}{p}} = E[|X|^p]^{\frac{1}{p}}$$

and in case of $p = \infty$ the norm is given by

$$\|X\|_\infty = \max_{n \in \{1, \ldots, N\}} \{X(\omega_n) \,|\, P[\omega_n] > 0\} = \max_{n \in \{1, \ldots, N\}} \{x_n \,|\, p_n > 0\}.$$

Since all these norms are always finite in the case of finite Ω, it follows that for every $p \in [1, \infty]$, $L^p(\Omega, \mathcal{F}, P)$ contains the same random variables, i.e. vectors in $\mathbb{R}^N$. If we do not specify a specific norm on these space we thus simply write

$$L(\Omega, \mathcal{F}, P) = \{[X] := X + \mathcal{N} \,|\, X : \Omega \to \mathbb{R}, X \text{ is } \mathcal{F}\text{-measurable}\} \tag{1.1}$$

for the space of equivalence classes of $\mathcal{F}$-measurable random variables, which also corresponds to $L^0(\Omega, \mathcal{F}, P)$ in the above notation. If $\mathcal{F} = \mathcal{P}(\Omega)$, $L(\Omega, \mathcal{F}, P)$ can be identified with $\mathbb{R}^N$.

1.3 The conditional expectation in the case of finite Ω

Let $(\Omega, \mathcal{F}, P)$ be a probability space as above where Ω only consists of finitely many elements, i.e.

$$\Omega = \{\omega_1, \dots, \omega_N\}$$

for some $N \in \mathbb{N}$ and a probability measure P such that

$$P[\omega_n] = p_n \geq 0, \text{ for } n = \{1, \dots, N\}.$$

As above let $L(\Omega, \mathcal{F}, P)$ denote the space of equivalence classes of $\mathcal{F}$-measurable random variables.

Let $B \in \mathcal{F}$ be an event with $P[B] > 0$, then the conditional probability $P[A \,|\, B] = P[A \cap B]/P[B]$ is a measure for the probability of event A given event B. Accordingly the conditional expectation

$$E[X \,|\, B] = \frac{E[1_B X]}{P[B]} \tag{1.2}$$

is a measure for the mean of the random variable X given the information concerning the occurrence of B. This elementary notion of conditional expectations is however not always sufficient. Indeed, we are more interested in conditional expectations of the form

$$E[X \,|\, \mathcal{G}],$$

i.e. in case where we have information concerning the occurrence of a set of events (a σ-algebra) $\mathcal{G} \subset \mathcal{F}$. In contrast to (1.2) this expression is again a random variable. As we will see $E[X \,|\, B]$ for $B \in \mathcal{G} \subset \mathcal{F}$ is the evaluation of the random variable $E[X \,|\, \mathcal{G}](\omega)$ for $\omega \in B$.

Definition 1.3.1. *Für $X, Y \in L(\Omega, \mathcal{F}, P)$ (for general Ω this would be $L^2(\Omega, \mathcal{F}, P)$) we define the scalar product*

$$\langle X, Y \rangle := \sum_{n=1}^{N} X(\omega_n) Y(\omega_n) p(\omega_n) = \sum_{n=1}^{K} x_n y_n p_n = E[XY].$$

The induced norm $\|\cdot\|_2$ corresponds to $\|x\| = \sqrt{\langle x, x \rangle}$, whence $(L(\Omega, \mathcal{F}, P), \langle \cdot, \cdot \rangle)$ is a finite dimensional Hilbert space.

Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$, then $L(\Omega, \mathcal{G}, P)$ is a linear subspace of $L(\Omega, \mathcal{F}, P)$. The conditional expectation is the random variable $Y \in L(\Omega, \mathcal{G}, P)$, which has the shortest distance to $X \in L(\Omega, \mathcal{F}, P)$, i.e. Y is the solution to the following minimizing problem:

$$\begin{aligned} \|X - Y\|_2 &= E\left[|X - Y|^2\right]^{\frac{1}{2}} \\ &= \min\left\{ E\left[|X - Z|^2\right]^{\frac{1}{2}} = \|X - Z\|_2 \,|\, Z \in L(\Omega, \mathcal{G}, P) \right\}. \end{aligned}$$

This minimizing problem has a unique solution, namely the orthogonal projection of X on $L(\Omega, \mathcal{G}, P)$. Thus $X - Y$ is orthogonal to all $Z \in L^2(\Omega, \mathcal{G}, P)$, i.e.,

$$\langle X - Y, Z \rangle = E[(X - Y)Z] = 0.$$

In other words, Y satisfies

$$E[XZ] = E[YZ]$$

for all $Z \in L(\Omega, \mathcal{G}, P)$. The following definition of the conditional expectation thus makes sense:

Definition 1.3.2. *Let $X \in L(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-Algebra. Then we call the orthogonal projection on $L(\Omega, \mathcal{G}, P)$ the conditional expectation of X given $\mathcal{G}$. We write $E[X \,|\, \mathcal{G}]$. In other words, $E[X \,|\, \mathcal{G}]$ is the unique element in $L(\Omega, \mathcal{G}, P)$, such that*

$$E[XZ] = E[E[X \,|\, \mathcal{G}]Z] \tag{1.3}$$

holds for all $Z \in L(\Omega, \mathcal{G}, P)$.

Since Ω is finite dimensional, we get a more explicit expression for the conditional expectation. Indeed, for every σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ there exists a partition $(B_i)_{i \in I}$ of Ω, i.e., a decomposition of Ω in disjoint, non-empty sets, where I denotes a finite index set. The functions

$$\frac{1_{B_i}}{\|1_{B_i}\|} = \frac{1_{B_i}}{\sqrt{\langle 1_{B_i}, 1_{B_i} \rangle}} = \frac{1_{B_i}}{\sqrt{P[B_i]}}$$

form an orthonormal basis of $L(\Omega, \mathcal{G}, P)$ and the orthogonal projection of a random variable $X \in L(\Omega, \mathcal{F}, P)$ on $L(\Omega, \mathcal{G}, P)$ is thus given by

$$E[X \,|\, \mathcal{G}] = \sum_{i \in I, P(B_i) > 0} \left\langle X, \frac{1_{B_i}}{\sqrt{P[B_i]}} \right\rangle \frac{1_{B_i}}{\sqrt{P[B_i]}}. = \sum_{i \in I, P(B_i) > 0} E[X 1_{B_i}] \frac{1_{B_i}}{P[B_i]}. \tag{1.4}$$

For all $\omega \in B_i$ and $A \in \mathcal{F}$ the value of $E[1_A \,|\, \mathcal{G}](\omega) = P[A \,|\, \mathcal{G}](\omega)$ is given by

$$P[A \,|\, \mathcal{G}](\omega) = E[1_A \,|\, \mathcal{G}](\omega) = \frac{E[1_A 1_{B_i}]}{P[B_i]} = \frac{P[A \cap B_i]}{P[B_i]}, \quad \omega \in B_i$$

and on events B_i with $P(B_i) > 0$ this corresponds to the definition of the conditional probability.

Example 1.3.3. Let $N = 3$ such that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and

$$\mathcal{G} = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2, \omega_3\}\}.$$

Consider the uniform distribution, i.e., $P(\omega_i) = \frac{1}{3}$ for all $i = 1, 2, 3$ and the random variable $X : \Omega \to \mathbb{R}, \omega_i \mapsto X(\omega_i) = i$. Then inserting in (1.4) yields

$$E[X|\mathcal{G}](\omega_1) = E[X 1_{\omega_1}] \frac{1_{\omega_1}}{P[\omega_1]} = 1 \cdot \frac{1}{3} \cdot 13 = 1,$$

$$E[X|\mathcal{G}](\omega_i) = E[X 1_{\{\omega_2, \omega_3\}}] \frac{1_{\omega_2, \omega_3}}{P[\{\omega_2, \omega_3\}]} = (2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3}) \cdot \frac{3}{2} = \frac{5}{2}, \quad i = 2, 3.$$

Subsequently we state some important notions of the conditional expectation. For this purpose recall the notion of *independence* of a random variable X and a σ-algebra $\mathcal{G}$, which means that the σ-algebra generated by X, denoted by $\sigma(X)$ is independent of $\mathcal{G}$. Two σ-algebras $\mathcal{G}, \mathcal{H}$ are independent if for all events $A \in \mathcal{G}$ and $B \in \mathcal{H}$, $P(A \cap B) = P(A)P(B)$.

Proposition 1.3.4. *Let $X \in L(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then we have:*

i) *The map $X \mapsto E[X \,|\, \mathcal{G}]$ is linear.*

ii) *If $X \geq 0$, then $E[X \,|\, \mathcal{G}] \geq 0$.*

iii) $E[E[X \,|\, \mathcal{G}]] = E[X]$.

iv) *Let $\mathcal{H} \subset \mathcal{G}$ be a sub-σ-algebra of $\mathcal{G}$. Then $E[X \,|\, \mathcal{H}] = E[E[X \,|\, \mathcal{G}] \,|\, \mathcal{H}]$.*

v) *If X is independent of $\mathcal{G}$, then $E[X \,|\, \mathcal{G}] = E[X]$*

vi) *If $Y \in L(\Omega, \mathcal{G}, P)$, then $E[XY \,|\, \mathcal{G}] = Y E[X \,|\, \mathcal{G}]$.*

vii) *Let $Y \in L(\Omega, \mathcal{G}, P)$ and X be independent of $\mathcal{G}$. Then we have for all measurable functions $f : \mathbb{R} \to \mathbb{R}$*

$$E[f(X+Y) \,|\, \mathcal{G}](\omega) = E[f(X+Y) \,|\, \sigma(Y)](\omega),$$

where $\sigma(Y)$ is the σ-algebra generated by Y. For the evaluation at $Y(\omega) = y$ we have

$$E[f(X+Y) \,|\, Y = y] = E[f(X+y)].$$

1.4 Martingales

The main reason for introducing the concept of the conditional expectation is to define the notion of a *martingale*, which will play a particular role for asset prices in financial markets.

Definition 1.4.1. *An adapted process $(X)_{t \in \{1,\ldots,T\}}$ is called martingale, if for all $t \in \{1, \ldots, T\}$*

$$E[X_t | \mathcal{F}_{t-1}] = X_{t-1}.$$

Similarly we have the notion of a super- and sub-martingale defined below:

Definition 1.4.2.

- *An adapted process $(X)_{t \in \{1,\ldots,T\}}$ is called* supermartingale, *if for all $t \in \{1, \ldots, T\}$,*

$$E[X_t | \mathcal{F}_{t-1}] \leq X_{t-1}.$$

- *An adapted process $(X)_{t \in \{1,\ldots,T\}}$ is called* submartingale, *if for all $t \in \{1, \ldots, T\}$,*

$$E[X_t | \mathcal{F}_{t-1}] \geq X_{t-1}.$$

Let us now state the Doob-decomposition of an adapted process which plays a crucial role in the representation of a supermartingale.

Theorem 1.4.3 (Doob-Decomposition)**.** *Let X be an adapted process defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Then there exists a unique decomposition*

$$X = M - A, \tag{1.5}$$

where M is a martingale and A is a process such that $A_0 = 0$ and $(A_t)_{t=1,\ldots,T}$ is predictable. The decomposition (1.5) *is called the Doob decomposition of X.*

Proof. Define

$$A_t - A_{t-1} = -E[X_t - X_{t-1}|\mathcal{F}_{t-1}], \quad t = 1, \ldots, T.$$

Then A is predictable, i.e. A_t is $\mathcal{F}_{t-1}$ measurable (by definition of the conditional expectation) and $M_t := X_t + A_t$ is a martingale. Indeed

$$E[M_t|\mathcal{F}_{t-1}] = E[X_t - E[X_t - X_{t-1}|\mathcal{F}_{t-1}] + A_{t-1}|\mathcal{F}_{t-1}] = X_{t-1} + A_{t-1} = M_{t-1}.$$

Concerning uniqueness, suppose that there are two representations of X, i.e.,

$$X_t = M_t - A_t = M'_t - A'_t,$$

from which we get $A_t - A'_t = M_t - M'_t$. Taking conditional expectations it follows that

$$A_t - A'_t = M_{t-1} - M'_{t-1}$$

and by setting $t = 1$ we have $A_1 - A'_1 = M_0 - M'_0 = X_0 - X_0 = 0$. Hence $M_1 = X_1 + A_1 = X_1 + A'_1 = M'_1$. Uniqueness then follows by induction. □

Proposition 1.4.4. *Let X be an adapted process. Then the following assertions are equivalent.*

1. *X is a supermartingale.*
2. *The predictable process A in the Doob decomposition is increasing.*

An analogous statement holds for submartingales.

Proof. Let X be a supermartingale. Then by definition of the process A in the Doob decomposition and the supermartingale property we have

$$A_t - A_{t-1} = -E[X_t - X_{t-1}|\mathcal{F}_{t-1}] \geq 0,$$

which implies that A is increasing. Conversely, suppose that A is increasing. Then again by the definition of A we obtain

$$0 \leq A_t - A_{t-1} = -E[X_t - X_{t-1}|\mathcal{F}_{t-1}],$$

from which we obtain the supermartingale property

$$E[X_t|\mathcal{F}_{t-1}] \leq X_{t-1}.$$

For submartingales the proof works analogously. □

Chapter 2

Models of financial markets on finite probability spaces

We consider a financial market with $1 \leq T \in \mathbb{N}$ periods and $d+1$ financial instruments. More precisely, the modeling framework consists of

- discrete trading times $t = 0, 1, \dots, T$;
- $d+1$ financial instruments (often a riskless bank account and d risky assets), whose modeling requires a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t \in \{0,1,\dots,T\}}$ and the notion of stochastic processes as introduced in the previous chapter.

2.1 Description of the model

This section is mainly based on [1, Chapter 2].

Adapted stochastic processes are used to model asset price processes. The idea is that $\mathcal{F}_t$ represents the information up to time t and the asset price is measurable with respect to $\mathcal{F}_t$, i.e., its value can be inferred from the knowledge of $\mathcal{F}_t$.

Definition 2.1.1. *A multi-period model of a financial market in discrete time $t \in \{0, 1, \dots, T\}$, $T \in \mathbb{N}$, consists of an $\mathbb{R}^{d+1}$-valued adapted stochastic process $\widehat{S} = (\widehat{S}^0, \widehat{S}^1, \dots, \widehat{S}^d)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where*

- *$\widehat{S}^0$ is the so-called numéraire asset used as denomination basis, which is supposed to be strictly positive, i.e. $\widehat{S}^0_t > 0$ for all $t \in \{0, 1, \dots, T\}$;*
- *$(\widehat{S}^1, \dots, \widehat{S}^d)$ are $\mathbb{R}^d$-valued adapted stochastic processes for the risky assets.*

The interpretation is as follows: The prices of the assets $0, \dots, d$ are measured in a fixed money unit, say Euro. The 0^{th} asset plays a special role, it is supposed to be strictly positive and will be used as numéraire. It allows to compare money (Euros) at time 0 to money at time $t > 0$. In many elementary models, $\widehat{S}^0$ is simply the bank account, which is in case of constant interest rates given by $\widehat{S}^0_t = (1+r)^t$.

Definition 2.1.2.

- *A trading strategy for the d risky assets $(\widehat{S}^1, \dots, \widehat{S}^d)$ is an $\mathbb{R}^d$-valued predictable process $H_t = (H^1_t, \dots, H^d_t)_{t\in\{1,\dots,T\}}$. The set of all such trading strategies is denoted by $\mathcal{H}$. (In other words $\mathcal{H}$ corresponds to all $\mathbb{R}^d$-valued predictable processes.)*
- *Similarly, a trading strategy for the $d+1$ assets $(\widehat{S}^0, \dots, \widehat{S}^d)$ is an $\mathbb{R}^{d+1}$-valued* predictable *process, which we denote as follows*

$$(\widehat{H}_t)_{t\in\{1,\dots,T\}} = (H^0_t, H^1_t, \dots, H^d_t)_{t\in\{1,\dots,T\}} = (H^0_t, H_t)_{t\in\{1,\dots,T\}}.$$

Remark 2.1.3. The component H^i_t corresponds to the number of shares invested in asset i from period $t-1$ up to t. This means $H^i_t S^i_{t-1}$ is the invested amount at time $t-1$ and $H^i_t S^i_t$ is the resulting wealth at time t. Predictability of $\widehat{H}$ means that an investment can only be made without knowledge of future asset price movements.

Definition 2.1.4. *A trading strategy for the $d+1$ assets $(\widehat{S}^0, \dots, \widehat{S}^d)$ is* self-financing *if for every $t = 1, \dots, T-1$, we have*

$$\widehat{H}_t^\top \widehat{S}_t = \widehat{H}_{t+1}^\top \widehat{S}_t$$

or more explicitly $\sum_{i=0}^d H^i_t \widehat{S}^i_t = \sum_{i=0}^d H^i_{t+1} \widehat{S}^i_t$.

The self-financing condition means that the portfolio is always adjusted in such a way that the current wealth remains the same (one does not remove or add wealth). Accumulated gains or losses are only achieved through changes in the asset prices.

Definition 2.1.5. *The undiscounted wealth process $(\widehat{V}_t)_{\{t\in\{0,1,\dots,T\}\}}$ with respect to a trading strategy $\widehat{H}$ is given by*

$$\widehat{V}_0 = \widehat{H}_1^\top \widehat{S}_0 = \sum_{i=0}^d H^i_1 \widehat{S}^i_0,$$
$$\widehat{V}_t = \widehat{H}_t^\top \widehat{S}_t = \sum_{i=0}^d H^i_t \widehat{S}^i_t, \quad t \in \{1, \dots, T\}. \tag{2.1}$$

The $\mathcal{F}_t$-measurable random variable $\widehat{V}_t$ defined in (2.1) is interpreted as the value of the portfolio at time t defined by the trading strategy $\widehat{H}$.

Remark 2.1.6. Note that if $\widehat{H}$ is self-financing, we have $\widehat{V}_t = \widehat{H}_t^\top \widehat{S}_t = \widehat{H}_{t+1}^\top \widehat{S}_t$.

In the sequel we shall work with discounted price and wealth processes, that means we consider everything in terms of units of the numéraire asset S^0.

Definition 2.1.7. *The* discounted *asset prices are given by*

$$S_t^i := \frac{\widehat{S}_t^i}{\widehat{S}_t^0}, \quad i \in \{1, \dots, d\}, \quad t \in \{0, 1, \dots, T\},$$

and we write $S = (S^1, \dots, S^d)$. The discounted wealth process is given by

$$V_t = \frac{\widehat{V}_t}{\widehat{S}_t^0}, \quad t \in \{0, 1, \dots, T\}.$$

Remark 2.1.8. Note that the discounted numéraire asset $S_t^0 \equiv 1$ for all $t \in \{0, \dots, T\}$.

The self-financing property can be characterized by the following proposition, where we use the notation $\Delta S_u = S_u - S_{u-1}$.

Proposition 2.1.9. *Let $\widehat{S}$ be a model of a financial market as of Definition 2.1.1 and consider an $\mathbb{R}^{d+1}$-valued trading strategy $\widehat{H} = (H^0, H)$ for $\widehat{S}$. Then the following are equivalent:*

1. *$\widehat{H}$ is self-financing.*
2. *The (undiscounted) wealth process satisfies*

$$\widehat{V}_t = \widehat{V}_0 + \sum_{j=1}^{t} \widehat{H}_j^\top \Delta \widehat{S}_j, \quad t = 0, \dots, T.$$

3. *We have*

$$H_t^0 + H_t^\top S_t = H_{t+1}^0 + H_{t+1}^\top S_t, \quad t = 1, \dots, T-1,$$

where S denotes the discounted price process as of Definition 2.1.7.

4. *The discounted wealth process satisfies*

$$V_t = V_0 + \sum_{j=1}^{t} H_j^\top \Delta S_j, \quad t = 0, \dots, T, \tag{2.2}$$

where S denotes the discounted price process as of Definition 2.1.7 and $V_0 = \frac{\widehat{V}_0}{\widehat{S}_0^0} = \frac{\widehat{H}_1^\top \widehat{S}_0}{\widehat{S}_0^0} = H_1^0 + H_1^\top S_0$.

Moreover, there is a bijection between self-financing $\mathbb{R}^{d+1}$-valued trading strategies $\widehat{H} = (H^0, H)$ and pairs (V_0, H), where V_0 is a $\mathcal{F}_0$-measurable random variable and H an $\mathbb{R}^d$-valued trading strategies for the risky assets. Explicitly, $H_t^0 = V_0 + \sum_{u=1}^{t} H_u^\top \Delta S_u - H_t^\top S_t$.

Proof. 1) $\Leftrightarrow$ 2): $\widehat{H}$ is self-financing if and only if

$$\widehat{V}_{j+1} - \widehat{V}_j = \widehat{H}_{j+1}^\top \widehat{S}_{j+1} - \widehat{H}_j^\top \widehat{S}_j = \widehat{H}_{j+1}(\widehat{S}_{j+1} - \widehat{S}_j), \quad j = 0, \dots, T-1$$

which in turn is equivalent to

$$\widehat{V}_t = \widehat{V}_0 + \sum_{j=1}^{t}(\widehat{V}_j - \widehat{V}_{j-1}) = \widehat{V}_0 + \sum_{j=1}^{t} \widehat{H}_j(\widehat{S}_j - \widehat{S}_{j-1}).$$

1) $\Leftrightarrow$ 3) 3) is obtained from 1) by dividing through S_t^0 and conversely 1) is obtained from 3) by multiplying with S_t^0.
3) $\Leftrightarrow$ 4): 3) holds if and only if

$$V_{j+1} - V_j = H_{j+1}^0 + H_{j+1}^\top S_{j+1} - H_j^0 - H_j^\top S_j = H_{j+1}^\top (S_{j+1} - S_j), \quad j = 0, \dots, T-1,$$

which in turn is equivalent to

$$V_t = V_0 + \sum_{j=1}^{t}(V_j - V_{j-1}) = V_0 + \sum_{j=0}^{t-1} H_j^\top (S_j - S_{j-1}).$$

For the last statement let (V_0, H) be given. Since the self-financing property of $\widehat{H}$ is equivalent to (2.2), we can determine H^0 from (V_0, H) via

$$V_0 + \sum_{j=1}^{t} H_j^\top (S_j - S_{j-1}) = V_t = H_t^0 + H_t^\top S_t,$$

where the last equality is simply the definition of the discounted wealth process. Thus

$$H_t^0 = V_0 + \sum_{j=1}^{t} H_j^\top (S_j - S_{j-1}) - H_t^\top S_t = V_0 + \sum_{j=1}^{t-1} H_j^\top (S_j - S_{j-1}) - H_t^\top S_{t-1}$$

which is predictable. Conversely, for a given self-financing $\mathbb{R}^{d+1}$-valued strategy (H^0, H), V_0 is determined via $H_1^0 + H_1^\top S_0$.

$\square$

Definition 2.1.10. *Let $S = (S^1, \dots, S^d)$ be a model of a financial market in discounted terms (as of Definition 2.1.7) and consider an $\mathbb{R}^d$-valued trading strategy $H \in \mathcal{H}$. The discounted* gains process *with respect to H is defined through the* stochastic integral *(in discrete time)*

$$G_t := (H \bullet S)_t := \sum_{j=1}^{t} H_j^\top (S_j - S_{j-1}) =: \sum_{j=1}^{t} H_j^\top \Delta S_j$$

and corresponds to the gains or losses accumulated up to time t in discounted terms.

Remark 2.1.11. Note that by Proposition 2.1.9 the discounted wealth process V of a self-financing strategy is given as the sum of the discounted initial wealth V_0 and the discounted gains process. Moreover due to the second part of 2.1.9, for any $\mathbb{R}^d$-valued trading strategy $H \in \mathcal{H}$ and initial wealth V_0 we can define $V_t := V_0 + (H \bullet S)_t$ which then corresponds to the discounted wealth processes of a self-financing $\mathbb{R}^{d+1}$-valued trading strategy $\widehat{H} = (H^0, H)$ where $H_t^0 = V_0 + \sum_{u=1}^t H_u^\top \Delta S_u - H_t^\top S_t$.

From now on we shall work in terms of the *discounted* $\mathbb{R}^d$*-valued process denoted by* S and *discounted wealth process* V.

2.2 No-arbitrage and the fundamental theorem of asset pricing

This section is mainly based on [1, Chapter 2].

Definition 2.2.1. *Let $S = (S^1, \dots, S^d)$ be a model of a financial market in discounted terms.*

- *An $\mathbb{R}^d$-valued trading strategy $H \in \mathcal{H}$ is called* arbitrage opportunity *if*

$$(H \bullet S)_T \geq 0 \quad P\text{-a.s.} \quad \text{and} \quad P[(H \bullet S)_T > 0] > 0.$$

- *We call a model* arbitrage-free *or satisfies the* no-arbitrage *condition (NA) if there exists no arbitrage strategy.*

Remark 2.2.2. The notion of arbitrage can equivalently be formulated as follows: A self-financing $\mathbb{R}^{d+1}$-valued strategy $\widehat{H}$ is called arbitrage opportunity if the associated wealth process $\widehat{V}$ satisfies $\widehat{V}_0 = 0$ and $\widehat{V}_T \geq 0$ P-a.s and $P[\widehat{V}_T > 0] > 0$.

Assumption 1. *From now on we assume that the probability space Ω underlying our model is finite.*

$$\Omega = \{\omega_1, \dots, \omega_N\}$$

for some $N \in \mathbb{N}$ and a probability measure P such that

$$P[\omega_n] = p_n > 0, \ \text{for } n = \{1, \dots, N\}$$

and that $\mathcal{F} = \mathcal{F}_T = \mathcal{P}(\Omega)$.

Recall the notation $L(\Omega, \mathcal{F}, P)$ from (1.1) which denotes in the present case (as $p_n > 0$ for all n) the space of random variables (which are under the above assumption on $\mathcal{F}$ *all* functions from $\Omega \to \mathbb{R}$).

Definition 2.2.3. *A (discounted)* European contingent claim *(derivative/option) f is an element of $L(\Omega, \mathcal{F}, P)$.*

Remark 2.2.4. The random variable f corresponds to the (discounted) payoff function at time T. For instance, in a model with bank account $S_t^0 = (1+r)^t$ where r denotes the constant interest rate, we have in the case of a *European call option* on the first asset with strike K,

$$f = \frac{(\widehat{S}_T^1 - K)^+}{(1+r)^T} = (S_T^1 - \widetilde{K})^+,$$

where $\widetilde{K} = \frac{K}{(1+r)^T}$.

Definition 2.2.5.
- *We call the subspace $K \subset L(\Omega, \mathcal{F}, P)$*

$$K = \{(H \bullet S)_T \,|\, H \in \mathcal{H}\}$$

the vector space of contingent claims attainable (replicable) at price 0.

- *For $a \in \mathbb{R}$, we call $K_a := a + K$ the set of contingent claims attainable (replicable) at price a.*

The economic interpretation is the following: If $f \in K$, then there exists a trading strategy $H \in \mathcal{H}$ such that $f = (H \bullet S)_T$, i.e. we can replicate f with 0 initial capital and trading accordingly to H. Similarly $f \in K_a$ means that it can be replicated with initial capital a and trading accordingly to some strategy H such that $f = a + (H \bullet S)_T$.

Definition 2.2.6.
- *We call the set $C \in L(\Omega, \mathcal{F}, P)$ defined by*

$$C = \{g \in L(\Omega, \mathcal{F}, P) \,|\, \exists f \in K \text{ with } f \geq g\}$$

the set of contingent claims super-replicable at price 0.

- *For $a \in \mathbb{R}$, we call $C_a := a + C$ the set of contingent claims super-replicable at price a.*

The economic interpretation is as follows: If $g \in C$, it can be super-replicated with 0 initial capital and trading accordingly to some strategy H such that we arrive at some contingent claim $f = (H \bullet S)_T \in K$ which satisfies $f(\omega) \geq g(\omega)$ for every $\omega \in \Omega$ (for general probability space it would be P-almost every ω). For ω where $f(\omega) > g(\omega)$ we consume or "throw away money".

Remark 2.2.7.
- The no-arbitrage condition (NA) is equivalent to

$$K \cap L_+(\Omega, \mathcal{F}, P) = \{0\},$$

where $L_+(\Omega, \mathcal{F}, P)$ denotes in our case (as $p_n > 0$ for all n) the space of nonnegative random variables, i.e. $L_+(\Omega, \mathcal{F}, P) = \{f \in L(\Omega, \mathcal{F}, P) \,|\, f \geq 0\}$ and 0 denotes the random variable which is identically equal to zero.

- (NA) is also equivalent to

$$C \cap L_+(\Omega, \mathcal{F}, P) = \{0\}.$$

- (NA) implies $C \cap (-C) = K$.

Lemma 2.2.8. *C is a closed convex cone.*

Proof. For C to be a convex cone, we have to verify that for any positive scalars λ_1, λ_2 and elements $g_1, g_2 \in C$, $\lambda_1 g_1 + \lambda_2 g_2 \in C$. Denote by f_1, f_2 the elements in K which dominate $g_1, g_2 \in C$. Then $\lambda_1 f_1 + \lambda_2 f_2 \in K$ and $\lambda_1 f_1 + \lambda_2 f_2 \geq \lambda_1 g_1 + \lambda_2 g_2$. Concerning closedness, let $(g_k) \in C$ be a convergent sequence with $g = \lim_k g_k$. Denote by $f_k \in K$ the elements dominating g_k. Then $g \in C$ since $g \leq \limsup f_k \in K$. □

The goal is now to characterize models for which (NA) holds. The answer is given by the so-called *Fundamental Theorem of Asset Pricing* of which we state a first version:

Theorem 2.2.9 (FTAP (first formulation))**.** *Let $S = (S^1, \dots, S^d)$ be a model of a financial market in discounted terms. Suppose that Assumption 1 holds true. Then the following assertions are equivalent:*

1. *S satisfies (NA).*
2. *There exists a measure $Q \sim P$ such that $E_Q[g] \leq 0$ for all $g \in C$.*

Remark 2.2.10. A measure $Q \sim P$ which satisfies $E_Q[g] \leq 0$ for all $g \in C$ is usually called *separating measure.*

For the proof of this theorem (direction (1) $\Rightarrow$ (2)) we need a version of the separating hyperplane theorem. Basically, this theorem tells that, if we have two convex sets, one closed and the other one compact (in the version we state) then it is possible to stick a hyperplane between them. This should be intuitively clear in $\mathbb{R}^2$, where a hyperplane is simply a line.

Theorem 2.2.11 (Separating Hyperplane Theorem, Hahn-Banach)**.** *Let $A \subseteq \mathbb{R}^N$ be convex and closed and $B \subseteq \mathbb{R}^N$ convex and compact such that $A \cap B = \emptyset$. Then there exists some non zero linear functional $l : \mathbb{R}^N \to \mathbb{R}$, i.e. a non-zero vector $y \in \mathbb{R}^N$, and numbers $\alpha < \beta$ such that*

$$
\begin{aligned}
l(a) = y^\top a \leq \alpha \quad &\text{for all } a \in A, \\
l(b) = y^\top b \geq \beta \quad &\text{for all } b \in B.
\end{aligned}
$$

Moreover, if A is a closed convex cone such that $A \supset \mathbb{R}^N_-$, $\alpha = 0$.

The proof is based on the following lemma.

Lemma 2.2.12. *Let $D \subseteq \mathbb{R}^N$ be a closed convex set which does not contain the origin 0. Then there exists a non-zero linear functional $l : \mathbb{R}^N \to \mathbb{R}$, i.e. a non-zero vector $y \in \mathbb{R}^N$, such that for all $x \in D$, $l(x) = y^\top x \geq \|y\|^2 > 0$.*

Proof. Consider a closed ball of radius r which intersects the set D. Then the function $x \mapsto \|x\|$ achieves its minimum at $B(r) \cap D$ at some $x_0 \neq 0$ as $B(r) \cap D$ is compact and we have for all $x \in D$, $\|x\| \geq \|x_0\|$. As D is convex, we have for $\lambda \in [0,1]$

$$\lambda x + (1-\lambda)x_0 = x_0 + \lambda(x - x_0) \in D.$$

Hence $\|\lambda x + (1-\lambda)x_0\|^2 \geq \|x\|^2$. Expanding the left hand side yields

$$2\lambda x_0^\top (x - x_0) + \lambda^2 \|x - x_0\|^2 \geq 0,$$

from which we obtain $x_0^\top (x - x_0) \geq 0$ (indeed, take λ small enough and suppose that $x_0^\top (x - x_0) < 0$, then there appears a contradiction in the above inequality). Hence and since $x_0 \neq 0$, we obtain

$$x_0^\top x \geq \|x_0\|^2 > 0.$$

The assertion follows by choosing $y = x_0$. □

We now apply this lemma to prove the Separating Hyperplane Theorem:

Proof. Proof of Theorem 2.2.11. Define $D = B - A$. Then D is closed and convex and does not contain the origin as $A \cap B = \emptyset$. Therefore we can apply Lemma 2.2.12, stating that there exists some non-zero vector y, such that for every $D \ni x = b - a$

$$y^\top x = y^\top (b - a) \geq \|y\|^2.$$

This implies that

$$\inf_{b \in B} y^\top b \geq \|y\|^2 + \sup_{a \in A} y^\top a,$$

whence $\inf_{b \in B} y^\top b > \sup_{a \in A} y^\top a$ and defining $\beta := \inf_{b \in B} y^\top b$ and $\alpha := \sup_{a \in A} y^\top a$ yields the assertion.

Finally we prove that for a closed convex cone A with $A \supset \mathbb{R}^N_-$, $\alpha = 0$. As $a = 0 \in A$ in this case, we certainly have $\alpha \geq 0$. Assume that α can not be chosen 0. Then there exists some a such that $y^\top a > \alpha' > 0$. Since $ka \in A$ for every $k \in \mathbb{R}_+$ we obtain $y^\top ka > k\alpha'$ and for k large enough $k\alpha' > \alpha$ which contradicts the fact $y^\top a \leq \alpha$ for all $a \in A$. □

We are now ready to give the proof of the FTAP:

Proof. Proof of Theorem 2.2.9. (2) $\Rightarrow$ (1): This is the obvious implication. Assume by contradiction that (NA) does not hold. Then by Remark 2.2.7 there exists some $g \in C \cap L_+(\Omega, \mathcal{F}, P)$ with $g \neq 0$. Then since $Q \sim P$, we would have

$$E_Q[g] > 0$$

which contradicts (2). (1) $\Rightarrow$ (2): We apply the Separating Hyperplane Theorem 2.2.11, with $A := C$, which is convex and closed by Lemma 2.2.8. Define the set

$$B := \{b \in L_+(\Omega, \mathcal{F}, P) \mid E_P[b] = 1\}.$$

Then B is convex and compact. Indeed, concerning convexity we have for all $\lambda \in [0,1]$ and elements $b_1, b_2 \in B$

$$E_P[\lambda b_1 + (1-\lambda)b_2] = \lambda E_P[b_1] + (1-\lambda)E_P[b_2] = 1.$$

Concerning compactness, we prove that B is closed and bounded. Indeed let $(b_k)_k \in B$ such that $\bar{b} = \lim_k b_k$ then

$$E_P[\bar{b}] = E_P[\lim_k b_k] = \sum_{n=1}^{N} \lim_k b_k(\omega_n)p_n = \lim_k \sum_{n=1}^{N} b_k(\omega_n)p_n = \lim_k E[b_k] = 1,$$

whence B is closed. Concerning boundedness we have $\max_{n\in\{1,\dots,N\}} b(\omega_n) \leq \frac{1}{\min_{n\in\{1,\dots,N\}} p_n}$, which proves the claim. By Theorem 2.2.11 there exists some functional $l : L(\Omega, \mathcal{F}, P) \to \mathbb{R}$, i.e., a random variable Y, and numbers $\alpha < \beta$, such that

$$l(g) = \sum_{n=1}^{N} Y(\omega_n)g(\omega_n) \leq \alpha \quad \text{for all } g \in C, \tag{2.3}$$

$$l(b) = \sum_{n=1}^{N} Y(\omega_n)b(\omega_n) \geq \beta \quad \text{for all } b \in B. \tag{2.4}$$

As C is a closed convex cone containing $L_-(\Omega, \mathcal{F}, P)$, $\alpha = 0$ by the second assertion of Theorem 2.2.11. For every n we define now

$$Q[\omega_n] = \frac{l(1_{\omega_n})}{l(1_\Omega)} = \frac{Y(\omega_n)}{\sum_{i=1}^{N} Y(\omega_i)},$$

which is strictly positive since $l(1_{\omega_n}) = p_n l(\frac{1_{\omega_n}}{p_n}) > 0$ due to the fact that the random variable $b = \frac{1_{\omega_n}}{p_n} \in B$. Due to (2.3), we thus have for all $g \in C$

$$E_Q[g] = \sum_{n=1}^{N} g(\omega_n)Q[\omega_n] = \sum_{n=1}^{N} g(\omega_n)\frac{Y(\omega_n)}{\sum_{n=1}^{N} Y(\omega_n)} \leq 0,$$

which proves the assertion. □

In order to formulate a second version of the fundamental theorem, let us introduce the notion of an *equivalent martingale measure.*

Definition 2.2.13. *A probability measure Q on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure for the discounted assets $S = (S^1, \dots, S^d)$ if $Q \sim P$ and if S is a martingale under Q, i.e.*

$$E[S_t|\mathcal{F}_{t-1}] = S_{t-1}, \quad t \in 1, \dots, T.$$

We write $\mathcal{M}^e(S)$ for the set of equivalent martingale measures and $\mathcal{M}^a(S)$ for the set of absolutely continuous martingale measures (which are, due to the fact that $P(\omega) > 0$ for all $\omega \in \Omega$, all *measures under which S is a martingale).*

The following lemma is left as an exercise to the reader.

Lemma 2.2.14. *Let S be an $\mathbb{R}^d$-valued martingale. Consider the stochastic integral $(H \bullet S)$, where H denotes a predictable $\mathbb{R}^d$-valued process. Then $(H \bullet S)$ is a martingale, i.e.*

$$E[(H \bullet S)_T | \mathcal{F}_t] = (H \bullet S)_t, \quad t = 0, \dots, T,$$

and in particular $E[(H \bullet S)_T] = 0$.

Lemma 2.2.15. *For a probability measure Q on $(\Omega, \mathcal{F})$ the following are equivalent:*

1. $Q \in \mathcal{M}^a$,
2. $E_Q[f] = 0$ *for all* $f \in K$,
3. $E_Q[g] \leq 0$ *for all* $g \in C$.

Proof. 1) $\Rightarrow$ 2) This follows from Lemma 2.2.14.
2) $\Rightarrow$ 1) We have to show that S is a Q martingale: S is adapted by definition, thus it remains to prove $E_Q[S_t|\mathcal{F}_{t-1}] = S_{t-1}$. By the definition of the conditional expectation we have for all $Z \in L(\Omega, \mathcal{F}_{t-1}, Q)$

$$E_Q[E_Q[S_t|\mathcal{F}_{t-1}]Z] = E_Q[S_t Z].$$

We thus have to prove that

$$E_Q[S_{t-1}Z] = E_Q[S_t Z]$$

for all $Z \in L(\Omega, \mathcal{F}_{t-1}, Q)$, which is equivalent to

$$E_Q[Z(S_{t-1} - S_t)] = 0. \tag{2.5}$$

By choosing $H_u = Z1_{\{t=u\}}$, we can write $Z(S_{t-1} - S_t) = (H \bullet S)_T$ which lies in K and therefore proves (2.5),
2) $\Rightarrow$ 3) Let $g \in C$. Then there exists some $K \ni f \geq g$ and we know $0 = E_Q[f] \geq E_Q[g]$.
3) $\Rightarrow$ 2) Let $f \in K$. Then f and $-f \in C$. Thus $E_Q[f] \leq 0$ and $E_Q[-f] \leq 0$, whence $E_Q[f] = 0$. □

By the above lemma, we now get the following formulation of the FTAP, which is the statement commonly used in the literature.

Theorem 2.2.16 (FTAP (usual formulation))**.** *Let $S = (S^1, \dots, S^d)$ be a model of a financial market in discounted terms. Suppose that Assumption 1 holds true. Then the following assertions are equivalent:*

1. *S satisfies (NA)*
2. *There exists an equivalent martingale measure $Q \sim P$ for S, i.e. $\mathcal{M}^e(S) \neq \emptyset$.*

Proof. The assertion follows from the first formulation of FTAP in Theorem 2.2.9 and the equivalence of (1) and (3) in the Lemma 2.2.15. □

Remark 2.2.17. The intuitive interpretation of this result is as follows: A martingale S is a mathematical model for a perfectly fair game. Applying any strategy $H \in \mathcal{H}$ we always have $E[(H \bullet S)_T] = 0$, i.e., an investor can neither lose or win in expectation. The above theorem tells that in the case of No-arbitrage we can always pass to an equivalent measure $Q \sim P$ under which S is a martingale, i.e. a perfectly fair game. Note that the passage from P to Q may change the probabilities but not the impossible events. This means that through a change of the probabilities the market becomes totally fair.

On the other hand a process allowing for arbitrage is a model for an utterly unfair game. Choosing an appropriate strategy H, the investor is sure not to lose but has strictly positive probability to gain something. Note that the possibility of making an arbitrage is not affected by passing to an equivalent probability Q.

Corollary 2.2.18. *Let S satisfy (NA) and let $f \in K_a$ be an attainable claim at price a for some $a \in \mathbb{R}$. In other words, f is of the form*

$$f = a + (H \bullet S)_T$$

for some trading strategy H. Then the constant a and the process $(H \bullet S)_t$ are uniquely determined and satisfy for every $Q \in \mathcal{M}^e(S)$

$$a = E_Q[f], \quad a + (H \bullet S)_t = E_Q[f|\mathcal{F}_t], \quad 0 \le t \le T. \tag{2.6}$$

Proof. The equations in (2.6) arise from Lemma 2.2.14. Indeed under every $Q \in \mathcal{M}^e(S)$ we have $E_Q[(H \bullet S)_T] = 0$, whence $a = E_Q[f]$ and $E_Q[f|\mathcal{F}_t] = a + (H \bullet S)_t$. Concerning uniqueness, assume that there are two representations, namely $f = a + (H \bullet S)_T$ and $f = \widetilde{a} + (\widetilde{H} \bullet S)_T$. By taking expectations under some $Q \in \mathcal{M}^e(S)$ we have

$$E_Q[a + (H \bullet S)_T] = E_Q[\widetilde{a} + (\widetilde{H} \bullet S)_T],$$

whence $a = \widetilde{a}$. This implies $(H \bullet S)_T = (\widetilde{H} \bullet S)_T$ and taking conditional expectations yields $(H \bullet S)_t = (\widetilde{H} \bullet S)_t$. □

Remark 2.2.19.
- Note that the process $H \bullet S$ is unique, but there could be strategies $H \neq \widetilde{H}$ such that $(H \bullet S)_t = (\widetilde{H} \bullet S)_t$.
- The representation $f = a + (H \bullet S)_T$ means, that a is the fair price at which we should buy or sell the contingent claim f. The strategy H is the hedging strategy to perfectly replicate the claim.

The goal of the following propositions is to obtain a characterization of the sets C and K in terms of $\mathcal{M}^a(S)$ and $\mathcal{M}^e(S)$. Let us start with the following proposition:

Proposition 2.2.20. *Suppose S satisfies (NA). Then the set $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$.*

Proof. By Theorem 2.2.16, there is at least one $Q^* \in \mathcal{M}^e(S)$. For any $Q \in \mathcal{M}^a(S)$ and $0 < \alpha < 1$, we have that $\alpha Q^* + (1-\alpha)Q \in \mathcal{M}^e(S)$, which clearly implies the density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$. □

In the following we introduce the notion of a polar set:

Definition 2.2.21. *Let $A \subseteq \mathbb{R}^N$. Then the* polar set *is defined through*

$$A^o = \{b \in \mathbb{R}^N \,|\, b^\top a \leq 1 \textit{ for all } a \in A\}.$$

If A is a cone, then the above definition is equivalent to

$$A^o = \{b \in \mathbb{R}^N \,|\, b^\top a \leq 0 \textit{ for all } a \in A\}.$$

Remark 2.2.22. The following properties are satisfied:

- If $A \subseteq B$, then $A^{\mathrm{o}} \supseteq B^{\mathrm{o}}$.
- If A is a cone, then A^{o} is a cone.

The so-called Bipolar-Theorem which we state without proof plays an important role in the sequel:

Theorem 2.2.23 (Bipolar-Theorem)**.** *We always have $A \subseteq A^{oo}$ and $A = A^{oo}$ holds if and only if A is convex and closed.*

In analogy with the above definition (by a slight adaption of the scalar product), we define the polar set of a set $A \in L(\Omega, \mathcal{F}, P)$ as follows:

$$A^{\mathrm{o}} = \{b \in L(\Omega, \mathcal{F}, P) \,|\, E[ba] \leq 1 \text{ for all } a \in A\}.$$

and for a cone A as

$$A^{\mathrm{o}} = \{b \in L(\Omega, \mathcal{F}, P) \,|\, E[ba] \leq 0 \text{ for all } a \in A\}.$$

In the present context the polar cone of C plays a particular role and we have the announced characterization of the cone C.

Proposition 2.2.24. *Suppose S satisfies (NA). Then we have*

$$C^o = \textit{cone}(\mathcal{M}^a(S)),$$

where cone denotes the conic hull of $\mathcal{M}^a(S)$. Moreover, the following assertions are equivalent:

1. *$g \in C$.*
2. *$E_Q[g] \leq 0$ for all $Q \in \mathcal{M}^a(S)$.*

3. $E_Q[g] \leq 0$ *for all* $Q \in \mathcal{M}^e(S)$.

Proof. The assertions are a consequence of Lemma 2.2.15, Proposition 2.2.20 and the Bipolar-Theorem. See [1, Proposition 2.2.9] for details. □

Similarly we get a characterization of the vector space K:

Corollary 2.2.25. *Suppose* S *satisfies (NA). Then the following assertions are equivalent:*

1. $f \in K$.
2. $E_Q[f] = 0$ *for all* $Q \in \mathcal{M}^a(S)$.
3. $E_Q[f] = 0$ *for all* $Q \in \mathcal{M}^e(S)$.

Proof. We have that $f \in K$ if and only if $f \in C \cap (-C)$. Hence the result follows from the preceding Proposition 2.2.24 □

This corollary has the following consequence:

Corollary 2.2.26. *Suppose* S *satisfies (NA) and* $E_Q[f] = a$ *for all* $Q \in \mathcal{M}^e(S)$. *Then there exists some* $H \in \mathcal{H}$ *such that*

$$f = a + (H \bullet S)_T,$$

i.e. f *is attainable at price* a.

2.3 Complete models and their properties

Definition 2.3.1. *A model of a financial market* S *(in discounted terms) is called* complete *if every contingent claim* $f \in L(\Omega, \mathcal{F}, P)$ *is attainable at some price* a, *i.e for every* $f \in L(\Omega, \mathcal{F}, P)$ *there exists some* $a \in \mathbb{R}$ *and* $H \in \mathcal{H}$ *such that*

$$f = a + (H \bullet S)_T.$$

From Corollary 2.2.26, we therefore obtain the so-called *Second Fundamental Theorem of Asset pricing*, which states that an arbitrage-free model is complete if an only if the equivalent martingale measure is unique.

Corollary 2.3.2 (Second Fundamental Theorem of Asset pricing)**.** *Suppose* S *satisfies* (NA). *The following assertions are equivalent:*

1. $\mathcal{M}^e(S)$ *consists of one single element.*
2. *The model is complete.*

Remark 2.3.3. Examples of complete models which are used in practice are the Binomial model (see Chapter 3) and (in continuous time) the Black-Scholes model.

For the following proposition, recall the notion of an atom as given in Definition 1.1.5.

Proposition 2.3.4. *Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Suppose that S satisfies (NA) and that it is a complete model. Then the number of atoms in $(\Omega, \mathcal{F}_T, P)$ is bounded from above by $(d+1)^T$.*

Proof. By proceed by induction on T. For $T = 1$ the assertion holds, since solvability of the following linear system for any atom $A_i \in \mathcal{F}_1$

$$f(A_i) = a + \sum_{j=1}^{d} H_1^j (S_1(A_i) - S_0)$$

for $a \in \mathbb{R}$ and $H_1 \in \mathbb{R}^d$, requires the number of atoms in Ω to be at most $d+1$. Suppose the assertions holds for $T-1$. By assumption any claim $f \in L(\Omega, \mathcal{F}, P)$ can be written as

$$f = a + (H \bullet S)_T = V_{T-1} + H_T(S_T - S_{T-1}).$$

V_{T-1} and H_T are $\mathcal{F}_{T-1}$ measurable and hence constant (i.e. elements in $\mathbb{R}$ and $\mathbb{R}^d$ respectively) on every atom A of $(\Omega, \mathcal{F}_{T-1}, P)$. Thus $(\Omega, \mathcal{F}_T, P[\cdot|A])$ has at most $d+1$ atoms. Applying the induction hypothesis where we supposed that $(\Omega, \mathcal{F}_{T-1}, P)$ has $(d+1)^{T-1}$ atoms concludes the proof. □

For the formulation of the subsequent theorem, recall that $\mathcal{M}^e(S)$ and $\mathcal{M}^a(S)$ are convex sets. An element of a convex set is called an *extreme point* if it cannot be written as a non-trivial convex combination of members of this set.

Theorem 2.3.5. *Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$. For $Q \in \mathcal{M}^e(S)$, the following conditions are equivalent:*

- $\mathcal{M}^e(S) = \{Q\}$.
- *Q is an extreme point of $\mathcal{M}^e(S)$.*
- *Q is an extreme point of $\mathcal{M}^a(S)$.*
- *Every Q-martingale M can be represented as stochastic integral, i.e.*

$$M_t = M_0 + (H \bullet S)_t$$

The latter property is called predictable representation property *or* martingale representation property.

Proof. (1)$\Rightarrow$ (3): Suppose by contradiction that Q can be represented by a non-trivial convex combination of elements in $\mathcal{M}^a(S)$, i.e. there exist some $\lambda \in (0,1)$ such that $Q = \lambda Q_1 + (1-\lambda)Q_2$ for $Q_1, Q_2 \in \mathcal{M}^a(S)$. By defining

$$P_i = \frac{1}{2}(Q + Q_i)$$

we obtain two martingale measures which are equivalent to Q. Since $\mathcal{M}^e(S)$ contains only one element it follows that $P_1 = P_2 = Q$ and thus also $Q_1 = Q_2 = Q$.
(3)$\Rightarrow$ (2): This is obvious since $\mathcal{M}^e(S) \subset \mathcal{M}^a(S)$.
(2) $\Rightarrow$ (1): Suppose there exists some $Q^* \in \mathcal{M}^e(S)$ which is different from Q. Moreover, there exists a constant c such that $\frac{Q^*}{Q}$ is bounded by c. For $0 < \varepsilon < \frac{1}{c}$, we can define

$$Q' = (1+\varepsilon)Q - \varepsilon Q^*,$$

which defines another measure in $\mathcal{M}^e(S)$. Then Q can be represented by

$$Q = \frac{1}{1+\varepsilon}Q' + \frac{\varepsilon}{1+\varepsilon}Q^*$$

which contradicts (2).
(1) $\Rightarrow$ (2): The terminal value of the martingale X, i.e. X_T is a claim in $L(\Omega, \mathcal{F}, P)$ which is attainable by the second fundamental theorem. Hence there exists some a and $H \in \mathcal{H}$ such that

$$X_T = a + (H \bullet S)_T$$

By the martingale property of X and since $a + (H \bullet S)_t$ is a martingale it follows that

$$X_t = E_Q[X_T|\mathcal{F}_t] = E_Q[a + (H \bullet S)_T|\mathcal{F}_t] = a + (H \bullet S)_t,$$

which proves the desired representation.
(4)$\Rightarrow$ (1): Let $f \in L(\Omega, \mathcal{F}, P)$ be any claim. Define the martingale $X_t = E_Q[f|\mathcal{F}_t]$. Then $X_T = f$ can be written as

$$X_T = X_0 + (H \bullet S)_T,$$

and is thus attainable at price X_0. Since f was arbitrary, we obtain that every claim is attainable, and by the second fundamental theorem $\mathcal{M}^e$ consists only one element Q. □

2.4 Pricing by No-arbitrage

In the general case when the market is not complete, the subsequent theorem tells us what the principle of no-arbitrage implies about the possible prices for a contingent claim f.

Let $f \in L(\Omega, \mathcal{F}, P)$. Then we define an *enlarged market* by introducing a new financial instrument S^{d+1} which can be bought (or sold) at price a at

$t = 0$ and generates a random payment f at time $t = T$. We do not postulate anything about the price of this financial instrument at the intermediate times $t = 1, \dots, T-1$.

Definition 2.4.1. *For a given discounted claim $f \in L(\Omega, \mathcal{F}, P)$ we call $a \in \mathbb{R}$ an* arbitrage-free price*, if there exists an adapted stochastic process S^{d+1} such that*

$$S_0^{d+1} = a \quad \text{and} \quad S_T^{d+1} = f \tag{2.7}$$

and such that the enlarged market model $(S^1, \dots, S^{d+1})$ is arbitrage-free.

Theorem 2.4.2. *Suppose $S = (S^1, \dots, S^d)$ satisfies (NA) and let $f \in L(\Omega, \mathcal{F}, P)$ and suppose $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Define*

$$\underline{\pi}(f) = \inf_{Q \in \mathcal{M}^e(S)} E_Q[f],$$
$$\overline{\pi}(f) = \sup_{Q \in \mathcal{M}^e(S)} E_Q[f].$$

Then either (1) or (2) is satisfied:

1. *$\underline{\pi}(f) = \overline{\pi}(f)$. Then f is attainable at price $a := \underline{\pi}(f) = \overline{\pi}(f)$, i.e.*

   $$f = a + (H \bullet S)_T$$

   for some $H \in \mathcal{H}$ and a is the unique arbitrage-free price.

2. *$\underline{\pi}(f) < \overline{\pi}(f)$. Then we have*

   $$(\underline{\pi}(f), \overline{\pi}(f)) = \{E_Q[f] \,|\, Q \in \mathcal{M}^e(S)\}$$

   and a is an arbitrage-free price for f if and only if $a \in (\underline{\pi}(f), \overline{\pi}(f))$.

Remark 2.4.3. From the above theorem we see that the arbitrage-free prices are in any case of the form $E_Q[f]$ for some $Q \in \mathcal{M}^e(S)$.

Proof.
- Case 1: $\underline{\pi}(f) = \overline{\pi}(f) =: a$ implies $a = E_Q[f]$ for all $Q \in \mathcal{M}^e(S)$ and by Corollary 2.2.26 there exists some $H \in \mathcal{H}$ such that $f = a + (H \bullet S)_T$. If one could buy f for another price this would generate an arbitrage opportunity.

- Case 2: First observe that $I := \{E_Q[f] \,|\, Q \in \mathcal{M}^e(S)\}$ is an interval, since it is convex and bounded. Concerning convexity let $a_1, a_2 \in I$. Then there exists some Q_1 and Q_2 with $a_1 = E_{Q_1}[f]$ and $a_2 = E_{Q_2}[f]$. For $\lambda \in [0,1]$, we then have $a = \lambda a_1 + (1-\lambda) a_2 = \lambda E_{Q_1}[f] + (1-\lambda) E_{Q_2}[f] = E_Q[f]$ where $Q = \lambda Q_1 + (1-\lambda) Q_2 \in \mathcal{M}^e(S)$ since $\mathcal{M}^e(S)$ is convex. Concerning boundedness, we have $\min_i f(\omega_i) \le E_Q[f] \le \max_i f(\omega_i)$.

  We now claim that $a \in I$ if and only if a is an arbitrage-free price. First, let $a \in I$. Then there exists some $Q \in \mathcal{M}^e(S)$ with $a = E_Q[f]$. Let us define the stochastic process $S_t^{d+1} := E_Q[f|\mathcal{F}_t]$, which satisfies all requirements of (2.7). Note that Q is an equivalent martingale measure for the extended market $(S^1, \dots, S^{d+1})$, since $E_Q[S_{t+1}^{d+1}|\mathcal{F}_t] = E_Q[E_Q[f|\mathcal{F}_{t+1}]|\mathcal{F}_t] = S_t^{d+1}$. Hence by the FTAP the extended market satisfies (NA).

Let a be an arbitrage-free price, i.e. the extended market $(S^1, \dots, S^{d+1})$ satisfies (NA). By FTAP (Theorem 2.2.16) there exists some $\widehat{Q}$ such that $(S^1, \dots, S^{d+1})$ is a martingale, i.e.

$$E_{\widehat{Q}}[S_T^i|\mathcal{F}_t] = S_t^i, \quad t = 0, \dots, T, \quad , i = 1, \dots, d+1.$$

This implies that $\widehat{Q} \in \mathcal{M}^e(S)$ and $E_{\widehat{Q}}[f] = a$ and thus $a \in I$.

It remains to prove that I is an open interval: This means that we have to show that $\overline{\pi}(f) \notin I$ (and analogously for $\underline{\pi}(f) \notin I$). Note first that $E_Q[f - \overline{\pi}(f)] \leq 0$ for all $Q \in \mathcal{M}^e(S)$, which implies by Proposition 2.2.24 that $f - \overline{\pi}(f) \in C$. Therefore there exists some $g \in K$ such that $g \geq f - \overline{\pi}(f)$. If $\overline{\pi}(f) \in I$, i.e. if there exists some Q^* such that $E_{Q^*}[f] = \overline{\pi}(f)$, then we have

$$0 = E_{Q^*}[g] \geq E_{Q^*}[f - \overline{\pi}(f)] = 0,$$

and thus $E_{Q^*}[g - (f - \overline{\pi}(f))] = 0$, which implies in view of $g \geq f - \overline{\pi}(f)$ that $K \ni g \equiv f - \overline{\pi}(f)$. Therefore $f \in K_{\overline{\pi}}(f)$, i.e. f is attainable at price $\overline{\pi}(f)$, which in turn implies that $E_Q[f] = \overline{\pi}(f)$ for all $Q \in \mathcal{M}^e(S)$ and I is therefore reduced to the singleton $\{\overline{\pi}(f)\}$ and we are back in case 1, which is a contradiction.

The analog proof works for $\underline{\pi}(f)$ and it follows that the I is the open interval $(\underline{\pi}(f), \overline{\pi}(f))$.

□

Corollary 2.4.4 (Superreplication)**.** *Suppose $S = (S^1, \dots, S^d)$ satisfies (NA). Then we have for $f \in L(\Omega, \mathcal{F}, P)$*

$$\begin{aligned} \overline{\pi}(f) &= \sup_{Q \in \mathcal{M}^e(S)} E_Q[f] = \max_{Q \in \mathcal{M}^a(S)} E_Q[f] \\ &= \min\{a \in \mathbb{R} \,|\, \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\}, \end{aligned}$$

and

$$\begin{aligned} \underline{\pi}(f) &= \inf_{Q \in \mathcal{M}^e(S)} E_Q[f] = \min_{Q \in \mathcal{M}^a(S)} E_Q[f] \\ &= \max\{a \in \mathbb{R} \,|\, \text{ there exists } H \in \mathcal{H} \text{ with } f \geq a + (H \bullet S)_T\}. \end{aligned}$$

Proof. The set $\mathcal{M}^a(S)$ is closed and bounded (in the topology of $\mathbb{R}^N$), thus compact. The function $Q \mapsto E_Q[f]$ is continuous. A continuous function on a compact set takes its maximum/minimum. We only prove the first assertion, the second one follows analogously. We first prove

$$\max_{Q \in \mathcal{M}^a(S)} E_Q[f] \leq \inf\{a \in \mathbb{R} \,|\, \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\}.$$

Take some a such that there exists some $H \in \mathcal{H}$ such that $f \leq a + (H \bullet S)_T$. Taking $Q \in \mathcal{M}^a(S)$ yields

$$E_Q[f] \leq E_Q[a + (H \bullet S)_T].$$

and thus $E_Q[f] \leq a$ as $Q \in \mathcal{M}^a(S)$. Since this holds for all Q, it follows that

$$\max_{Q \in \mathcal{M}^a(S)} E_Q[f] \leq \inf\{a \in \mathbb{R} \mid \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\}.$$

In order to prove the other direction, we have that $f - \overline{\pi}(f) \in C$ since $E_Q[f - \overline{\pi}(f)] \leq 0$ for all $Q \in \mathcal{M}^a(S)$. Thus there exists an element $g \in K$ such that $f - \overline{\pi}(f) \leq g$. As $g \in K$ there exists some $H \in \mathcal{H}$ such that $g = (H \bullet S)_T$ and we obtain

$$f - \overline{\pi}(f) \leq (H \bullet S)_T \Leftrightarrow f \leq \overline{\pi}(f) + (H \bullet S)_T,$$

from which we obtain

$$\overline{\pi}(f) \in \{a \in \mathbb{R} \mid \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\}$$

and thus

$$\inf\{a \in \mathbb{R} \mid \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\} \leq \overline{\pi}(f) = \max_{Q \in \mathcal{M}^a(S)} E_Q[f].$$

All together we have

$$\inf\{a \in \mathbb{R} \mid \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\} = \overline{\pi}(f),$$

which implies that the infimum is actually a minimum as

$$\overline{\pi}(f) \in \{a \in \mathbb{R} \mid \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\}.$$

□

Remark 2.4.5.
- The expression $\{a \in \mathbb{R} \mid \text{ there exists } H \in \mathcal{H} \text{ with } f \leq a + (H \bullet S)_T\}$ is called superhedging price. The interpretation of the above theorem is $\overline{\pi}(f)$ is exactly the minimal capital which is needed to superhedge the claim.
- In the case $\underline{\pi}(f) \leq \overline{\pi}(f)$, the interval $(\underline{\pi}(f), \overline{\pi}(f))$ is exactly the set of arbitrage-free prices. This means if one buys or sells f at price $\overline{\pi}(f)$, then there exists an arbitrage opportunity.

2.5 The optional decomposition theorem

We now present a dynamic version of the superreplication result due to Dimitry Kramkov, who proved this result in a much more general context (continuous time).

Theorem 2.5.1 (Optional decomposition). *Assume that S satisfies (NA) and let V be an adapted process. Then the following assertions are equivalent:*

1. *V is a supermartingale for all $Q \in \mathcal{M}^e(S)$.*
2. *V is a supermartingale for all $Q \in \mathcal{M}^a(S)$.*
3. *V can be decomposed into $V_t = V_0 + (H \bullet S)_t - C_t$ where $H \in \mathcal{H}$ and C is an increasing adapted process starting at $C_0 = 0$.*

Remark 2.5.2. 1. Let us compare the assertion of the above theorem with the Doob composition of supermartingales. Indeed the latter asserts that there is equivalence between a supermartingale and the fact that a process can be written in terms of $V = V_0 + M - A$ where M is a martingale and A an increasing *predictable* process. The above theorem is similar in spirit, but there are significant differences:

   - The supermartingale property pertains to all (absolutely continuous/equivalent) martingale measures Q and the role of the martingale is played by the stochastic integral $H \bullet S$.
   - Another difference is the fact that the decomposition is no longer unique and one cannot choose, in general, C to be predictable but only *adapted.* In continuous time it is a so-called optional process (which is equivalent to adapted in discrete time), which explains the name optional decomposition.

2. In the case of complete models when $|\mathcal{M}^e(S)| = 1$, the above theorem is essentially the Doob decomposition.
3. The economic interpretation of the optional decomposition theorem reads as follows: A process of the form $V = V_0 + H \bullet S - C$ describes the wealth process. Starting with initial capital V_0, trading according to the strategy H and consuming according to the process C which models the accumulated consumption, one obtains V_t. The assertion of the optional decomposition theorem is that these wealth processes are characterized by condition (i) (or (ii)), namely the supermartingale property for all martingale measures.
4. It is possible to obtain the superhedging result from the optional decomposition. Indeed assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let $f \in L(\Omega, \mathcal{F}, P)$ Define

$$V_t = \sup_{Q \in \mathcal{M}^e(S)} E_Q[f | \mathcal{F}_t].$$

   Then it can be shown that V_t is a supermartingale for all $Q \in \mathcal{M}^e$. Indeed, it holds that

$$V_t = \sup_{Q \in \mathcal{M}^e(S)} E_Q[V_{t+1} | \mathcal{F}_t].$$

   Hence for every specific $Q \in \mathcal{M}^e(S)$ we have

$$V_t \geq E_Q[V_{t+1} | \mathcal{F}_t],$$

implying that V is a supermartingale. By the optional decomposition theorem there exists H, C with $V_t = V_0 + (H \bullet S)_t - C_t$ and

$$\begin{aligned} f = V_T &= V_0 + (H \bullet S)_T - C_T \leq V_0 + (H \bullet S)_T \\ &= \sup_{Q \in \mathcal{M}^e(S)} E_Q[f] + (H \bullet S)_T = \overline{\pi}(f) + (H \bullet S)_T \end{aligned}$$

Proof. First assume $T = 1$. Then V is a supermartingale for all $Q \in \mathcal{M}^e(S)$ is equivalent to

$$E_Q[V_1] \leq V_0 \text{ for all } Q \in \mathcal{M}^e(S) \ \Leftrightarrow E_Q[V_1 - V_0] \leq 0 \text{ for all } Q \in \mathcal{M}^e(S) \ .$$

This means that $V_1 - V_0 \in C$, hence there exists a strategy H such that $(H \bullet S)_1 \geq V_1 - V_0$. Letting $C_0 = 0$ and writing $C_1 = C_1 - C_0 = -V_1 + V_0 + (H \bullet S)_1 \geq 0$ yields the desired decomposition in the case $T = 1$.

For general T, consider for every $t \in \{1, \dots, T\}$ the one period market from $t-1$ to t. The supermartingale property for all $Q \in \mathcal{M}^e(S)$ means that

$$E_Q[V_t - V_{t-1} | \mathcal{F}_{t-1}] \leq 0 \text{ for all } Q \in \mathcal{M}^e(S),$$

and implies $E_Q[V_t - V_{t-1}] \leq 0$ for all $Q \in \mathcal{M}^e(S)$. Therefore there exists a $\mathcal{F}_{t-1}$ measurable random variable H_t such that $V_t - V_{t-1} \leq H_t^\top (S_t - S_{t-1})$ and $C_t - C_{t-1} := H_t^\top (S_t - S_{t-1}) - (V_t - V_{t-1}) \geq 0$. Note that C is inductively defined if we start with $C_0 = 0$ and we have

$$C_t = C_t - C_0 = \sum_{i=1}^{t} (C_i - C_{i-1}) = \sum_{i=1}^{t} (H_i^\top (S_i - S_{i-1}) - (V_i - V_{i-1})) = (H \bullet S)_t + V_t - V_0,$$

which is an increasing adapted process. Hence $V_t = V_0 + (H \bullet S)_t - C_t$, which proves (1) $\Rightarrow$ (3). Concerning the direction (3) $\Rightarrow$ (2) we have for all $Q \in \mathcal{M}^a(S)$

$$\begin{aligned} E_Q[V_t | \mathcal{F}_{t-1}] &= E_Q[V_0 + (H \bullet S)_t - C_t | \mathcal{F}_{t-1}] \\ &= V_0 + (H \bullet S)_{t-1} - C_{t-1} - E_Q[C_t - C_{t-1} | \mathcal{F}_{t-1}] \\ &\leq V_0 + (H \bullet S)_{t-1} - C_{t-1}, \end{aligned}$$

which proves (2). The direction (2) $\Rightarrow$ (1) is obvious. □

Chapter 3

The Binomial model (Cox-Ross-Rubinstein model)

The Binomial model or Cox-Ross-Rubinstein model is a particular complete model. Therefore all properties derived in Section 2.3 pertain to this model.

3.1 Definition of the Binomial model and first properties

This section is mainly based on [2, Chapter 5.5].

By Proposition 2.3.4 a complete market with only one asset must have a binary tree structure (the number of atoms is bounded from above by 2^T). Under an additional homogeneity assumption this reduces to the following particularly simple model, which was introduced by Cox, Ross, Rubinstein. The model consist of only one (discounted) asset $S = S^1$, whose return

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}$$

can take two values $a, b \in \mathbb{R}$ such that $-1 < a < b$. Therefore the asset prices can be expressed via

$$S_t = S_{t-1}(1 + R_t) = \begin{cases} S_{t-1}(1 + b), \\ S_{t-1}(1 + a). \end{cases}$$

We construct the model on the following probability space:

$$\Omega := \{-1, 1\}^T = \{\omega = (y_1, \dots, y_T) \,|\, y_i \in \{-1, 1\}\}$$

The canonical process is defined as follows:

$$Y_t : \Omega \to \mathbb{R}, \quad Y_t(\omega) = Y_t(y_1, \dots, y_T) = y_t$$

Formally, we define

$$R_t(\omega) = \begin{cases} b & \text{if } Y_t(\omega) = y_t = 1, \\ a & \text{if } Y_t(\omega) = y_t = -1. \end{cases}$$

The price process of S is modeled as

$$S_t = S_0 \Pi_{k=1}^t (1 + R_k), \quad S_0 \in \mathbb{R}_+.$$

As filtration we take

$$\mathcal{F}_t = \sigma(S_0, \ldots, S_t) = \sigma(Y_0, \ldots, Y_t)) = \sigma(R_0, \ldots, R_t)$$

and note that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ as S_0 is constant and $\mathcal{F}_T = \mathcal{P}(\Omega)$. Let us fix a probability measure P such that $P[\omega] > 0$ for every $\omega \in \Omega$. Such a model is called *Binomial or CRR model.*

The following theorem characterizes the parameter values for which the model is arbitrage free.

Theorem 3.1.1. *The CRR model satisfies (NA) if and only if $a < 0 < b$. In this case, the CRR model is complete and $\mathcal{M}^e(S) = Q$. The martingale measure is characterized by the fact that*

1. *$R_1, \ldots, R_T$ are independent.*
2. *$Q[R_t = b] = q = \frac{-a}{b-a}$.*

Remark 3.1.2. Recall the independence of random variables $X_1, \ldots, X_N$. Indeed $X_1, \ldots, X_N$ are independent if the σ-algebras generated by them, i.e. $\sigma(X_1), \ldots, \sigma(X_N)$ are independent. This in turn means that for any combination of $A_i \in \sigma(X_i)$, $i = 1, \ldots, n \leq N$, we have $P[\bigcap A_i] = \prod P[A_i]$.

Remark 3.1.3. Recall the following notions of independence:

- A set A is independent of a σ-algebra $\mathcal{G}$ if for all sets $B \in \mathcal{G}$, $P[A \cap B] = P[A]P[B]$.
- A random variable X is called independent of a sigma algebra $\mathcal{G}$ if all sets of $\sigma(X)$ are independent of $\mathcal{G}$.

Lemma 3.1.4. *A set A is independent of a σ-algebra $\mathcal{G}$ if and only if*

$$P[A|\mathcal{G}] = P[A]$$

Proof. Let $B_1, \ldots B_m$ denote the atoms of $\mathcal{G}$. Then

$$\begin{aligned} P[A|\mathcal{G}] = P[A] &\Leftrightarrow P[A|B_i] = P[A] \text{ for all } i, \\ &\Leftrightarrow \frac{P[A \cap B_i]}{P[B_i]} = P[A] \text{ for all } i, \\ &\Leftrightarrow P[A \cap B_i] = P[A]P[B_i] \text{ for all } i. \end{aligned}$$

$\square$

Proof. Proof of Theorem 3.1.1: S satisfies (NA) if and only if there exists some $Q \in \mathcal{M}^e(S)$, i.e.

$$E_Q[S_t|\mathcal{F}_{t-1}] = S_{t-1}, \quad t = 1, \dots, T.$$

This is turn equivalent to

$$\begin{aligned}
&E_Q[S_{t-1}(1+R_t)|\mathcal{F}_{t-1}] = S_{t-1} \\
\Leftrightarrow &S_{t-1}E_Q[(1+R_t)|\mathcal{F}_{t-1}] = S_{t-1} \\
\Leftrightarrow &E_Q[(1+R_t)|\mathcal{F}_{t-1}] = 1 \\
\Leftrightarrow &E_Q[R_t|\mathcal{F}_{t-1}] = 0 \\
\Leftrightarrow &bQ[R_t = b|\mathcal{F}_{t-1}] + aQ[R_t = a|\mathcal{F}_{t-1}] = 0 \\
\Leftrightarrow &bQ[R_t = b|\mathcal{F}_{t-1}] + a(1 - Q[R_t = b|\mathcal{F}_{t-1}]) = 0 \\
\Leftrightarrow &(b-a)Q[R_t = b|\mathcal{F}_{t-1}] = -a \\
\Leftrightarrow &Q[R_t = b|\mathcal{F}_{t-1}](\omega) = \frac{-a}{b-a} =: q, \qquad \text{for all } \omega \in \Omega.
\end{aligned}$$

For notational reasons we leave ω away in all equations except the last one. This holds if and only if R_t is independent of $\mathcal{F}_{t-1}$ and we have $Q[R_t = b] = q$. Indeed by the above Lemma the set $\{R_t = b\}$ is independent of $\mathcal{F}_{t-1}$, thus also the set $\{R_t = a\}$ and thus also the random variable R_t since

$$\sigma(R_t) = \{\emptyset, \Omega, \{R_t = b\}, \{R_t = a\}\}.$$

As the sigma algebra $\mathcal{F}_t$ is generated by $R_1, \dots, R_{t-1}$ the independence of $R_1, \dots, R_T$ follows.

As $Q \in \mathcal{M}^e(S)$, the condition $P \sim Q$, implies that

$$Q[R_t = b] = q \in (0,1),$$

which holds if and only if $a < 0 < b$.

Conversely , if $a < 0 < b$ we can define a measure $Q \sim P$ such that

$$Q[\omega] = q^k(1-q)^{T-k} > 0,$$

where k denotes the number of occurrences of $+1$ in ω. Under Q, $Y_1, \dots, Y_T$ and thus $R_1, \dots, R_k$ are independent and $Q[Y_t = 1] = q = \frac{-a}{b-a}$. Independence holds because

$$Q[\bigcap_{i=1}^{n}\{Y_i = \pm 1\}] = q^k(1-q)^{n-k} = \prod Q[\{Y_i = \pm 1\}],$$

where k denotes the number of $+1$ in the sets $\{Y_i = \pm 1\}$. As $Q[Y_t = 1] = Q[R_t = b] = \frac{-a}{b-a}$, it follows from above that $Q \in \mathcal{M}^e(S)$. $\square$

Let us now turn to the problem of *pricing* and *hedging* a given contingent claim $f \in L(\Omega, \mathcal{F}_T, P)$. Note that since the σ-algebra $\mathcal{F}_T$ is generated by $S_0, \dots, S_T$ the claim f is of form

$$f = f(S_0, \dots, S_T)$$

for some suitable function f.

Proposition 3.1.5. *Let $f \in L(\Omega, \mathcal{F}_T, P)$ with $f = f(S_0, \dots, S_T) = a + (H \bullet S)_T$ for some $a \in \mathbb{R}$ and $H \in \mathcal{H}$. Then the value process of the replicating strategy, i.e.*

$$V_t = a + (H \bullet S)_t = E_Q[f|\mathcal{F}_t]$$

satisfies $V_t = v_t(S_0, S_1, \dots, S_t)$ where $v_t : \mathbb{R}^{t+1} \to \mathbb{R}$,

$$v_t(x_0, \dots, x_t) = E_Q[f(x_0, \dots, x_t, \frac{x_t}{S_0} S_1, \dots, \frac{x_t}{S_0} S_{T-t})].$$

For the proof of this assertion we apply the so-called Independence Lemma, which we here state without proof.

Lemma 3.1.6 (Independence Lemma)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Assume that the random variables $X_1, \dots, X_k$ are $\mathcal{G}$-measurable and that the random variables $Y_1, \dots, Y_m$ are independent of $\mathcal{G}$. Moreover, let $f : \mathbb{R}^{k+m} \to \mathbb{R}$ be a function and define $g : \mathbb{R}^k \to \mathbb{R}$ via*

$$g(x_1, \dots, x_k) = E[f(x_1, \dots, x_k, Y_1, \dots, Y_m)].$$

Then we have

$$E[f(X_1, \dots, X_k, Y_1, \dots, Y_m)|\mathcal{G}] = g(X_1, \dots, X_k).$$

Remark 3.1.7. The interpretation is as follows the realizations of the random variables $X_1, \dots, X_k$ are known due their $\mathcal{G}$ measurability. Therefore they can be treated like constants. The other random variables $Y_1, \dots, Y_m$ are independent of $\mathcal{G}$ and therefore we can simply build the usual expectation.

Proof. Proof of Proposition 3.1.5 We have to show that

$$V_t = E_Q[f(S_0, \dots, S_T)|\mathcal{F}_t] = v_t(S_0, \dots, S_t).$$

First note that $v_t(S_0, \dots, S_t)$ is $\mathcal{F}_t$-measurable. Let us write

$$E_Q[f(S_0, \dots, S_T)|\mathcal{F}_t] = E_Q[f(S_0, \dots, S_t, \frac{S_{t+1}}{S_t}, \dots S_t \frac{S_T}{S_t})|\mathcal{F}_t].$$

Since $S_0, \dots, S_t$ is $\mathcal{F}_t$-measurable and since $\frac{S_{t+s}}{S_t} = \prod_{k=t+1}^{t+s}(1 + R_k)$ is independent of $\mathcal{F}_t$ (under Q) and has the same distribution as $\frac{S_s}{S_0} = \prod_{k=1}^{s}(1 + R_k)$, the independence lemma yields the assertion.

□

Remark 3.1.8. 1. The above proof can be rephrased as follows: We have to find the conditional distribution of $(S_1, \dots, S_t)$ given $\mathcal{F}_t$, which means finding $Q[\cdots|\{S_0 = x_0, S_1 = x_1, \dots, S_t = x_t\}]$. This is again a Binomial model with $T - t$ periods and starts at x_t instead of S_0 and thus we have to rescale S_i with the factor $\frac{x_t}{S_0}$.

2. Note that $V_T = f$ and by the martingale property of $V_t = E_Q[f|\mathcal{F}_t]$ we have $V_t = E_Q[V_{t+1}|\mathcal{F}_t]$. Therefore we obtain the following recursion formula:

$$\begin{aligned}
v_T(x_0, \dots, x_t) &= f(x_0, \dots, x_T), \\
v_t(x_0, \dots, x_t) &= E_Q[V_{t+1}|\{S_0 = x_0, \dots, S_t = x_t\}] \\
&= E_Q[v_{t+1}(x_0, \dots, x_t, x_t \frac{S_{t+1}}{x_t})] \\
&= q v_{t+1}(x_0, \dots, x_t, x_t(1+b)) + (1-q) v_{t+1}(x_0, \dots, x_t, x_t(1+a)).
\end{aligned}$$

3. In particular for an option whose payoff f only depends on S_t we have

$$V_t = v_t(S_t) = q v_{t+1}(S_t(1+b)) + (1-q) v_{t+1}(S_t(1+a))$$

and formula of Proposition 3.1.5 simplifies to the expectation with respect to the binomial distribution $B(T-t, q)$.

$$v_t(x_t) = \sum_{k=0}^{T-t} f(x_t(1+a)^{T-t-k}(1+b)^k) \binom{T-t}{k} q^k (1-q)^{T-t-k}.$$

Note that for the conditional (with respect to $\mathcal{F}_t$) distribution of S_T we have

$$Q[S_T = x_t(1+a)^{T-t-k}(1+b)^k | S_t = x_t] = \binom{T-t}{k} q^k (1-q)^{T-t-k}.$$

In particular the unique arbitrage-free price of a derivative with payoff $f(S_T)$ is given by

$$v_0(S_0) = \sum_{k=0}^{T} f(S_0(1+a)^{T-t-k}(1+b)^k) \binom{T}{k} q^k (1-q)^{T-k}.$$

Example 3.1.9. Denote by $M_t := \max_{s \le t} S_s$, $t \le T$, the running maximum and consider a claim $f = f(S_T, M_T)$, for instance a lookback put with floating strike, i.e.

$$f(S_T, M_T) = M_T - S_T.$$

Then the value process V_t is of the form

$$V_t = v_t(S_t, M_t),$$

where $v_t(x_t, m_t) = E_Q[f(x_t \frac{S_{T-t}}{S_0}, m_t \vee x_t \frac{M_{T-t}}{S_0})]$. This follows from the fact that

$$M_T = M_t \vee S_t \max_{t \le u \le T} \frac{S_u}{S_t},$$

where $\max_{t \le u \le T} \frac{S_u}{S_t}$ is independent of $\mathcal{F}_t$ and has the same law as M_{T-t}/S_0 under Q.

Let us now derive the following hedging formula in the Binomial model.

Proposition 3.1.10. *In order to replicate an option $f(S_0, S_1, \ldots, S_T)$, one has to trade according to the strategy H given by*

$$H_{t+1}(\omega) = \Delta_{t+1}(S_0, S_1(\omega), \ldots, S_t(\omega)), \tag{3.1}$$

where

$$\Delta_{t+1}(x_0, x_1, \ldots, x_t) = \frac{v_{t+1}(x_0, x_1, \ldots, x_t(1+b)) - v_{t+1}(x_0, x_1, \ldots, x_t(1+a))}{x_t(b-a)},$$

i.e., the strategy H in the representation

$$V_t = a + (H \bullet S)_t$$

is given by (3.1)

Proof. Subtracting $V_t = a + (H \bullet S)_t$ from $V_{t+1} = a + (H \bullet S)_{t+1}$ yields

$$V_{t+1} - V_t = H_{t+1}(S_{t+1} - S_t).$$

The left hand side is given by

$$V_{t+1} - V_t = \begin{cases} v_{t+1}(S_0, S_1, \ldots, S_t(1+b)) - V_t & \text{if } R_{t+1} = b \\ v_{t+1}(S_0, S_1, \ldots, S_t(1+a)) - V_t & \text{if } R_{t+1} = a \end{cases} \tag{3.2}$$

$$= \begin{cases} H_{t+1}(S_t(1+b) - S_t) & \text{if } R_{t+1} = b \\ H_{t+1}(S_t(1+a) - S_t) & \text{if } R_{t+1} = a \end{cases} \tag{3.3}$$

Solving for H_{t+1} yields the claim. □

3.2 Exotic derivatives

This section is taken from [2, Section 5.6].

The above recursion formula can be used for the numeric computation of the value process of any contingent claim. For the value processes of certain exotic derivatives which depend on the maximum of the stock price, it is possible to obtain simple closed-form solutions if we make the additional assumption that

$$1 + a = \frac{1}{1+b}.$$

In this case we have

$$S_t = S_0(1+b)^{Z_t},$$

where

$$Z_0 = 0, \quad Z_t := \sum_{i=1}^{t} Y_i, \quad t = 1, \ldots, T.$$

Let P denote the uniform distribution on Ω, i.e.,

$$P[\{\omega\}] = \frac{1}{|\Omega|} = 2^{-T}.$$

Under P, the random variables Y_t are independent with common distribution

$$P[Y_t = +1] = \frac{1}{2}.$$

Thus, the stochastic process Z becomes a *standard random walk* under P. Therefore,

$$P[Z_t = k] = \begin{cases} 2^{-t}\binom{t}{\frac{t+k}{2}} & \text{if } t+k \text{ even} \\ 0 & \text{if } t+k \text{ odd.} \end{cases} \tag{3.4}$$

For further use, we denote the running maximum of Z by $M_t := \max_{0 \leq s \leq t} Z_s$.

3.2.1 Reflection principle

The following proposition is a classical result for the standard random walk and can be proved by the reflecting the paths of the random walk at some level k.

Lemma 3.2.1 (Reflection principle)**.** *For all $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, we have*

$$P[M_T \geq k \text{ and } Z_T = k - l] = P[Z_T = k + l]$$

and

$$P[M_T = k \text{ and } Z_T = k - l] = 2\frac{k+l+1}{T+1} P[Z_{T+1} = 1 + k + l]$$

Proof. For the proof we refer to [2, Lemma 5.48]. □

We are now interested in getting a similar result under the martingale measure Q. Under the martingale measure Q with $Q[Y_t = 1] = \frac{-a}{b-a} = q$, we have for $n \in \mathbb{N}$

$$Q[Z_t = n - (t-n)] = Q[Z_t = 2n - t] = \binom{t}{n} q^n (1-q)^{t-n}.$$

Thus for $k = 2n - t$, i.e. for $k + t$ an even number it follows that

$$Q[Z_t = k] = \binom{t}{\frac{t+k}{2}} q^{\frac{t+k}{2}} (1-q)^{\frac{t-k}{2}}$$

otherwise $Q[Z_t = k] = 0$.

The reflection principle under Q now reads as follows:

Lemma 3.2.2 (Reflection principle under Q)**.** *For all $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, we have*

$$Q[M_T \geq k \text{ and } Z_T = k-l] = \left(\frac{1-q}{q}\right)^l Q[Z_T = k+l] = \left(\frac{q}{1-q}\right)^k Q[Z_T = -k-l]$$

and

$$Q[M_T = k \text{ and } Z_T = k-l] = \left(\frac{1-q}{q}\right)^l \frac{1}{q}\frac{k+l+1}{T+1} Q[Z_{T+1} = 1+k+l].$$

Proof. We show first that the Radon Nykodym derivative is given by

$$\frac{Q[\{\omega\}]}{P[\{\omega\}]} = 2^T q^{\frac{T+Z_T}{2}}(1-q)^{\frac{T-Z_T}{2}}.$$

Indeed, we have

$$Q[\{\omega\}] = q^k(1-q)^{T-k}, \quad P[\{\omega\}] = (\frac{1}{2})^T$$

for each $\omega = (y_1, \dots, Y_T)$ which contains exactly k components with $y_i = 1$. Since for such an ω, $Z_T = k-(T-k) = 2k-T$ the above formula follows. Take now

$$Q[M_T \geq k, Z_T = k-l] = 2^T q^{\frac{T+k-l}{2}}(1-q)^{\frac{T+l-k}{2}} P[M_T \geq k, Z_T = k-l]$$

Applying the reflection principle using again the density formula we have

$$P[Z_T = k+l] = 2^{-T} q^{\frac{-(T+k+l)}{2}}(1-q)^{\frac{-(T-k-l)}{2}} Q[Z_T = k+l],$$

which yields the claim for the first identity. The rest follows analogously. □

The above reflection principle can now be applied in order to compute the prices of exotic options in the CRR model, i.e. options whose payoff depends on the path of the asset price (in contrast to plain vanilla options whose payoff only depends on the terminal value).

An example is the so-called *Up-and-in call option* whose price is computed in [2, Example 5.50].

3.3 Convergence to the Black Scholes Price

The goal of this section is to prove the convergence of pricing formulas in the CRR model in discrete time to a limit in continuous time. More precisely, we will show that the CRR model converges to the Black Scholes model and that we obtain the famous Black Scholes formula for the price of a call option.

Let T be a fixed time horizon (not the number of trading periods as usual) and divide the interval $[0, T]$ in N equidistant time steps $\frac{T}{N}, \frac{2T}{N}, \dots, \frac{NT}{N}$ for some $N \in \mathbb{N}$.

For each market N we consider a CRR model with

$$1 + a_N = \frac{1}{1 + b_N}.$$

Recall that in this case we can write

$$S_{\frac{kT}{N}}(\omega) = S_0(1 + b_N)^{Z_k(\omega)},$$

where $Z_k(\omega) = \sum_{i=1}^{k} Y_i(\omega)$. Set now $1 + b_N = e^{\sigma\sqrt{\frac{T}{N}}}$ and $1 + a_N = e^{-\sigma\sqrt{\frac{T}{N}}}$.
We denote the risk neutral measure by Q_N for which $Q_N[R_t = b] = \frac{-a_N}{b_N - a_N} =: q_N$.

We have the following lemma whose proof is left to the reader.

Lemma 3.3.1. *We have*

$$q_N = \frac{1}{2} - \frac{\sigma}{4}\sqrt{\frac{T}{N}} + o(\frac{1}{\sqrt{N}})$$
$$1 - q_N = \frac{1}{2} + \frac{\sigma}{4}\sqrt{\frac{T}{N}} + o(\frac{1}{\sqrt{N}}),$$

where the notation $o(\frac{1}{\sqrt{N}})$ stands for a term which goes ot 0 faster than $\frac{1}{\sqrt{N}}$.

Lemma 3.3.2. *Under the risk-neutral measure Q_N we have*

$$E_{Q_N}[Y_j] = -\frac{\sigma}{2}\sqrt{\frac{T}{N}} + o\left(\frac{1}{\sqrt{N}}\right),$$
$$\mathsf{Var}_{Q_N}[Y_j] = 1 + o\left(\frac{1}{\sqrt{N}}\right).$$

Proof.

$$E_{Q_N}[Y_j] = q_N 1 + (1 - q_N)(-1) = \frac{1}{2} - \frac{\sigma}{4}\sqrt{\frac{T}{N}} - \frac{1}{2} - \frac{\sigma}{4}\sqrt{\frac{T}{N}} = -\frac{\sigma}{2}\sqrt{\frac{T}{N}} + o\left(\frac{1}{\sqrt{N}}\right).$$

For the variance we have

$$\mathsf{Var}_{Q_N}[Y_j] = E_{Q_N}[Y_j^2] - (EQ_N[Y_j])^2 = 1 - (\frac{\sigma}{2}\sqrt{\frac{T}{N}} + o\left(\frac{1}{\sqrt{N}}\right))^2 = 1 + o(\frac{1}{\sqrt{N}}).$$

□

We now apply the following version of the Central Limit Theorem which we state without proof:

Theorem 3.3.3. *Suppose for each $N \in \mathbb{N}$ we are given N independent random variables $X_1^{(N)}, \dots, X_N^{(N)}$ on $(\Omega_N, \mathcal{F}_N, Q_N)$ which satisfy the following conditions*

- *There exist constants* γ_N *such that* $\gamma_N \to 0$ *and* $|X_k^{(N)}| \leq \gamma_N$*;*
- $\sum_{k=1}^N E_{Q_N}[X_k^{(N)}] \to m$*;*
- $\sum_{k=1}^N Var_{Q_N}[X_k^{(N)}] \to \sigma^2$*.*

Then the distribution of

$$\sum_{k=1}^N X_k^{(N)},$$

converges weakly to the normal distribution with mean m *and variance* σ^2*.*

Theorem 3.3.4. *Under the above assumptions, the distribution of* $S_T^{(N)} = S_{\frac{TN}{N}}^{(N)} = S_0(1+b_N)^{Z_N}$ *under* Q_N *converge weakly to the log-normal distribution with parameters* $\log S_0 - \frac{1}{2}\sigma^2 T$ *and* $\sigma^2 T$*, i.e. to the distribution of*

$$S_T = S_0 \exp(\sigma\sqrt{T}Y - \frac{1}{2}\sigma^2 T), \tag{3.5}$$

where Y *is standard normally distributed.*

Proof. We apply the above CLT to the random variable $X_i^{(N)} = \frac{Y_i}{\sqrt{N}}$. Then we have $|X_k^{(N)}| = |\frac{Y_i}{\sqrt{N}}| \leq \frac{1}{\sqrt{N}}$. Moreover,

$$\sum_{k=1}^N E_{Q_N}[\frac{Y_k}{\sqrt{N}}] = NE_{Q_N}[\frac{Y_k}{\sqrt{N}}] = \sqrt{N}E_{Q_N}[Y_k] = -\frac{\sigma}{2}\sqrt{T} + o(1)$$

and for the variance we have due to the independence

$$\sum_{k=1}^N \mathsf{Var}_{Q_N}[\frac{Y_k}{\sqrt{N}}] = N\mathsf{Var}_{Q_N}[\frac{Y_k}{\sqrt{N}}] = N\frac{1}{N}\mathsf{Var}_{Q_N}[Y_k] = 1 + o(\frac{1}{\sqrt{N}}).$$

Therefore

$$\frac{Z_N}{\sqrt{N}} = \sum_{i=1}^N \frac{Y_i}{\sqrt{N}}$$

converges weakly to the normal distribution with mean $\frac{\sigma}{2}\sqrt{T}$ and variance 1. Then in turn $S_T^{(N)} = S_0(e^{\sigma\sqrt{\frac{T}{N}}})^{Z_N}$ converges weakly to

$$S_T = S_0 e^{\sigma\sqrt{T}(Y - \frac{\sigma}{2}\sqrt{T})}$$

where Y is a standard normal distributed random variable. This proves the assertion. □

3.3.1 Derivative pricing and limits

Consider a payoff $C^{(N)} = f(S_T^{(N)})$. Then the following corollary allows to obtain its price in the Black & Scholes model as limit of the arbitrage-free prices in the Binomial model.

Corollary 3.3.5. *Let f be continuous and bounded, then the arbitrage-free price of the derivative with payoff $C^{(N)}$ computed as expectation under the risk neutral measure Q_N converges to the expectation under a log-normal distribution, which is often called the Black & Scholes price. More precisely,*

$$\lim_{N\to\infty} E_{Q_N}\left[C^{(N)}\right] = E_Q[f(S_T)] \qquad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(S_0 e^{\sigma\sqrt{T}y-\sigma^2\frac{T}{2}})e^{\frac{y^2}{2}}\,dy,$$

where S_T has the form (3.5) *under Q.*

Proof. This is a consequence of the definition of weak convergence and the continuous mapping theorem. □

This result applies in particular to the bounded payoff of a put $(K-x)^+$. The *Put-Call Parity* relation then allows to transfer the convergence also to the call, which then yields the famous *Black & Scholes* formula. We refer to [2, Example 5.57] and the discussions below.

Chapter 4

American Options

This chapter is based to a great extend on [2, Section 6.1 and 6.2].

4.1 Pricing and Hedging from the perspective of the seller

We refer to [2, Section 6.1].

4.2 Stopping strategies for the buyer

This section is based on [2, Section 6.2].

As in [2, Section 6.1] we assume that the model is a complete model of a financial market in discrete time with finite time horizon T and martingale measure Q. We consider here the perspective of the buyer. Her goal is to optimize the exercise strategy. The natural assumption is that the decision to exercise the option only depends on the information at time t (modeled via the filtration $(\mathcal{F})_t$). This leads to the concept of a stopping time.

Definition 4.2.1. *A random variable $\tau : \Omega \to \{0, 1, \ldots, T\} \cup \{\infty\}$ is called* stopping time *if $\{\tau = t\}$ is $\mathcal{F}_t$-measurable for $t = 0, \ldots, T$.*

The buyer's problem is to find the optimal exercise time within the following set of stopping times:

$$\mathcal{T} := \{\tau \,|\, \tau \text{ is a stopping time with } \tau \leq T\},$$

i.e. all stopping times which do not take the value ∞.

Let (f_t) be the payoff of the American option. Then the buyer's goal is to choose τ such that

$$E_Q[f_\tau] \text{ maximal among all } \tau \in \mathcal{T}.$$

This is called *optimal stopping problem.* The stopping time being a solution of such an optimal stopping problem is called optimal, which is precisely defined below.

Definition 4.2.2. *A stopping time τ^* is called* optimal *for an American option with payoff (f_t) if*

$$E[f_{\tau^*}] = \sup_{\tau \in \mathcal{T}} E_Q[f_\tau].$$

We are now interested in finding optimal stopping times. Denote the Snell envelope of (f_t) by (V_t) and define the stopping time

$$\tau_{\min} := \min\{t \geq 0 \,|\, V_t = f_t\}.$$

Note that $\tau_{\min} \leq T$ since $V_T = f_T$. In the theorem below we prove that $\tau_{\min}$ is optimal. For the formulation of the theorem let us introduce

$$\tau_{\min}^{(t)} := \min\{u \geq t \,| V_u = f_u\}$$

which is a member of the set $\mathcal{T}_t := \{\tau \in \mathcal{T} \,|\, \tau \geq t\}$.

Theorem 4.2.3. *The Snell envelope V of f satisfies*

$$V_t = E_Q[f_{\tau_{min}^{(t)}} | \mathcal{F}_t] = \sup_{\tau \in \mathcal{T}_t} E_Q[f_\tau | \mathcal{F}_t].$$

In particular

$$V_0 = E_Q[f_{\tau_{min}}] = \sup_{\tau \in \mathcal{T}} E_Q[f_\tau],$$

i.e. τ_{min} is optimal.

Proof. Since V is a supermartingale, we know that for every stopping time τ, $(V_{t\wedge\tau})_t$ is a supermartingale. In particular, for $\tau \in \mathcal{T}_t$, we have

$$V_t = V_{t\wedge\tau} \geq E_Q[V_{T\wedge\tau} | \mathcal{F}_t] = E_Q[V_\tau | \mathcal{F}_t] \geq E_Q[f_\tau | \mathcal{F}_t].$$

Therefore, $V_t \geq \sup_{\tau \in \mathcal{T}_t} E_Q[f_\tau | \mathcal{F}_t]$. Since $E_Q[f_{\tau_{\min}^{(t)}} | \mathcal{F}_t] \leq \sup_{\tau \in \mathcal{T}_t} E_Q[f_\tau | \mathcal{F}_t]$ holds trivially, the theorem is proved if we can show that $V_t = E_Q[f_{\tau_{\min}^{(t)}} | \mathcal{F}_t]$. By the definition of $\tau_{\min}^{(t)}$, we have $f_{\tau_{\min}^{(t)}} = V_{\tau_{\min}^{(t)}}$. Thus this is implied by

$$V_t = E_Q[V_{\tau_{\min}^{(t)}} | \mathcal{F}_t]. \tag{4.1}$$

In order to prove (4.1), denote

$$V_s^{(t)} = V_{s\wedge\tau_{\min}^{(t)}}, \quad s \in [t, T].$$

On the set $\{t_{\min}^{(t)} > s\}$, $V_s > f_s$. Hence

$$\begin{aligned} V_s^{(t)} 1_{\{t_{\min}^{(t)} > s\}} &= V_s 1_{\{t_{\min}^{(t)} > s\}} = (f_s \vee E_Q[V_{s+1} | \mathcal{F}_s]) 1_{\{t_{\min}^{(t)} > s\}} \\ &= E_Q[V_{s+1} | \mathcal{F}_s] 1_{\{t_{\min}^{(t)} > s\}} = E_Q[V_{s+1}^{(t)} | \mathcal{F}_s] 1_{\{t_{\min}^{(t)} > s\}}. \end{aligned}$$

On the set $\{t^{(t)}_{\min} \le s\}$, we have $V^{(t)}_{s+1} = V_{t^{(t)}_{\min}} = V^{(t)}_s$. Hence

$$1_{\{t^{(t)}_{\min} \le s\}} V^{(t)}_s = E_Q[V^{(t)}_{s+1}|\mathcal{F}_s] 1_{\{t^{(t)}_{\min} \le s\}}.$$

Hence $V^{(t)}_s = E_Q[V^{(t)}_{s+1}|\mathcal{F}_s]$ is a martingale for $s \in [t, T-1]$ and in particular

$$E_Q[V_{\tau^{(t)}_{\min}}|\mathcal{F}_t] = E_Q[V^{(t)}_T|\mathcal{F}_t] = V^{(t)}_t = V_t.$$

which proves (4.1). □

Remark 4.2.4.

- The stopping time $\tau_{\min}$ is not the only optimal stopping time, but the minimal one.
- The above theorem states that the buyer can in fact meet the value of the seller's hedging portfolio and this happens if and only if the option is exercised at an optimal stopping time. In this sense the *arbitrage-free or fair price* of an American option with payoff f is given by V_0, where V is defined as the Snell envelope of f or equivalently by $\sup_{\tau \in \mathcal{T}} E_Q[f_\tau]$.

Let us now compare American claims with the corresponding European. In particular we are intersted in the relation of American and European Call or Put options. Let

$$V^E_t := E_Q[f_T|\mathcal{F}_t]$$

be the unique arbitrage free price of the European claim f_T.

Proposition 4.2.5. *Let V denote the Snell envelope of f. Then $V_t \ge V^E_t$. Moreover if $V_t \ge f_t$ then $V^E_t = V_t$.*

Proof. The first assertion follows from the supermartingale property.

$$V_t \ge E_Q[V_T|\mathcal{F}_t] = E_Q[f_T|\mathcal{F}_t] = V^E_t.$$

If for the martingale V_t we have $V^E_t \ge f_t$ then it also dominates V_t, since V is the smallest supermartingale dominating f. Thus $V^E = V$. □

Remark 4.2.6.

- The situation where $V^E_t \ge f$ occurs, when (f_t) is a submartingale. This happens if $f = g(S_t)$ for a convex function $g : \mathbb{R}^d \to \mathbb{R}_+$. Indeed, in this case we have by Jensen's inequality

$$V^E_t = E_Q[f_{t+1}|\mathcal{F}_t] = E_Q[g(S_{t+1})|\mathcal{F}_t] \ge g(E_Q[S_{t+1}|\mathcal{F}_t]) = g(S_t) = f_t.$$

- As $x \mapsto (x-K)^+$ and $x \mapsto (K-x)^+$ are convex, the American call and the American put price are equal to their European counterparts in the absence of interest rates. This changes when we have interest rates. In this case the price of the American call is still equal to the price of the European, but this does not hold true for the put any longer.

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