The Asymptotic Theory of Transaction Costs

Lecture Notes by
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1 Models on Finite Probability Spaces

In this section we consider a stock price process $S = (S_t)_{t=0}^T$ in finite, discrete time, based on and adapted to a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, where $\mathcal{F} = \mathcal{F}_T$). Similarly as in the introductory section 2 of [21] we want to develop the basic ideas of the duality theory in this technically easy setting. The extra ingredient will be the role of transaction costs. To avoid trivialities we assume $\mathbb{P}(\omega) > 0$, for all $\omega \in \Omega$.

To keep things simple we assume that there is only one stock. It takes strictly positive values, i.e., $S = (S_t)_{t=0}^T$ is an $\mathbb{R}_+$-valued process. In addition, there is a bond, denoted by $B = (B_t)_{t=0}^T$; by choosing B as numéraire we may assume that $B_t \equiv 1$.

Next we introduce transaction costs $\lambda \geq 0$: that is, the process $((1-\lambda)S_t, S_t)_{t=0}^T$ models the bid and ask price of the stock S respectively. Of course, we assume $\lambda < 1$ for obvious economic reasons.

We have chosen a very simple setting here. For a much more general framework we refer, e.g., to [41], [42], [43], [46] and [68]. These authors investigate the setting given by finitely many stocks $S^1, \dots, S^n$, where the prices $(\pi_{ij})_{1 \leq i,j \leq n}$ of exchanging stock j into stock i are general adapted processes. A good economic interpretation for this situation is the case of n currencies where the bid and ask prices $\pi_{i,j}$ and $\pi_{j,i}$ depend on the pair (i, j) of currencies.

Here we do not strive for this generality. We do this on the one hand for didactic reasons to keep things as non-technical as possible. On the other hand we shall mainly be interested in the asymptotic theory for $\lambda \to 0$, for which our present simple setting is more natural.

Definition 1.1. *For given* $S = (S_t)_{t=0}^T$ *and* $0 \leq \lambda < 1$, *we associate the process of* solvency cones

$$K_t = \left\{(\varphi_t^0, \varphi_t^1) \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^2) : \varphi_t^0 \geq \max(-\varphi_t^1 S_t, -\varphi_t^1(1-\lambda)S_t\right\} \quad (1)$$

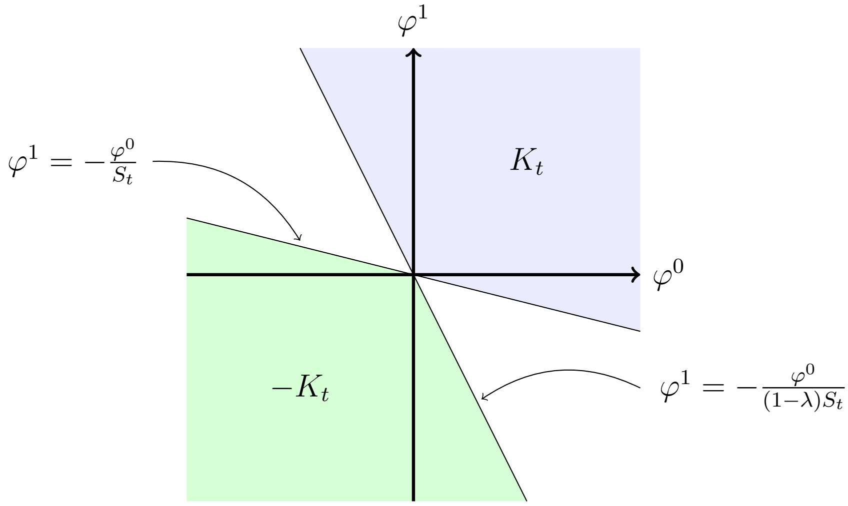


Figure 1: The solvency cone

The interpretation is that an economic agent holding φ_t^0 units of bond, and φ_t^1 units of stock is *solvent* for a given stock price S_t if, after liquidating the position in stocks, the resulting position in bonds is non-negative. "Liquidating the stock" means selling φ_t^1 stocks (at price $(1-\lambda)S_t$) if $\varphi_t^1 > 0$ and buying $-\varphi_t^1$ stocks (at price S_t) if $\varphi_t^1 < 0$.

Definition 1.2. *For given* $S = (S_t)_{t=0}^T$ *and* $0 \leq \lambda < 1$, *an adapted process* $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ *starting at* $(\varphi_{-1}^0, \varphi_{-1}^1) = (0,0)$ *is called* self-financing *if*

$$(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1) \in -K_t, \qquad t = 0, \ldots, T. \quad (2)$$

The relation (2) *is understood to hold* $\mathbb{P}$*-a.s., which in the present setting simply means: for each* $\omega \in \Omega$.

To motivate this definition note that the change at time t of positions in the portfolio $(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1)$ can be carried out for the bid-ask prices $\big((1-\lambda)S_t, S_t\big)$ iff $(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1) \in -K_t$.

For $(x^1, x^2) \in \mathbb{R}^2$, we call $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ self-financing and starting at (x^1, x^2) if $(\varphi_t^0 - x^1, \varphi_t^1 - x^2)_{t=-1}^T$ is self-financing and starting at $(0,0)$. We also note that it is natural in the context of transaction costs to allow for T trades (i.e. exchanges of bonds against stocks) in the T-period model $(S_t)_{t=0}^T$, which leads to the initial condition in terms of $(\varphi_{-1}^0, \varphi_{-1}^1)$.

Definition 1.3. *The process $S = (S_t)_{t=0}^T$ admits for* arbitrage under transaction costs $0 \leq \lambda < 1$ *if there is a self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$, starting at $\varphi_{-1}^0 = \varphi_{-1}^1 = 0$, and such that*

$$(\varphi_T^0, \varphi_T^1) \geq (0,0), \qquad \mathbb{P}\text{-}a.s.$$

and

$$\mathbb{P}\left[(\varphi_T^0, \varphi_T^1) > (0,0)\right] > 0.$$

We say that S satisfies the no arbitrage condition (NA^λ) if it does not allow for an arbitrage under transaction costs $0 \leq \lambda < 1$.

We introduce the following notation. For fixed S and $\lambda > 0$, denote by $\mathcal{A}^\lambda$ the set of $\mathbb{R}^2$-valued $\mathcal{F}$-measurable random variables (φ^0, φ^1), such that there exists a self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at $(\varphi_{-1}^0, \varphi_{-1}^1) = (0,0)$, and such that $(\varphi^0, \varphi^1) \leq (\varphi_T^0, \varphi_T^1)$.

Proposition 1.4. *Suppose that S satisfies (NA^λ), for fixed $0 \leq \lambda < 1$. Then $\mathcal{A}^\lambda$ is a closed polyhedral cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, containing $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_-)$ and such that $\mathcal{A}^\lambda \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_+) = \{0\}$.*

Proof: Fix $0 \leq t \leq T$ and an atom $F \in \mathcal{F}_t$. Recall that F is an *atom* of the finite sigma-algebra $\mathcal{F}_t$ if $E \in \mathcal{F}_t, E \subseteq F$ implies that either $E = F$ or $E = \emptyset$. Define the *ask* and *bid* elements a_F and b_F in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ as

$$a_F = \left(-S_{t|F}, 1\right) \mathbb{1}_F, \qquad b_F = \left((1-\lambda)S_{t|F}, -1\right) \mathbb{1}_F. \tag{3}$$

Note that $S_{t|F}$ is a well-defined positive number, as S_t is $\mathcal{F}_t$-measurable and F an atom of $\mathcal{F}_t$.

The elements a_F and b_F are in $\mathcal{A}^\lambda$. They correspond to the trading strategy $(\varphi_s^0, \varphi_s^1)_{s=-1}^T$ such that $(\varphi_s^0, \varphi_s^1) = (0,0)$, for $-1 \leq s < t$, and $(\varphi_s^0, \varphi_s^1)$ equals a_F (resp. b_F), for $t \leq s \leq T$. The interpretation is that an agent does nothing until time t. Then, conditionally on the event $F \in \mathcal{F}_t$, she buys (resp. sells) one unit of stock and holds it until time T.

Note that an element (φ^0, φ^1) in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ is in $\mathcal{A}^\lambda$ iff there are non-negative numbers $\mu_F \geq 0$ and $\nu_F \geq 0$, where F runs through the atoms of $\mathcal{F}_t$, and $t = 0, \ldots, T$, such that

$$(\varphi^0, \varphi^1) \leq \sum_F (\mu_F a_F + \nu_F b_F).$$

In other words, the elements of the form (3), together with the vectors $(-1,0)\mathbb{1}_\omega$ and $(0,-1)\mathbb{1}_\omega$, where ω runs through Ω, generate the cone $\mathcal{A}^\lambda$. It follows that $\mathcal{A}^\lambda$ is a closed polyhedral cone (see Appendix A).

The inclusion $\mathcal{A}^\lambda \supseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_-)$ is obvious, and (NA^λ) is tantamount to the assertion $\mathcal{A}^\lambda \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_+) = \{0\}$. ■

Definition 1.5. *An element* $(\varphi^0, \varphi^1) \in \mathcal{A}^\lambda$ *is called* maximal *if, for* $((\varphi^0)', (\varphi^1)') \in \mathcal{A}^\lambda$ *satisfying* $((\varphi^0)', (\varphi^1)') \geq (\varphi^0, \varphi^1)$ *a.s., we have* $((\varphi^0)', (\varphi^1)') = (\varphi^0, \varphi^1)$ *a.s.*

Definition 1.6. *Fix the process* $S = (S_t)_{t=0}^T$ *and transaction costs* $0 \leq \lambda < 1$. *A* consistent price system *is a pair* $(\tilde{S}, Q)$*, such that* Q *is a probability measure on* Ω *equivalent to* $\mathbb{P}$*, and* $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ *is a martingale under* Q *taking its values in the bid-ask spread* $[(1-\lambda)S, S]$*, i.e.*

$$(1-\lambda)S_t \leq \tilde{S}_t \leq S_t, \qquad \mathbb{P}\text{-}a.s. \tag{4}$$

We denote by $\mathcal{S}^\lambda$ *the set of consistent price systems.*

Remark 1.7. *For* $\lambda = 0$ *we have* $\tilde{S} = S$*, so that we find the classical equivalent martingale measures* $Q \in \mathcal{M}^e$*. We shall see that the set of real numbers* $\mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T]$*, where* $(\tilde{S}, Q)$ *ranges in* $\mathcal{S}^\lambda$*, yields precisely the arbitrage-free prices (in terms of bonds) for the contingent claim* $(\varphi^0_T, \varphi^1_T) \in L^\infty(\mathbb{R}^2)$*.*

Theorem 1.8. *(Fundamental Theorem of Asset Pricing): Fixing the process* $S = (S_t)_{t=0}^T$ *and transaction costs* $0 \leq \lambda < 1$*, the following are equivalent:*
(i) *The no arbitrage condition* (NA^λ) *is satisfied.*
(ii) *There is a consistent price system* $(\tilde{S}, Q) \in \mathcal{S}^\lambda$*.*
(iii) *There is an* $\mathbb{R}^2$*-valued* $\mathbb{P}$*-martingale* $(Z_t)_{t=0}^T = (Z_t^0, Z_t^1)_{t=0}^T$ *such that* $Z_t^0 > 0, Z_t^1 > 0$*, and*

$$\frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t], \qquad t = 0, \ldots, T. \tag{5}$$

Remark 1.9. *The basic idea of the above version of the Fundamental Theorem of Asset Pricing goes back to the paper [38] of Jouini and Kallal from 1995. The present version dealing with finite probability space* Ω *is due to Kabanov and Stricker [44].*

In the case $\lambda = 0$ *condition (iii) allows for the following interpretation: in this case (5) means that*

$$Z_t^1 = Z_t^0 S_t. \tag{6}$$

Interpret Z_T^0 *as a probability measure by letting* $\frac{dQ}{d\mathbb{P}} := Z_T^0$*, where we assume (without loss of generality) that* $Z_0^0 = \mathbb{E}_\mathbb{P}[Z_T^0] = 1$*. Condition (6) and the* $\mathbb{P}$*-martingale property of* Z^1 *then is tantamount to the assertion that* S *is a* Q*-martingale.*

Proof: $(ii) \Rightarrow (i)$ As usual in the context of the Fundamental Theorem of Asset Pricing, this is the easy implication, allowing for a rather obvious economic interpretation. Suppose that $(\tilde{S}, Q)$ is a consistent price system.

Let us first give the intuition: as the process $\tilde{S}$ is a martingale under Q, it is free of arbitrage (without transaction costs). Trading in S under transaction costs λ only allows for less favorable terms of trade than trading in $\tilde{S}$ without transaction costs (see (4)). Hence we find that S under transaction costs λ satisfies (NA^λ) *a fortiori.*

Here is the formalization of this economically obvious reasoning.

Note that, for every self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at $(\varphi_{-1}^0, \varphi_{-1}^1) = (0,0)$ we have

$$\begin{aligned}\varphi_t^0 - \varphi_{t-1}^0 &\leq \min\big(-(\varphi_t^1 - \varphi_{t-1}^1)S_t, -(\varphi_t^1 - \varphi_{t-1}^1)(1-\lambda)S_t\big) \\ &\leq -(\varphi_t^1 - \varphi_{t-1}^1)\tilde{S}_t,\end{aligned}$$

by (4), as $(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1) \in -K_t$.

Hence

$$\begin{aligned}(\varphi_T^0 - \varphi_{-1}^0) &= \sum_{t=0}^T (\varphi_t^0 - \varphi_{t-1}^0) \\ &\leq -\sum_{t=0}^T (\varphi_t^1 - \varphi_{t-1}^1)\tilde{S}_t \\ &= \sum_{t=1}^T \varphi_{t-1}^1(\tilde{S}_t - \tilde{S}_{t-1}) + \varphi_{-1}^1\tilde{S}_0 - \varphi_T^1\tilde{S}_T.\end{aligned}$$

Taking expectations under Q, and using that $\varphi_{-1}^0 = \varphi_{-1}^1 = 0$, we get

$$\mathbb{E}_Q[\varphi_T^0] + \mathbb{E}_Q[\varphi_T^1\tilde{S}_T] \leq \mathbb{E}_Q\left[\sum_{t=1}^T \varphi_{t-1}^1(\tilde{S}_t - \tilde{S}_{t-1})\right] = 0. \tag{7}$$

Now suppose that $\varphi_T^0 \geq 0$ and $\varphi_T^1 \geq 0$, $\mathbb{P}$-a.s., i.e. $\varphi_T^0(\omega) \geq 0$ and $\varphi_T^1(\omega) \geq 0$, for all ω in the finite probability space Ω.

Using the fact that Q is equivalent to $\mathbb{P}$, i.e. $Q(\omega) > 0$ for all $\omega \in \Omega$, we conclude from (7) that $\varphi_T^0(\omega) = 0$ and $\varphi_T^1(\omega)\tilde{S}_T(\omega) = 0$, for all $\omega \in \Omega$. Observing that $\tilde{S}_T$ is strictly positive by the assumption $\lambda < 1$, for each $\omega \in \Omega$, we also have $\varphi_T^1(\omega) = 0$ so that S satisfies (NA^λ).

$(i) \Rightarrow (iii)$ Now suppose that S satisfies (NA^λ). By Proposition 1.4 we know that $\mathcal{A}^\lambda$ is a closed convex cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ such that

$$\mathcal{A}^\lambda \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}_+^2) = \{0\}.$$

Claim: There is an element $Z = (Z^0, Z^1) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, verifying $Z^0(\omega) > 0$ and $Z^1(\omega) \geq 0$, for all $\omega \in \Omega$, and normalized by $\mathbb{E}[Z^0] = 1$, such that

$$\langle (\varphi_T^0, \varphi_T^1), (Z^0, Z^1) \rangle = \mathbb{E}_{\mathbb{P}}[\varphi_T^0 Z^0 + \varphi_T^1 Z^1] \leq 0, \quad \text{for all } (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda. \tag{8}$$

Indeed, fix $\omega \in \Omega$, and consider the element $(\mathbb{1}_\omega, 0) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ which is not an element of $\mathcal{A}^\lambda$.

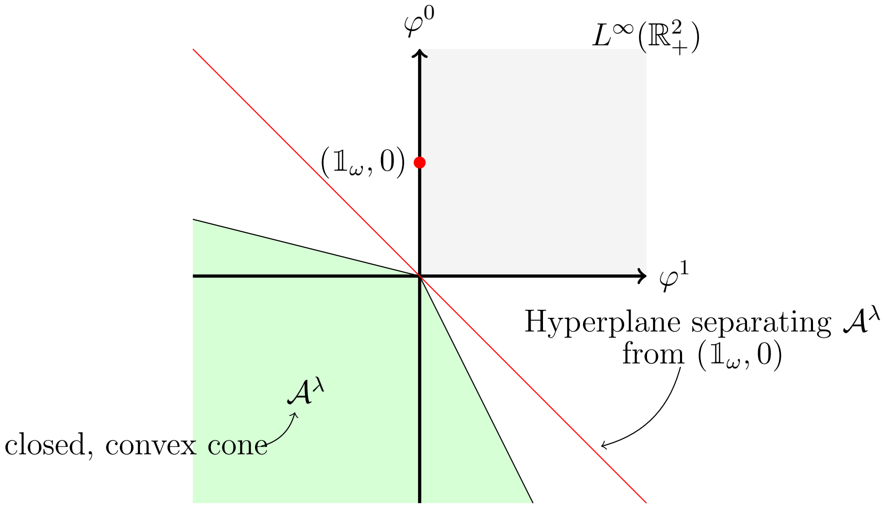


Figure 2: Regarding the proof of Thm 1.10

By Hahn-Banach and the fact that $\mathcal{A}^\lambda$ is closed and convex (Prop. 1.4), we may find, for fixed $\omega \in \Omega$, an element $Z_\omega \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ separating $\mathcal{A}^\lambda$ from $(\mathbb{1}_\omega, 0)$. As $\mathcal{A}^\lambda$ is a cone, we may find Z_ω such that

$$\langle (\mathbb{1}_\omega, 0), (Z_\omega^0, Z_\omega^1) \rangle > 0,$$

while

$$Z_{\omega | \mathcal{A}^\lambda} \leq 0.$$

The first inequality simply means that the element $Z_\omega^0 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ takes a strictly positive value on ω, i.e.

$$Z_\omega^0(\omega) > 0.$$

As $\mathcal{A}^\lambda$ contains the negative orthant $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; -\mathbb{R}^2_+)$, the second inequality implies that

$$Z_\omega^0 \geq 0, \qquad Z_\omega^1 \geq 0.$$

Doing this construction for each $\omega \in \Omega$ and defining

$$Z = \sum_{\omega \in \Omega} \mu_\omega Z_\omega,$$

where $(\mu_\omega)_{\omega \in \Omega}$ are arbitrary strictly positive scalars, we obtain an element $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ such that

$$Z_{|\mathcal{A}^\lambda} \leq 0, \tag{9}$$

which is tantamount to (8), and

$$Z^0 > 0, \quad Z^1 \geq 0,$$

proving the claim.

We associate to Z the $\mathbb{R}^2$-valued martingale $(Z_t)_{t=0}^T$ by

$$Z_t = \mathbb{E}[Z|\mathcal{F}_t], \qquad\qquad t = 0, \dots, T.$$

We have to show that $\frac{Z_t^1}{Z_t^0}$ takes its values in the bid-ask-spread $[(1-\lambda)S_t, S_t]$ of S_t. Applying (9) to the element a_F defined in (3), for an atom $F \in \mathcal{F}_t$, we obtain

$$\left\langle (Z_t^0, Z_t^1), (-S_{t|F}, 1)\mathbb{1}_F \right\rangle = \mathbb{E}\left[(-S_{t|F} Z_t^0 + Z_t^1)\mathbb{1}_F\right] \leq 0.$$

In the last line we have used the $\mathcal{F}_t$-measurability of $S_t \mathbb{1}_F$ to conclude that $0 \geq \mathbb{E}[(-S_{t|F} Z_t^0 + Z_t^1)\mathbb{1}_F] = \mathbb{E}[(-S_{t|F} Z_{t|F}^0 + Z_{t|F}^1)\mathbb{1}_F]$. As $S_{t|F}, Z_{t|F}^0$, and $Z_{t|F}^1$ are constants, we conclude that

$$-S_{t|F} Z_{t|F}^0 + Z_{t|F}^1 \leq 0,$$

i.e.

$$\frac{Z_{t|F}^1}{Z_{t|F}^0} \leq S_{t|F}.$$

As this inequality holds true for each $t = 0, \dots, T$ and each atom $F \in \mathcal{F}_t$ we have shown that

$$\frac{Z_t^1}{Z_t^0} \in \left]-\infty, S_t\right] \qquad\qquad t = 0, \dots, T.$$

Applying the above argument to the element b_F in (3) instead of to a_F we obtain

$$\frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, \infty[, \qquad\qquad t = 0, \dots, T, \tag{10}$$

which yields (5).

Finally we obtain from (10), and the fact that $\lambda < 1$, that $(Z_t^1)_{t=0}^T$ also takes strictly positive values.

$(iii) \Rightarrow (ii)$ Defining the measure Q on $\mathcal{F}$ by

$$\frac{dQ}{d\mathbb{P}} = \frac{Z^0}{\mathbb{E}[Z^0]}$$

we obtain a probability measure equivalent to $\mathbb{P}$.

Define the process $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ by

$$\tilde{S}_t = \frac{Z_t^1}{Z_t^0}.$$

By (5) the process $\tilde{S}$ takes its values in the bid-ask-spread of S. To verify that $\tilde{S}$ is a Q-martingale it suffices to note that this property is equivalent to $\tilde{S}Z^0$ being a $\mathbb{P}$-martingale. As $Z^1 = \tilde{S}Z^0$ this is indeed the case. ■

We denote by $\mathcal{B}^\lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ the *polar* of $\mathcal{A}^\lambda$ (see Definition A.3 in the Appendix), i.e.

$$\mathcal{B}^\lambda := (\mathcal{A}^\lambda)^\circ = \{Z = (Z^0, Z^1) : \langle(\varphi^0, \varphi^1), (Z^0, Z^1)\rangle = \mathbb{E}_\mathbb{P}\left[\varphi^0 Z^0 + \varphi^1 Z^1\right] \leq 0, \\ \text{for all } (\varphi^0, \varphi^1) \in \mathcal{A}^\lambda\}.$$

As $\mathcal{A}^\lambda$ is a closed polyhedral cone in a finite-dimensional space, its polar $\mathcal{B}^\lambda$ is so too (Proposition A.3). As $\mathcal{A}^\lambda$ contains the negative orthant $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; -\mathbb{R}_+^2) = \{(\varphi_T^0, \varphi_T^1) : \varphi_T^0 \leq 0, \varphi_T^1 \leq 0\}$, we have that $\mathcal{B}^\lambda$ is contained in the positive orthant $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}_+^2)$.

Corollary 1.10. *Suppose that S satisfies (NA^λ) under transaction costs $0 \leq \lambda < 1$. Let $Z = (Z^0, Z^1) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ and associate to Z the martingale $Z_t = \mathbb{E}_\mathbb{P}[Z|\mathcal{F}_t]$, where $t = 0, \ldots, T$.*

Then $Z \in \mathcal{B}^\lambda$ iff $Z \geq 0$ and $\tilde{S}_t := \frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t]$ on $\{Z_t^0 \neq 0\}$ and $Z_t^1 = 0$ on $\{Z_t^0 = 0\}$, for every $t = 0, \ldots, T$.

Dually, an element $(\varphi^0, \varphi^1) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ is in $\mathcal{A}^\lambda$ iff for every consistent price system $(\tilde{S}, Q)$ we have

$$\mathbb{E}_Q[\varphi^0 + \varphi^1 \tilde{S}_T] \leq 0. \tag{11}$$

Proof: If $Z = (Z^0, Z^1)$ is in $\mathcal{B}^\lambda$ we have $Z \geq 0$ by the paragraph preceding the corollary. Repeating the argument in the proof of the Fundamental Theorem 1.8, conditionally on $\{Z_t^0 \neq 0\}$, we obtain that $\tilde{S}_t := \frac{Z_t^1}{Z_t^0}$ indeed takes values in the bid-ask interval $[(1-\lambda)S_t, S_t]$ on $\{Z_t^0 \neq 0\}$ for $t = 0, \ldots, T$.

As regards the set $\{Z^0_t = 0\}$ fix an atom $F_t \in \mathcal{F}_t$, with $F_t \subseteq \{Z^0_t = 0\}$. Observe again that $(-S_{t|F_t}, 1)\mathbb{1}_{F_t} \in \mathcal{A}^\lambda$ as in the preceding proof. As $Z \in (\mathcal{A}^\lambda)^\circ$ we get

$$\langle (-S_t, 1)\mathbb{1}_{F_t}, (0, Z^1_t) \rangle \leq 0,$$

which readily implies that Z^1_t also vanishes on F_t.

Conversely, if $Z = (Z^0, Z^1)$ satisfies $Z \geq 0$ and $\frac{Z^1_t}{Z^0_t} \in [(1-\lambda)S_t, S_t]$ (with the above caveat for the case $Z^0_t = 0$), we have that

$$\langle \mathbb{1}_{F_t}(-S_t, 1), (Z^0, Z^1) \rangle \leq 0,$$

and

$$\langle \mathbb{1}_{F_t}\left((1-\lambda)S_t, -1\right), (Z^0, Z^1) \rangle \leq 0,$$

for every atom $F_t \in \mathcal{F}_t$. As the elements on the left hand side generate the cone $\mathcal{A}^\lambda$ we conclude that $Z \in \mathcal{B}^\lambda$.

Let us now pass to the dual point of view. By the bipolar theorem (see Proposition A.1 in the appendix) and the fact that $\mathcal{A}^\lambda$ is closed and convex in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, we have $(\mathcal{A}^\lambda)^{\circ\circ} = (\mathcal{B}^\lambda)^\circ = \mathcal{A}^\lambda$. Hence $(\varphi^0_T, \varphi^1_T) \in \mathcal{A}^\lambda = (\mathcal{A}^\lambda)^{\circ\circ}$ iff for every $Z = (Z^0, Z^1) \in \mathcal{B}^\lambda$ we have

$$\mathbb{E}_{\mathbb{P}}[Z^0\varphi^0_T + Z^1\varphi^1_T] \leq 0. \tag{12}$$

This is equivalent to (11) if we have that Z^0 is strictly positive as in this case $\frac{dQ}{d\mathbb{P}} := Z^0/\mathbb{E}_{\mathbb{P}}[Z^0]$ and $\tilde{S}_t = E_{\mathbb{P}}[Z^1|\mathcal{F}_t]/\mathbb{E}_{\mathbb{P}}[Z^0|\mathcal{F}_t]$ well-defines a consistent price system.

In the case when Z^0 may also assume the value zero, a little extra care is needed to deduce (11) from (12). By assumption (NA^λ) and the Fundamental Theorem 1.8 we know that there is $\bar{Z} = (\bar{Z}^0, \bar{Z}^1) \in \mathcal{B}^\lambda$ such that $\bar{Z}^0$ and $\bar{Z}^1$ are strictly positive. Given an arbitrary $Z = (Z^0, Z^1) \in \mathcal{B}^\lambda$ and $\mu \in]0, 1]$ we have that the convex combination $\mu\bar{Z} + (1-\mu)Z$ is in $\mathcal{B}^\lambda$ and strictly positive. Hence we may deduce the validity of (12) from (11) for $\mu\bar{Z} + (1-\mu)Z$. Sending μ to zero we conclude that $(\varphi^0_T, \varphi^1_T) \in \mathcal{A}^\lambda$ iff (11) is satisfied, for all consistent price systems $(\tilde{S}, Q)$. ■

Corollary 1.11. *(Superreplication Theorem): Fix the process* $S = (S_t)^T_{t=0}$, *transaction costs* $0 \leq \lambda < 1$, *and suppose that* (NA^λ) *is satisfied. Let* $(\varphi^0, \varphi^1) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ *and* $(x^0, x^1) \in \mathbb{R}^2$ *be given. The following are equivalent.*

- *(i)* $(\varphi^0, \varphi^1) = (\varphi^0_T, \varphi^1_T)$ *for some self-financing trading strategy* $(\varphi^0_t, \varphi^1_t)^T_{t=0}$ *starting at* $(\varphi^0_{-1}, \varphi^1_{-1}) = (x^0, x^1)$.

(ii) $\mathbb{E}_Q[\varphi^0+\varphi^1\tilde{S}_T] \leq x^0+x^1\tilde{S}_0$, *for every consistent price system* $(\tilde{S}, Q) \in \mathcal{S}^\lambda$.

Proof: Condition (i) is equivalent to $(\varphi^0 - x^0, \varphi^1 - x^1)$ being in $\mathcal{A}^\lambda$. By Corollary 1.10 this is equivalent to the inequality

$$\mathbb{E}_Q\left[(\varphi^0 - x^0) + (\varphi^1 - x^1)\tilde{S}_T\right] \leq 0,$$

for every $(\tilde{S}, Q) \in \mathcal{S}^\lambda$ which in turn is tantamount to (ii). ■

We now specialize the above result, considering only the case of trading strategies $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at some $(\varphi_{-1}^0, \varphi_{-1}^1) = (x, 0)$, i.e. without initial holdings in stock. Similarly we require that at terminal time T the position in stock is liquidated, i.e., $(\varphi_T^0, \varphi_T^1)$ satisfies $\varphi_T^1 = 0$.

We call $\mathcal{C}^\lambda$ the cone of claims (in units of bonds), attainable from initial endowment $(0, 0)$:

$$\begin{aligned} \mathcal{C}^\lambda &= \{\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \text{ there is } (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda \text{ s.t. } \varphi_T^0 \geq \varphi^0, \varphi_T^1 \geq 0\} \\ &= \mathcal{A}^\lambda \cap L_0^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2). \end{aligned} \tag{13}$$

In the last line we denote by $L_0^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ the subspace of $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ formed by the elements (φ^0, φ^1) with $\varphi^1 = 0$. We may and shall identify $L_0^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ with $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

The present notation $\mathcal{C}^\lambda$ corresponds, for $\lambda = 0$, to the notation of [21], where the cone of contingent claims attainable at prize 0 (without transaction costs) is denoted by C.

By (13) and Proposition 1.4 we conclude that $\mathcal{C}^\lambda$ is a closed polyhedral cone. Using analogous notation as in [57], we denote by $\mathcal{D}^\lambda$ the polar of $\mathcal{C}^\lambda$. By elementary linear algebra we obtain from (13) the representation

$$\mathcal{D}^\lambda = \{Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : \text{ there is } Z = (Z^0, Z^1) \in \mathcal{B}^\lambda \text{ with } Y = Z^0\}, \tag{14}$$

which is a polyhedral cone in $L_+^1(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{M}^\lambda$ the probability measures in $\mathcal{D}^\lambda$, i.e.

$$\mathcal{M}^\lambda = \mathcal{D}^\lambda \cap \{Y : \|Y\|_1 = \mathbb{E}_\mathbb{P}[Y] = 1\}.$$

The Superreplication Theorem 1.11 now specializes into a very familiar form, if we start with initial endowment $(x, 0)$ consisting only of bonds, and liquidate all positions in stock at terminal date T.

Corollary 1.12. *(one-dimensional Superreplication Theorem): Fix the process* $S = (S_t)_{t=0}^T$*, transaction costs* $0 \leq \lambda < 1$*, and suppose that* (NA^λ) *is satisfied. Let* $\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *and* $x \in \mathbb{R}$ *be given. The following are equivalent.*
(i) $\varphi^0 - x$ *is in* $\mathcal{C}^\lambda$.
(ii) $\mathbb{E}_Q[\varphi^0] \leq x$*, for every* $Q \in \mathcal{M}^\lambda$.

Proof: Condition (i) is equivalent to $(\varphi^0 - x, 0)$ being in $\mathcal{A}^\lambda$. This in turn is equivalent to $\mathbb{E}_Q[\varphi^0 - x] \leq 0$, for every $Q \in \mathcal{M}^\lambda$, which is the same as (ii). ■

Formally, the above corollary is in perfect analogy to the superreplication theorem in the frictionless case (see, e.g., ([21], Th. 2.4.2)). The reader may wonder whether – in the context of this corollary – there is any difference at all between the frictionless and the transaction cost case.

In fact, there is a subtle difference: in the frictionless case the set $\mathcal{M} = \mathcal{M}^0$ of martingale probability measures Q for the process S has the following remarkable *concatenation property*: let $Q', Q'' \in \mathcal{M}$ and associate the density process $Y'_t = \mathbb{E}[\frac{dQ'}{d\mathbb{P}}|\mathcal{F}_t]$, and $Y''_t = \mathbb{E}[\frac{dQ''}{d\mathbb{P}}|\mathcal{F}_t]$. For a stopping time τ we define the concatenated process

$$Y_t = \begin{cases} Y'_t, & \text{for } 0 \leq t \leq \tau, \\ Y'_\tau \frac{Y''_t}{Y''_\tau} & \text{for } \tau \leq t \leq T. \end{cases} \tag{15}$$

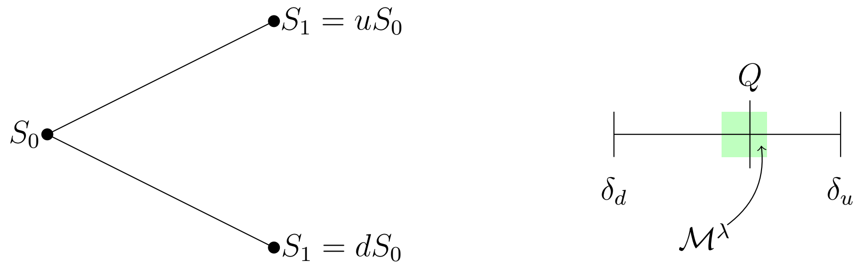
We then have that $\frac{dQ}{d\mathbb{P}} = Y_T$ again defines a probability measure under which S is a martingale, as one easily checks. This concatenation property turns out to be crucial for several aspects of the frictionless theory.

For $\lambda > 0$ the sets $\mathcal{M}^\lambda$ do not have this property any more. But apart from this drawback the sets $\mathcal{M}^\lambda$ share the properties of $\mathcal{M}$ of being a closed polyhedral subset of the simplex of probability measures on $(\Omega, \mathcal{F})$. Hence all the results pertaining only to the latter aspect, e.g. much of the duality theory, carry over from the frictionless to the transaction cost case, at least in the present setting of finite Ω. This applies in particular to the theory of utility maximization treated in the next section.

We end this section by illustrating the above result for two very elementary examples.

Example 1.13. *(One Period Binomial Model; for notation see, e.g., [21, Ex.3.3.1]):* In the traditional case, without transaction costs, we have which amounts to

$$\mathbb{E}_{\mathbb{Q}}[S_1] = S_0 = uS_0q_u + dS_0q_d = uS_0q_u + dS_0(1 - q_u) \tag{16}$$
$$1 = uq_u + d(1 - q_u), \tag{17}$$



or the well-known formulas for the risk less probability Q.

$$q_u = \tfrac{1-d}{u-d} \quad \text{and} \quad q_d = 1 - q_u = \tfrac{u-1}{u-d}. \tag{18}$$

Introducing proportional transaction costs, we are looking for a consistent price system $(\tilde{S}, Q)$, where $\tilde{S}$ is a Q-martingale and

$$(1-\lambda)S_t \leq \tilde{S}_t \leq S_t, \quad t \in \{0,1\}. \tag{19}$$

We therefore have:

$$\mathbb{E}_Q[\tilde{S}_1] = \underbrace{\tilde{S}_0}_{\geq (1-\lambda)S_0} = q_u \tilde{S}_1(u) + q_d \tilde{S}_1(d) \leq q_u u S_0 + q_d d S_0, \tag{20}$$

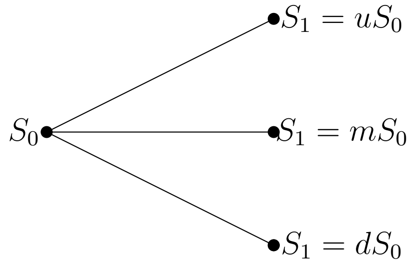
and therefore $q_u u + q_d d \geq (1-\lambda)$. Using analogue inequalities in the other direction and the fact that $q_u = 1 - q_d$ we obtain by elementary calculations lower and upper bounds for q_u:

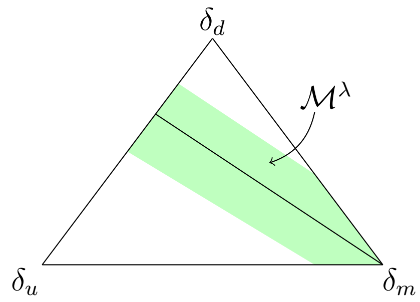
$$\max\left(\frac{\frac{1-\lambda}{1} - d}{u-d}, 0\right) \leq q_u \leq \min\left(\frac{\frac{1}{1-\lambda} - d}{u-d}, 1\right). \tag{21}$$

For $\lambda \mapsto 0$, this interval shrinks to the point $q_u = \frac{1-d}{u-d}$ which is the unique frictionless probability (18). For λ sufficiently close to 1, this interval equals $[0,1]$, i.e. $\mathcal{M}^\lambda$ consists of all convex, combinations of the Dirac measures δ_d and δ_u. In an intermediate range of λ, the set $\mathcal{M}^\lambda$ is an interval containing the measure $Q = q_u\delta_u + q_d\delta_d$ in its interior (see the above sketch).

Example 1.14. *(One period trinomial model):*

In this example (compare [21, Ex.3.3.4]) we consider three possible values for S_1: apart from the possibilities $S_1 = uS_0$ and $S_1 = dS_0$, where again $0 < d < 1 < u$, we also allow for an intermediate case $S_1 = mS_0$. For notational simplicity we let $m = 1$.





In the frictionless case we have, similar to the binomial model, for any martingale measure Q, that
which amounts to

$$\begin{aligned} \mathbb{E}_Q[S_1] = S_0 &= uS_0 q_u + S_0 q_m + dS_0 q_d && (22)\\ 1 &= u q_u + d q_d + (1 - q_u - q_d), && (23) \end{aligned}$$

which reduces one degree of freedom among all probabilities (q_u, q_m, q_d), for the cases of an *up, medium* or *down* movement of S_0. The corresponding set $\mathcal{M}$ of martingale measures for S in the set of convex combinations of the Dirac measures $\{\delta_u, \delta_m, \delta_q\}$ therefore is determined by the triples (q_u, q_m, q_d) of non-negative numbers where $0 \leq q_m \leq 1$ is arbitrary and where q_u and q_d are determined via

$$q_u + q_d = 1 - q_m, \qquad (u-1)q_u + (d-1)q_d = 0. \tag{24}$$

This corresponds to the line through δ_m in the above sketch. We now introduce transaction costs $0 \leq \lambda \leq 1$, and look for the set of consistent probability measures. In analogy to (20) we obtain the inequalities:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\tilde{S}_1] = \overbrace{\tilde{S}_0}^{\geq(1-\lambda)S_0} &= q_u \tilde{S}_1(u) + q_m \tilde{S}_1(m) + q_d \tilde{S}_1(d) && (25)\\ &\leq \left[q_u u S_0 + q_m S_0 + q_d d S_0\right]. && (26) \end{aligned}$$

Together with the other direction this gives us again a lower and upper bound:

$$-\lambda \leq q_u(u-1) + q_d(d-1) \leq \frac{\lambda}{1-\lambda}. \tag{27}$$

Hence $\mathcal{M}^\lambda$ is given by the shaded area in the above sketch which is confined by two lines, parallel to the line given by (24).