4 General Duality Theory

In this section we continue the line of research of Section 2 where we have refrained ourselves to the case of finite Ω.

We now consider a stock price process $S = (S_t)_{0 \leqslant t \leqslant T}$ in continuous time with a fixed horizon T. The process is assumed to be based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$, satisfying the usual conditions of completeness and right continuity. We assume that S is adapted and has *continuous*, strictly positive trajectories, i.e. the function $t \to S_t(\omega)$ is continuous, for almost each $\omega \in \Omega$. The extension to the case of càdlàg (right continuous, left limits) processes is more technical and we refer the reader to [18] for a thorough treatment.

To make life easier, we even assume that the filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ is generated by a d-dimensional Brownian motion $(W_t)_{0 \leqslant t \leqslant T}$. This convenient (but not really necessary, see [21]) assumption eases the presentation as it has the following pleasant consequence: if $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ is a local martingale under some measure $Q \sim \mathbb{P}$, then $\tilde{S}$ has $\mathbb{P}$-a.s. continuous paths.

Definition 4.1. *Fix* $\lambda > 0$. *A process* $S = (S_t)_{0 \leqslant t \leqslant T}$ *as above satisfies the condition* (CPS^λ) *of* having a consistent price system under transaction costs $\lambda > 0$, *if there is a process* $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$, *adapted to* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$ *such that*

$$(1 - \lambda) S_t \leqslant \tilde{S}_t \leqslant S_t, \qquad 0 \leqslant t \leqslant T,$$

as well as a probability measure Q *on* $\mathcal{F}$, *equivalent to* $\mathbb{P}$, *such that* $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ *is a local martingale under* Q.

We say that S admits consistent price systems for arbitrarily small transaction costs *if* (CPS^λ) *is satisfied, for all* $\lambda > 0$.

As in section 1 we observe that a λ-consistent price system can also be written as a pair $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$, where now Z^0 is a $\mathbb{P}$-martingale and Z^1 a local $\mathbb{P}$-martingale. The identification again is given by the formulas $Z_T^0 = \frac{dQ}{d\mathbb{P}}$ and $\tilde{S} = \frac{Z^1}{Z^0}$.

In [41] we related the condition of *admitting consistent price systems* for arbitrarily small transaction costs to a *no arbitrage condition* under arbitrarily small transaction costs, thus proving a version of the Fundamental Theorem of Asset Pricing under (small) transaction costs.

It is important to note that we *do not assume* that S is a semi-martingale as one is forced to do in the frictionless theory [26, Theorem 7.2]. However, the process $\tilde{S}$ appearing in Definition 4.1 always is a semi-martingale, as it becomes a local martingale after passing to an equivalent measure Q.

The notion of *self-financing trading strategies* $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$, *starting at* $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0,0)$ as well as the notion of *admissibility* have been given in Definition 3.8. For the convenience of the reader we recall it.

Definition 4.2. *Fix a stock price process* $S = (S_t)_{0 \leqslant t \leqslant T}$ *with continuous paths, as well as transaction costs* $\lambda > 0$.

A self-financing trading strategy starting with zero endowment *is a pair of right continuous, adapted finite variation processes* $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ *such that*
(i) $\varphi_{0-}^0 = \varphi_{0-}^1 = 0$
(ii) Denoting by $\varphi_t^0 = \varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}$ *and* $\varphi_t^1 = \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow}$, *the canonical decompositions of* φ^0 *and* φ^1 *into the difference of two increasing processes, starting at* $\varphi_{0-}^{0,\uparrow} = \varphi_{0-}^{0,\downarrow} = \varphi_{0-}^{1,\uparrow} = \varphi_{0-}^{1,\downarrow} = 0$, *these processes satisfy*

$$d\varphi_t^{0,\uparrow} \leqslant (1-\lambda) S_t d\varphi_t^{1,\downarrow}, \quad d\varphi_t^{0,\downarrow} \geqslant S_t d\varphi_t^{1,\uparrow}, \quad 0 \leqslant t \leqslant T. \tag{153}$$

The trading strategy $\varphi = (\varphi^0, \varphi^1)$ *is called* admissible *if there is* $M > 0$ *such that the liquidation value* V_t^{liq} *satisfies*

$$V_t^{liq}(\varphi^0, \varphi^1) := \varphi_t^0 + (\varphi_t^1)^+ (1-\lambda) S_t - (\varphi_t^1)^- S_t \geqslant -M, \tag{154}$$

a.s., for $0 \leqslant t \leqslant T$.

Remark 4.3. (1) We have chosen to define the trading strategies by explicitly specifying both accounts, the holdings in bond φ^0 as well as the holdings in stock φ^1. It would be sufficient to only specify one of the holdings, e.g. the number of stocks φ^1. Given a (right continuous, adapted) finite variation process $\varphi^1 = (\varphi_t^1)_{0 \leqslant t \leqslant T}$ starting at $\varphi_{0_-}^1 = 0$, which we canonically decompose as the difference $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$, we may define the process φ^0 by

$$d\varphi_t^0 = (1-\lambda) S_t d\varphi_t^{1,\downarrow} - S_t d\varphi_t^{1,\uparrow}.$$

The resulting pair (φ^0, φ^1) obviously satisfies (153) with equality holding true rather than inequality. However, it is convenient in (153) to consider trading strategies (φ^0, φ^1) which allow for an inequality, i.e. for "throwing away money". But it is clear from the preceding argument that we may always pass to a dominating pair (φ^0, φ^1) where equality holds true in (153).

We still note that we also might start from a (right continuous, adapted) process $(\varphi_t^0)_{0 \leqslant t \leqslant T} = (\varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow})_{0 \leqslant t \leqslant T}$ and define φ^1 via

$$d\varphi_t^1 = \frac{d\varphi_t^{0,\downarrow}}{S_t} - \frac{d\varphi_t^{0,\uparrow}}{(1-\lambda) S_t}.$$

(2) Now suppose that, in assumption (ii) above, the processes $\varphi^{0,\uparrow}, \varphi^{0,\downarrow}$, $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$ are right continuous, adapted, and starting at zero, but not necessarily the canonical decompositions of $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ (resp. $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$). In other words suppose that $\varphi^{0,\uparrow}$ and $\varphi^{0,\downarrow}$ (resp. $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$) may "move simultaneously". If the four processes satisfy the inequalities (153), then these inequalities are also satisfied for the canonical decompositions as one easily checks (and as is economically obvious). Summing up: in (ii) above the requirement that $\varphi^{0,\uparrow}, \varphi^{0,\downarrow}, \varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$ are the canonical decompositions could be dropped.

(3) We allow the finite variation process $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ to have jumps which we define to be of right continuous (i.e. càdlàg) type (note that a finite variation process automatically has left and right limits at every point $t \in [0,T]$). Unfortunately, we have a little problem[3] at $t = 0$. In fact, we have already encountered this problem in the discrete time setting in section 1 above. In order to model a possible (right continuous) jump at $t = 0$, we have to enlarge the time index set $[0,T]$ by adding the point 0_- which now takes the role of the point $t = -1$ in the discrete time setting of section 1. Hence whenever we write $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ we mean, strictly speaking, the process $(\varphi^0_t, \varphi^1_t)_{t \in \{0_-\} \cup [0,T]}$.

We could avoid the problem at $t = 0$ by passing to the *left* continuous modification $(\varphi^0_{t_-}, \varphi^1_{t_-})_{0 \leqslant t \leqslant T}$ where $(\varphi^0_{t_-}, \varphi^1_{t_-}) = \lim_{u \nearrow t} (\varphi^0_u, \varphi^1_u)$ denotes the left limits, for $0 < t \leqslant T$. In fact, this would be quite natural, as the adapted, càglàd (i.e. left continuous, right limits) process $(\varphi^0_{t_-}, \varphi^1_{t_-})_{0 \leqslant t \leqslant T}$ is *predictable*, while the càdlàg process $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ may in general fail to be predictable (it only is optional). In the general stochastic integration theory *predictable* processes are the natural class of integrands for general semi-martingales. However, this passage to the càglàd version shifts the "jump" problem at $t = 0$ to a similar problem at the end-point $t = T$, where we would be forced to add a point T_+ to $[0,T]$.

We have therefore decided to choose the càdlàg version $(\varphi^0_t, \varphi^1_t)_{t \in \{0_-\} \cup [0,T]}$ in the above definition for the following reasons:
(i) as long as we restrict ourself to the case of *continuous* processes $S = (S_t)_{0 \leqslant t \leqslant T}$, it does not make a difference whether we consider the integral $\int_0^T \varphi^1_t dS_t$ or $\int_0^T \varphi^1_{t_-} dS_t$,
(ii) most of the preceding literature uses the càdlàg versions $(\varphi^0_t, \varphi^1_t)$, and
(iii) the addition of a point T_+ to $[0,T]$ seems even more awkward than the addition of a point 0_-. We refer to [18] for a thorough discussion of these issues in the case of a general càdlàg process S.

[3] P. A. Meyer once observed that 0_- "plays the role of the devil" in stochastic integration theory.

(4) Finally, we observe for later use that in the definition (iii) of admissibility it does not matter whether we require (154), for all $0 \leqslant t \leqslant T$, or for all $[0, T]$-valued stopping times τ.

Similarly as in (3) the *simple strategies* are particularly easy cases.

Proposition 4.4. *Fix the continuous process S and $1 > \lambda > 0$. For a right continuous, adapted, finite variation process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ starting at $(\varphi_{0_-}^1, \varphi_{0_-}^0) = (0,0)$ we again denote by $\varphi_t^{0,\uparrow}, \varphi_t^{0,\downarrow}, \varphi_t^{1,\uparrow}, \varphi_t^{1,\downarrow}$ its canonical decomposition into differences of increasing processes.*

The following assertions are equivalent (in an almost sure sense):

(i) The process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ is self-financing, i.e.

$$d\varphi_t^0 \leqslant (1-\lambda) S_t d\varphi_t^{1,\downarrow} - S_t d\varphi_t^{1,\uparrow}, \quad \text{a.s. for } 0 \leqslant t \leqslant T. \tag{155}$$

(ii) For each pair of reals $0 \leqslant a < b \leqslant T$, as well as for $a = 0_-, b = 0$,

$$\varphi_b^{0,\uparrow} - \varphi_a^{0,\uparrow} \leqslant \int_a^b (1-\lambda) S_u d\varphi_u^{1,\downarrow}, \qquad \varphi_b^{0,\downarrow} - \varphi_a^{0,\downarrow} \geqslant \int_a^b S_u d\varphi_u^{1,\uparrow}. \tag{156}$$

(iii) For each pair of rationals $0 \leqslant a < b \leqslant T$, as well as for $a = 0_-$ and $b = 0$

$$\varphi_b^{0,\uparrow} - \varphi_a^{0,\uparrow} \leqslant (\varphi_b^{1,\downarrow} - \varphi_a^{1,\downarrow})(1-\lambda) \max_{a\leqslant u\leqslant b}\{S_u\}, \qquad \varphi_b^{0,\downarrow} - \varphi_a^{0,\downarrow} \geqslant (\varphi_b^{1,\uparrow} - \varphi_a^{1,\uparrow}) \min_{a\leqslant u\leqslant b}\{S_u\}. \tag{157}$$

Proof: $(i) \Leftrightarrow (ii)$: Inequality (155) states that the process

$$\left(\int_0^t [(1-\lambda) S_u d\varphi_u^{1,\downarrow} - S_u d\varphi_u^{1,\uparrow} - d\varphi_u^0]\right)_{0\leqslant t\leqslant T}$$

is non-decreasing; this statement is merely reformulated in (156). Note that the integrals in (156) make sense in a pointwise manner as Riemann-Stieltjes integrals.

$(ii) \Leftrightarrow (iii)$ We only have to proof $(iii) \Rightarrow (ii)$. Suppose that (ii) fails to be true, say,

$$\varphi_b^{0,\uparrow} - \varphi_a^{0,\uparrow} > \int_a^b (1-\lambda) S_u d\varphi_u^{1,\downarrow} + \delta(b-a)$$

for some real numbers $0 \leqslant a < b \leqslant T$ and $\delta > 0$ holds true with probability bigger than $\varepsilon > 0$. Then we can approximate a and b by rationals α, β such that the above inequality still holds true. Using the continuity of S we can break the integral $\int_\alpha^\beta$ into a sum of finitely many integrals $\int_{\alpha_i}^{\beta_i}$, with rational endpoints α_i, β_i, such that the oscillation of S on each $[\alpha_i, \beta_i]$ is smaller than $\delta/2$ on a set of probability bigger than $1 - \frac{\varepsilon}{2}$. Then (157) fails to hold true almost surely, for some pair (α_i, β_i). ■

Proposition 4.5. *Fix $S = (S_t)_{0 \leqslant t \leqslant T}$ and $\lambda > 0$ as above and suppose that (CPS^λ) holds true. Let $(\varphi^0, \varphi^1) = (\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ be a self-financing, admissible trading strategy, and $(\tilde{S}, Q)$ be a consistent price system.*

The process

$$\tilde{V}_t = \varphi^0_t + \varphi^1_t \tilde{S}_t, \qquad 0 \leqslant t \leqslant T \tag{158}$$

is a local Q-super-martingale which is uniformly bounded from below, and therefore a super-martingale.

Proof: As $(\varphi^1_t)_{0 \leqslant t \leqslant T}$ is of bounded variation and $\tilde{S}$ is continuous, the product rule applied to (158) yields

$$d\tilde{V}_t = d\varphi^0_t + \tilde{S}_t \, d\varphi^1_t + \varphi^1_t \, d\tilde{S}_t. \tag{159}$$

As $\tilde{S}$ takes values in $[(1-\lambda)S, S]$, we conclude from (155) that the process $(\int_0^t (d\varphi^0_u + \tilde{S}_u d\varphi^1_u))_{0 \leqslant t \leqslant T}$ is non-increasing. The second term in (159) defines the local Q-martingale $(\int_0^t \varphi^1_u d\tilde{S}_u)_{0 \leqslant t \leqslant T} = (\varphi^1 \cdot \tilde{S})_{0 \leqslant t \leqslant T}$. By (154) and the admissibility assumption, the process $\tilde{V}$ is uniformly bounded from below. It therefore is a super-martingale under Q. ■

Remark 4.6. The interpretation of the process $\tilde{V}$ is the value of the portfolio process (φ^0, φ^1) if we evaluate the position φ^1 in stock at price $\tilde{S}$. Note that $\tilde{V} \geqslant V^{\text{liq}}$, where V^{liq} is defined in (154).

Definition 4.7. *Let $S = (S_t)_{0 \leqslant t \leqslant T}$ and $1 > \lambda > 0$ be fixed as above.*

We denote by $\mathcal{A}$ the set of random variables $(\varphi^0_T, \varphi^1_T)$ in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ such that there is an admissible, self-financing, process $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$, as in Definition 4.2 starting at $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (0, 0)$, and ending at $(\varphi^0_T, \varphi^1_T)$.

We denote by $\mathcal{C}$ the set of random variables

$$\mathcal{C} = \{\varphi^0_T \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : (\varphi^0_T, 0) \in \mathcal{A}\}. \tag{160}$$

$$= \{V^{liq}(\varphi^0_T, \varphi^1_T) : (\varphi^0_T, \varphi^1_T) \in \mathcal{A}\} \tag{161}$$

We denote by $\mathcal{A}^M$, resp. $\mathcal{C}^M$ the corresponding subsets of M-admissible elements, i.e. for which there is a process $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ satisfying (154), for fixed $M > 0$.

Definition 4.8. *Fix S and $\lambda > 0$ as above, let $\tau : \Omega \to [0,T] \cup \{\infty\}$ be a stopping time, and let f_τ, g_τ be $\mathcal{F}_\tau$-measurable $\mathbb{R}_+$-valued functions. We define the corresponding* ask *and* bid *processes as the $\mathbb{R}^2$-valued processes*

$$a_t = (-S_\tau, 1) \qquad f_\tau \ \mathbb{1}_{[\![\tau,T]\!]}(t), \qquad 0 \leqslant t \leqslant T, \tag{162}$$

$$b_t = ((1-\lambda)S_\tau, -1) \quad g_\tau \ \mathbb{1}_{[\![\tau,T]\!]}(t), \qquad 0 \leqslant t \leqslant T. \tag{163}$$

We call a process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ a simple, self-financing *process, if it is a finite sum of ask and bid processes as above. Admissibility is defined as in Definition 4.2.*

The interpretation of a_t is the following: an investor does nothing until time τ and then decides to buy f_τ many stocks and to hold them until time T. The resulting holdings in bond and stock are $\varphi_t^0 = -S_\tau f_\tau \mathbb{1}_{[\![\tau,T]\!]}(t)$ and $\varphi_t^1 = f_\tau \mathbb{1}_{[\![\tau,T]\!]}(t)$. The case of b_t is analogous.

In the above definition we also allow for $\tau = 0$ in the above definition: this case models the trading between time $t = 0_-$ and time $t = 0$ at bid ask prices $\{(1-\lambda)S_0, S_0\}$. In this case we interpret the function $\mathbb{1}_{[\![0,T]\!]}$ as $\mathbb{1}_{[\![0,T]\!]}(0_-) = 0$, while $\mathbb{1}_{[\![0,T]\!]}(t) = 1$, for $0 \leqslant t \leqslant T$.

We denote by $\mathcal{A}^s$ the set of $\mathbb{R}^2$-valued random variables (φ^0, φ^1) such that there is a simple (see Definition 4.8, admissible, self-financing, process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ satisfying $(\varphi^0, \varphi^1) \leqslant (\varphi_T^0, \varphi_T^1)$.

Lemma 4.9. *Fix the continuous process S and $\lambda > 0$ as above. The set $\mathcal{A}^s$ is a convex cone in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ which is dense in $\mathcal{A}$ with respect to the topology of convergence in measure.*

More precisely, let $M > 0$ and $(\varphi^0, \varphi^1) = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ be a self-financing process as in Definition 4.7, starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0,0)$ which is M-admissible, i.e.

$$V_t(\varphi^0, \varphi^1) := \varphi_t^0 + (\varphi_t^1)^+(1-\lambda)S_t - (\varphi_t^1)^- S_t \geqslant -M, \qquad 0 \leqslant t \leqslant T.$$

Then there is a sequence $(\varphi^{0,n}, \varphi^{1,n})_{n=1}^\infty$ of simple, self-financing, M-admissible processes starting at $(\varphi_{0_-}^{0,n}, \varphi_{0_-}^{1,n}) = (0,0)$, such that $(\varphi_T^{0,n} \wedge \varphi_T^0, \varphi_T^{1,n} \wedge \varphi_T^1)$ converges to $(\varphi_T^0, \varphi_T^1)$ almost surely.

Proof: The idea of the approximation is simple: the strategy $(\varphi^{0,n}, \varphi^{1,n})$ does the same buying and selling operations as (φ^0, φ^1), but always waits until $(S_t)_{0\leqslant t\leqslant T}$ has moved by some $\delta > 0$; then the $(\varphi^{0,n}, \varphi^{1,n})$-strategy does the same buying/selling in one lump sum, which the strategy (φ^0, φ^1) has done during the preceding time interval. In this way the approximation $(\varphi^{0,n}, \varphi^{1,n})$ still is adapted to the filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ as it only uses past information; the terms of trade for the strategies (φ^0, φ^1) and $(\varphi^{0,n}, \varphi^{1,n})$ are close to each other, as the continuous process S has only moved by at most δ during the preceding (stochastic) time interval.

Here are the more formal details: fix the self-financing, M-admissible strategy (φ^0, φ^1) and $1 > \varepsilon > 0$. As (φ^0, φ^1) is of finite variation we may find a constant $V_\varepsilon > 1$ such that the probability of $(\varphi^0_t)_{0 \leqslant t \leqslant T}$ having total variation $\mathrm{Var}_T(\varphi^0)$ bigger than V_ε, has probability less than $\varepsilon > 0$.

Let σ be the stopping time

$$\sigma = \inf\{t \in [0, T] : \ \mathrm{Var}_t(\varphi^0) \geqslant V_\varepsilon\}, \tag{164}$$

so that $\mathbb{P}[\sigma < \infty] < \varepsilon$, and let $\delta = \min(\frac{\varepsilon}{V_\varepsilon}, \frac{\lambda}{3})$. Define a sequence of stopping times $(\tau_k)_{k=0}^{\infty}$ by $\tau_0 = 0$ and, for $k \geqslant 0$,

$$\tau_{k+1} = \inf\{t \in [\![\tau_k, T]\!] : \tfrac{S_t}{S_{\tau_k}} = (1 + \delta) \text{ or } (1 - \delta)\} \wedge \sigma, \tag{165}$$

where, as in (164), the inf over the empty set is infinity.

As the trajectories of $S = (S_t)_{0 \leqslant t \leqslant T}$ are continuous and strictly positive, the sequence $(\tau_k)_{k=0}^{\infty}$ increases to infinity a.s. on $\{\sigma = \infty\}$. Fix $K \in \mathbb{N}$ such that $\mathbb{P}[\tau_K < \infty] < 2\varepsilon$. Now construct inductively the approximating simple process $(\varphi^{0,n}, \varphi^{1,n})$, where $n \in \mathbb{N}$ will correspond to some $\varepsilon_n > 0$ and $\delta_n \leqslant \frac{\varepsilon_n}{V_{\varepsilon_n}}$ to be specified below.

At time $t = 0$ we observe that $(\varphi^0_0, \varphi^1_0)\mathbb{1}_{[\![0,T]\!]}(t)$ is the sum of the terms (162) and (163),i.e.

$$\begin{aligned}(\varphi^0_0, \varphi^1_0)\mathbb{1}_{[\![0,T]\!]}(t) &= a^0_t + b^0_t \\ &= ((-S_0, 1) f_{\tau_0} + ((1-\lambda)S_0, -1)\, g_{\tau_0})\, \mathbb{1}_{[\![0,T]\!]}(t),\end{aligned}$$

where $f_{\tau_0} = (\varphi^1_0 - \varphi^1_{0\_})^+ = (\varphi^1_0)^+$ and $g_{\tau_0} = (\varphi^1_0 - \varphi^1_{0\_})^- = (\varphi^1_0)^-$.

At time τ_1 we want to adjust our holdings in bond and stock to have $\varphi^{1,n}_{\tau_1} = \varphi^1_{\tau_1}$, i.e. that the holding in stock at time τ_1 are the same, for the strategy (φ^0, φ^1) and $(\varphi^{0,n}, \varphi^{1,n})$. This can be done by defining

$$a^1_t + b^1_t = [(-S_{\tau_1}, 1) f_{\tau_1} + ((1-\lambda)S_{\tau_1}, -1) g_{\tau_1}]\, \mathbb{1}_{[\![\tau_1,T]\!]}(t), \qquad 0 \leqslant t \leqslant T, \tag{166}$$

where $f_{\tau_1} = (\varphi^1_{\tau_1} - \varphi^1_{\tau_0})^+$ and $g_{\tau_1} = (\varphi^1_{\tau_1} - \varphi^1_{\tau_0})^-$, where $\tau_0 = 0$ so that $\varphi^1_{\tau_0} = \varphi^1_0$ (as opposed to $\varphi^1_{0\_}$). We add this process to $a^0_t + b^0_t$, i.e. we define

$$(\varphi^{0,n,1}_t, \varphi^{1,n,1}_t) = (a^0_t + b^0_t) + (a^1_t + b^1_t), \qquad 0 \leqslant t \leqslant T.$$

We then have that the process $(\varphi^{0,n,1}, \varphi^{1,n,1})$ jumps at times 0 and τ_1 only, and satisfies

$$\varphi^{1,n,1}_{\tau_1} = \varphi^1_{\tau_1}.$$

As regards the holdings $\varphi^{0,n,1}_{\tau_1}$ in bond at time τ_1, we cannot assert that $\varphi^{0,n,1}_{\tau_1} = \varphi^0_{\tau_1}$, but we are not far off the mark: speaking economically, the strategy (φ^0, φ^1) has changed the position in bond during the interval $[\![\tau_0, \tau_1]\!]$ from φ^0_0 to $\varphi^0_{\tau_1}$ by buying $(\varphi^1_{\tau_1} - \varphi^1_{\tau_0})_+$, resp. selling $(\varphi^1_{\tau_1} - \varphi^1_{\tau_0})_-$, numbers of stock. These are figures accumulated over the interval $[\![\tau_0, \tau_1]\!]$. As the stock price S_t is in the interval $[(1-\delta)S_0, (1+\delta)S_0]$ for $t \in]\!]\tau_0, \tau_1]\!]$ and $\delta < \frac{\lambda}{3}$, we may estimate by (155)

$$(\varphi^{0,n,1}_{\tau_1} - \varphi^{0,n,1}_{\tau_0}) - (\varphi^0_{\tau_1} - \varphi^0_{\tau_0}) = \varphi^{0,n,1}_{\tau_1} - \varphi^0_{\tau_1} \geqslant -3\delta \, |\varphi^0_{\tau_1} - \varphi^0_{\tau_0}|. \tag{167}$$

Now continue in an analogous way on the intervals $]\!]\tau_{k-1}, \tau_k]\!]$, for $k = 1, \ldots, K$, to find $a^k_t + b^k_t$ as in (166)

$$a^k_t + b^k_t = [(-S_{\tau_k}, 1) f_{\tau_k} + ((1-\lambda)S_{\tau_k}, -1) g_{\tau_k}] \, \mathbb{1}_{[\![\tau_k, T]\!]}(t), \qquad 0 \leqslant t \leqslant T, \tag{168}$$

so that the process

$$(\varphi^{0,n,k}_t, \varphi^{1,n,k}_t) = \sum_{j=0}^{k} (a^j_t + b^j_t), \qquad 0 \leqslant t \leqslant T,$$

satisfies $\varphi^{1,n,k}_{\tau_j} = \varphi^1_{\tau_j}$, for $j = 0, \ldots, k$, and

$$\varphi^{0,n,k}_{\tau_k} - \varphi^0_{\tau_k} \geqslant -3\delta \sum_{j=1}^{k} |\varphi^0_{\tau_j} - \varphi^0_{\tau_{j-1}}|. \tag{169}$$

Finally define the process $(\varphi^{0,n}, \varphi^{1,n}) := (\varphi^{0,n,K}, \varphi^{1,n,K})$.

We have not yet made precise what we do, when, for the first time $k = 1, \ldots, K$, we have $\tau_k = \infty$. In this case we interpret (168) by letting $\tau_k := T$ rather than $\tau_k = \infty$, i.e. as a final trade at time T, to make sure that $\varphi^{1,n,k}_T = \varphi^1_T$ on $\{\tau_k = \infty\}$.

Hence the process $(\varphi^{0,n}, \varphi^{1,n})$ is such that, on the set $\{\tau_K = \infty\}$, we have $\varphi^{1,n}_T = \varphi^1_T$ so that

$$\mathbb{P}[\varphi^{1,n}_T = \varphi^1_T] > 1 - 2\varepsilon. \tag{170}$$

By (169) we may also estimate on $\{\tau_K < \infty\} \subseteq \{\sigma < \infty\}$

$$\begin{aligned} \varphi^{0,n}_{\tau_K} - \varphi^0_{\tau_K} &\geqslant -3\delta \sum_{j=1}^{K} |\varphi^0_{\tau_j} - \varphi^0_{\tau_{j-1}}| \\ &\geqslant -3\delta \, [V_\varepsilon + 2\delta], \end{aligned}$$

so that

$$\mathbb{P}[\varphi^{0,n}_T \geqslant \varphi^0_T - 4\varepsilon] \geqslant 1 - 2\varepsilon. \tag{171}$$

As regards the admissibility of $(\varphi^{0,n}, \varphi^{1,n})$: this process is not yet M-admissible, but it is straightforward to check that it is $M + 3\delta(V_\varepsilon + 2\delta)$-admissible. Hence by multiplying $(\varphi^{0,n}, \varphi^{1,n})$ by the factor $c := \frac{M}{M+3\delta(V_\varepsilon+2\delta)}$ we obtain an M-admissible process $(c\varphi^{0,n}, c\varphi^{1,n})$ such that $(c\varphi_T^{0,n} \wedge \varphi_T^1, c\varphi_T^{1,n})$ is close to $(\varphi_T^0, \varphi_T^1)$ in probability.

Finally, we have to specify $\varepsilon = \varepsilon_n$: it now is clear that it will be sufficient to choose $\varepsilon_n = 2^{-n}$ in the above construction to obtain the a.s. convergence of $(\varphi_T^{0,n} \wedge \varphi_T^0, \varphi_T^{1,n})$ to $(\varphi_T^0, \varphi_T^1)$. ∎

The following lemma was proved by L. Campi and the author [11] in the more general framework of Kabanov's modeling of d-dimensional currency markets. Here we spell out the proof for a single risky asset model. In Definition 4.2 we postulated as a qualitative — a priori — assumption that the strategies (φ^0, φ^1) have *finite variation.* The next lemma provides — a posteriori — quantitative control on the size of the finite variation.

Lemma 4.10. *Let S and $\lambda > 0$ be as above, and suppose that $(CPS^{\lambda'})$ is satisfied, for some $0 < \lambda' < \lambda$, i.e., there is a consistent price system for transaction costs λ'. Fix $M > 0$. Then the total variation of the process $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ remains bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, when (φ^0, φ^1) runs through all M-admissible, self-financing strategies.*

More explicitly: for $M > 0$ and $\varepsilon > 0$, there is $C > 0$ such that, for all M-admissible, self-financing strategies (φ^0, φ^1), starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0, 0)$, and all partitions $0_- = t_0 < t_1 < \ldots < t_K = T$ we have

$$\mathbb{P}\left[\sum_{k=1}^{K} |\varphi_{t_k}^0 - \varphi_{t_{k-1}}^0| \ \geqslant C\right] < \varepsilon, \tag{172}$$

$$\mathbb{P}\left[\sum_{k=1}^{K} |\varphi_{t_k}^1 - \varphi_{t_{k-1}}^1| \ \geqslant C\right] < \varepsilon. \tag{173}$$

Proof: Fix $0 < \lambda' < \lambda$ as above. By the hypothesis $(CPS^{\lambda'})$ there is a probability measure $Q \sim \mathbb{P}$, and a local Q-martingale $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ such that $\tilde{S}_t \in [(1-\lambda')S_t, S_t]$.

Fix $M > 0$ and a self-financing (with respect to transaction costs λ), M-admissible process $(\varphi_t^0, \varphi_t^1)_{t \geqslant 0}$, starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0, 0)$. Write $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ as the canonical differences of increasing processes, as in Definition 4.2. We shall show that

$$\mathbb{E}_Q\left[\varphi_T^{0,\uparrow}\right] \leqslant \frac{M}{\lambda - \lambda'}. \tag{174}$$

Define the process $((\varphi^0)', (\varphi^1)')$ by

$$\left((\varphi^0)'_t, (\varphi^1)'_t\right) = \left(\varphi^0_t + \frac{\lambda - \lambda'}{1-\lambda}\varphi^{0,\uparrow}_t, \varphi^1_t\right), \qquad 0 \leqslant t \leqslant T.$$

This is a self-financing process under transaction costs λ': indeed, whenever $d\varphi^0_t > 0$ so that $d\varphi^0_t = d\varphi^{0,\uparrow}_t$, the agent sells stock and receives $d\varphi^{0,\uparrow}_t = (1-\lambda)S_t d\varphi^{1,\downarrow}_t$ (resp. $(1-\lambda')S_t d\varphi^{1,\downarrow}_t = \frac{1-\lambda'}{1-\lambda} d\varphi^{0,\uparrow}_t$) under transaction costs λ (resp. λ'). The difference between these two terms is $\frac{\lambda-\lambda'}{1-\lambda} d\varphi^{0,\uparrow}_t$; this is the amount by which the λ'-agent does better than the λ-agent. It is also clear that $((\varphi^0)', (\varphi^1)')$ under transaction costs λ' still is M-admissible.

By Proposition 4.5 the process $((\varphi^0)'_t + \varphi^1_t \tilde{S}_t)_{0 \leqslant t \leqslant T} = (\varphi^0_t + \frac{\lambda-\lambda'}{1-\lambda}\varphi^{0,\uparrow}_t + \varphi^1_t \tilde{S}_t)_{0 \leqslant t \leqslant T}$ is a Q-super-martingale. Hence $\mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T] + \frac{\lambda-\lambda'}{1-\lambda}\mathbb{E}_Q[\varphi^{0,\uparrow}_T] \leqslant 0$. As $\varphi^0_T + \varphi^1_T \tilde{S}_T \geqslant -M$ we have shown (174).

To obtain a control on $\varphi^{0,\downarrow}_T$ too, we may assume w.l.g. in the above reasoning that the strategy (φ^0, φ^1) is such that $\varphi^1_T = 0$, i.e. the position in stock is liquidated at time T. We then must have $\varphi^0_T \geqslant -M$ so that $\varphi^{0,\downarrow}_T \leqslant \varphi^{0,\uparrow}_T + M$. Therefore we obtain the following estimate for the total variation $\varphi^{0,\uparrow}_T + \varphi^{0,\downarrow}_T$ of φ^0

$$\mathbb{E}_Q\left[\varphi^{0,\uparrow}_T + \varphi^{0,\downarrow}_T\right] \leqslant M\left(\frac{2}{\lambda - \lambda'} + 1\right). \tag{175}$$

The passage from the $L^1(Q)$-estimate (175) to the $L^0(\mathbb{P})$-estimate (172) is standard: for $\varepsilon > 0$ there is $\delta > 0$ such that for a subset $a \in \mathcal{F}$ with $Q[A] < \delta$ we have $\mathbb{P}[A] < \varepsilon$. Letting $C = \frac{M}{\delta}(\frac{2}{\lambda-\lambda'} + 1)$ and applying Tschebyschoff to (175) we get

$$\mathbb{P}\left[\varphi^{0,\uparrow}_T + \varphi^{0,\downarrow}_T \geqslant C\right] < \varepsilon,$$

which implies (172).

As regards (173) we note that, by the continuity and strict positivity assumption on S, for $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\mathbb{P}\left[\inf_{0 \leqslant t \leqslant T} S_t < \delta\right] < \frac{\varepsilon}{2}.$$

Hence we may control $\varphi^{1,\uparrow}_T$ by using the second inequality in (157); then we can control $\varphi^{1,\downarrow}_T$ by a similar reasoning as above so that we obtain (173) for a suitably adapted constant C. ∎

Remark 4.11. In the above proof we have shown that the elements $\varphi_T^{0,\uparrow}$, $\varphi_T^{0,\downarrow}$, $\varphi_T^{1,\uparrow}$, $\varphi_T^{1,\downarrow}$ remain bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, when (φ^0, φ^1) runs through the M-admissible self-financing process and $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ denote the canonical decompositions. For later use we remark that the proof shows, in fact, that also the convex combinations of these functions $\varphi_T^{0,\uparrow}$ etc. remain bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Indeed the estimate (174) shows that the convex hull of the functions $\varphi_T^{0,\uparrow}$ is bounded in $L^1(Q)$ and (175) yields the same for $\varphi_T^{0,\downarrow}$. For $\varphi_T^{1,\uparrow}$ and $\varphi_T^{1,\downarrow}$ the argument is similar.

In order to prove the subsequent Theorem 4.13 we still need one more preparation (compare [79]).

Proposition 4.12. *Fix S and $1 > \lambda > 0$ as above, and suppose that S satisfies $(CPS^{\lambda'})$, for* each $\lambda' > 0$.

Let $(\varphi_t)_{0\leqslant t\leqslant T} = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ be a self-financing and admissible process under transaction costs λ, and suppose that there is $M > 0$ s.t. for the terminal value V_T we have

$$V_T(\varphi^0, \varphi^1) = \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \geqslant -M. \tag{176}$$

Then we also have

$$V_\tau(\varphi_\tau^0, \varphi_\tau^1) = \varphi_\tau^0 + (\varphi_\tau^1)^+(1-\lambda)S_\tau - (\varphi_\tau^1)^- S_\tau \geqslant -M, \tag{177}$$

a.s., for every stopping time $0 \leqslant \tau \leqslant T$, i.e. φ is M-admissible.

Proof: We start with the observation, that by liquidating the stock position at time T, we may assume in (176) w.l.g. that $\varphi_T^1 = 0$, so that $\varphi_T^0 \geqslant -M$.

Supposing that (177) fails, we may find $\frac{\lambda}{2} > \alpha > 0$, a stopping time $0 \leqslant \tau \leqslant T$, such that either $A = A_+$ or $A = A_-$ satisfies $\mathbb{P}[A] > 0$, where

$$A_+ = \{\varphi_\tau^1 \geqslant 0, \, \varphi_\tau^0 + \varphi_\tau^1 \tfrac{1-\lambda}{1-\alpha} S_\tau < -M\}, \tag{178}$$

$$A_- = \{\varphi_\tau^1 \leqslant 0, \, \varphi_\tau^0 + \varphi_\tau^1 (1-\alpha)^2 S_\tau < -M\}. \tag{179}$$

Choose $0 < \lambda' < \alpha$ and a λ'-consistent price system $(\tilde{S}, Q)$. As $\tilde{S}$ takes values in $[(1-\lambda')S, S]$, we have that $(1-\alpha)\tilde{S}$ as well as $\frac{1-\lambda}{1-\alpha}\tilde{S}$ take values in $[(1-\lambda)S, S]$ so that $((1-\alpha)\tilde{S}, Q)$ as well as $(\frac{1-\lambda}{1-\alpha}\tilde{S}, Q)$ are consistent price systems under transaction costs λ. By Proposition 4.5 we obtain that

$$\left(\varphi_t^0 + \varphi_t^1 (1-\alpha)\tilde{S}_t\right)_{0\leqslant t\leqslant T}, \text{ and } (\varphi_t^0 + \varphi_t^1 \tfrac{1-\lambda}{1-\alpha}\tilde{S}_t)_{0\leqslant t\leqslant T}$$

are Q-supermartingales. Arguing with the second process and using that $\tilde{S} \leqslant S$ we obtain from (178) the inequality

$$\mathbb{E}_Q \left[\varphi_T^0 + \varphi_T^1 \frac{1-\lambda}{1-\alpha} \tilde{S}_T | A_+ \right] \leqslant \mathbb{E}_Q \left[\varphi_\tau^0 + \varphi_\tau^1 \frac{1-\lambda}{1-\alpha} \tilde{S}_\tau | A_+ \right] < -M.$$

Arguing with the first process and using that $\tilde{S} \geqslant (1-\lambda')S \geqslant (1-\alpha)S$ (which implies that $\varphi_\tau^1(1-\alpha)\tilde{S}_\tau \leqslant \varphi_\tau^1(1-\alpha)^2 S_\tau$ on A_-) we obtain from (179) the inequality

$$\mathbb{E}_Q \left[\varphi_T^0 + \varphi_T^1 (1-\alpha) \tilde{S}_T | A_- \right] \leqslant \mathbb{E}_Q \left[\varphi_\tau^0 + \varphi_\tau^1 (1-\alpha) \tilde{S}_\tau | A_- \right] < -M.$$

Either A_+ or A_- has strictly positive probability; hence we arrive at a contradiction, as $\varphi_T^1 = 0$ and $\varphi_T^0 \geqslant -M$. ■

The assumption $CPS^{\lambda'}$, for *each* $\lambda' > 0$, cannot be dropped in Proposition 4.12 as shown by an explicit example in [79].

We now can state the central result from [11] in the present framework. Recall Definition 4.7 of the sets $\mathcal{A}^M$ and $\mathcal{C}^M$. Proposition 4.12 has the following important consequence concerning these definitions. We may equivalently define $\mathcal{A}^M$ as the set of random variables $(\varphi_T^0, \varphi_T^1)$ in $\mathcal{A}$ such that $V^{\text{liq}}(\varphi_T^0, \varphi_T^1) \geqslant -M$. The point is that the requirement $\varphi = (\varphi_T^0, \varphi_T^1) \in \mathcal{A}$ only implies that φ is the terminal value of an $\bar{M}$-admissible strategy, for some $\bar{M} > 0$ which – a priori – has nothing to do with M. But Proposition 4.12 tells us that $V^{\text{liq}}(\varphi_T^0, \varphi_T^1) \geqslant -M$ already implies that we may replace the a priori constant $\bar{M}$ by the constant M. In other words, if the liquidation value of an admissible φ is above the threshold $-M$ at the terminal time T, it also is so at all previous times $0 \leqslant t \leqslant T$.

Theorem 4.13. *Fix $S = (S_t)_{0 \leqslant t \leqslant T}$ and $\lambda > 0$ as above, and suppose that $(CPS^{\lambda'})$ is satisfied, for each $0 < \lambda' < \lambda$. For $M > 0$, the convex set $\mathcal{A}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ as well as the convex set $\mathcal{C}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ are closed with respect to the topology of convergence in measure.*

For the proof we use the following well-known variant of Komlos' theorem. This result ([26, Lemma A 1.1]) turned out to be very useful in the applications to Mathematical Finance.

For the convenience of the reader we reproduce the proof.

Lemma 4.14. *Let $(f_n)_{n=1}^\infty$ be a sequence of $\mathbb{R}_+$-valued, measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$.*

There is a sequence $g_n \in conv\,(f_n, f_{n+1}, \ldots)$ of convex combinations which converges a.s. to some $[0, \infty]$-valued function g_0.

If $(f_n)_{n=1}^{\infty}$ is such that the convex hull conv $(f_1, f_2, \ldots)$ is bounded in the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the function g_0 takes a.s. finite values.

Proof: Choose $g_n \in \text{conv}\,(f_n, f_{n+1}, \ldots)$ such that

$$\lim_{n\to\infty} \mathbb{E}[\exp(-g_n)] = \lim_{n\to\infty} \inf_{g \in \text{conv}\,(f_n, f_{n+1}, \ldots)} \mathbb{E}[\exp(-g)]. \tag{180}$$

For fixed $1 > \varepsilon > 0$ we claim that

$$\lim_{n,m\to\infty} \mathbb{P}[(A_n \cup A_m) \cap B_{n,m}] = 0, \tag{181}$$

where

$$\begin{aligned} A_n &= \{g_n \in [0, \tfrac{1}{\varepsilon}]\} \\ A_m &= \{g_m \in [0, \tfrac{1}{\varepsilon}]\} \\ B_{n,m} &= \{|g_n - g_m| \geqslant \tfrac{\varepsilon}{2}\}. \end{aligned}$$

Indeed, the function $x \to e^{-x}$ is strictly convex on $[0, \frac{1}{\varepsilon} + \frac{\varepsilon}{2}]$ so that, for given $\varepsilon > 0$, there is $\delta > 0$ such that, for $x, y \in [0, \frac{1}{\varepsilon} + \frac{\varepsilon}{2}]$ satisfying $(x - y) \geqslant \frac{\varepsilon}{2}$ we have

$$\exp(-\tfrac{x+y}{2}) \leqslant \tfrac{\exp(-x)+\exp(-y)}{2} - \delta.$$

For $\omega \in (A_n \cup A_m) \cap B_{n,m}$ we therefore have

$$\exp(-\tfrac{g_n(\omega)+g_m(\omega)}{2}) \leqslant \tfrac{\exp(-g_n(\omega))+\exp(-g_m(\omega))}{2} - \delta.$$

Using the convexity of $x \to e^{-x}$ on $[0, \infty[$ (this time without strictness) we get

$$\begin{aligned} \mathbb{E}\left[\exp(-\tfrac{g_n+g_m}{2})\right] \leqslant\ & \mathbb{E}[\tfrac{\exp(-g_n)+\exp(-g_m)}{2}] \\ & - \delta\mathbb{P}[(A_n \cup A_m) \cap B_{n,m}]. \end{aligned}$$

The negation of (181) reads as

$$\limsup_{n,m\to\infty} \mathbb{P}[(A_n \cup A_m) \cap B_{n,m}] = \alpha > 0.$$

This would imply that

$$\liminf_{n,m\to\infty} \mathbb{E}[\exp(-\tfrac{g_n+g_m}{2})] \leqslant \lim_{n\to\infty} \inf_{g \in \text{conv}\,(f_n, f_{n+1}, \ldots)} \mathbb{E}[\exp(-g)] - \alpha\delta,$$

in contradiction to (180), which shows (181).

By passing to a subsequence, still denoted by $(g_n)_{n=1}^{\infty}$, we may suppose that, for fixed $1 > \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}[(A_n \cup A_{n+1}) \cap B_{n,n+1}] < \infty, \tag{182}$$

and, by passing to a diagonal sequence, that this holds true for each $1 > \varepsilon > 0$. Taking a subsequence once more and applying Borel-Cantelli we get that, for almost each $\omega \in \Omega$, either $g_n(\omega) \to \infty$ or $(g_n(\omega))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}_+$.

As regards the second assertion, the condition on the L^0-boundedness states that, for $\eta > 0$, we may find $M > 0$ such that $\mathbb{P}[g \geqslant M] < \eta$, for each $g \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$. This L^0-boundedness condition prevents $(g_n)_{n=1}^{\infty}$ from converging to $+\infty$ with positive probability. ■

Convex combinations work very much like subsequences. For example, one may form sequences of convex combinations of sequences of convex combinations: if $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$ and $h_n \in \operatorname{conv}(g_n, g_{n+1}, \ldots)$, then h_n is a sequence of convex combinations of the original sequence $(f_n)_{n=1}^{\infty}$, i.e. $h_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$. Similarly, the concept of a diagonal subsequence carries over in an obvious way. This will repeatedly used in the subsequent proof.

Proof of Theorem 4.13: Fix $M > 0$ and let $(\varphi_T^n)_{n=1}^{\infty} = (\varphi_T^{0,n}, \varphi_T^{1,n})_{n=1}^{\infty}$ be a sequence in $\mathcal{A}^M$. We may find self-financing, M-admissible strategies $(\varphi_t^{0,n}, \varphi_t^{1,n})_{0 \leqslant t \leqslant T}$, starting at $(\varphi_{0_-}^{0,n}, \varphi_{0_-}^{1,n}) = (0,0)$, with given terminal values $(\varphi_T^{0,n}, \varphi_T^{1,n})$. As above, decompose canonically these processes as $\varphi_t^{0,n} = \varphi_t^{0,n,\uparrow} - \varphi_t^{0,n,\downarrow}$, and $\varphi_t^{1,n} = \varphi_t^{1,n,\uparrow} - \varphi_t^{1,n,\downarrow}$. By Lemma 4.10 and the subsequent remark we know that $(\varphi_T^{0,n,\uparrow})_{n=1}^{\infty}, (\varphi_T^{0,n,\downarrow})_{n=1}^{\infty}, (\varphi_T^{1,n,\uparrow})_{n=1}^{\infty}$, and $(\varphi_T^{1,n,\downarrow})_{n=1}^{\infty}$ as well as their convex combinations are bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ too, so that by Lemma 4.14 we may find convex combinations converging a.s. to some elements $\varphi_T^{0,\uparrow}, \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow}$, and $\varphi_T^{1,\downarrow} \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. To alleviate notation we denote the sequences of convex combinations still by the original sequences. We claim that $(\varphi_T^0, \varphi_T^1) = (\varphi_T^{0,\uparrow} - \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow} - \varphi_T^{1,\downarrow})$ is in $\mathcal{A}^M$ which will readily show the closedness of $\mathcal{A}^M$ with respect to the topology of convergence in measure.

By inductively passing to convex combinations, still denoted by the original sequences, we may, for each rational number $r \in [0, T[$, assume that $(\varphi_r^{0,n,\uparrow})_{n=1}^{\infty}, (\varphi_r^{0,n,\downarrow})_{n=1}^{\infty}, (\varphi_r^{1,n,\uparrow})_{n=1}^{\infty}$, and $(\varphi_r^{1,n,\downarrow})_{n=1}^{\infty}$ converge a.s. to some elements $\bar{\varphi}_r^{0,\uparrow}, \bar{\varphi}_r^{0,\downarrow}, \bar{\varphi}_r^{1,\uparrow}$, and $\bar{\varphi}_r^{1,\downarrow}$ in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. By passing to a diagonal subsequence, we may suppose that this convergence holds true for all rationals $r \in [0, T[$.

Clearly the four processes $\bar{\varphi}^{0,\uparrow}_{r\in\mathbb{Q}\cap[0,T[}$ etc, indexed by the rationals r in $[0,T[$, still are increasing and define an M-admissible process, indexed by $[0,T[\cap\mathbb{Q}$, in the sense of (154). They also satisfy (157), where we define $\bar{\varphi}^{0,\uparrow}_{0_-} = 0$ and $\bar{\varphi}^{0,\uparrow}_T = \varphi^{0,\uparrow}_T$ (etc. for the other three cases).

We still have to pass to a right continuous version and to extend the processes to all real numbers $t \in [0,T]$. This is done by letting

$$\varphi^{0,\uparrow}_t = \lim_{\substack{r\searrow t\\ r\in\mathbb{Q}}} \bar{\varphi}^{0,\uparrow}_r, \qquad 0 \leqslant t < T, \tag{183}$$

and $\varphi^{0,\uparrow}_{0_-} = 0$. Note that the terminal value $\varphi^{0,\uparrow}_T$ is still given by the first step of the construction. The three other cases, $\varphi^{0,\downarrow}$, $\varphi^{1,\uparrow}$, and $\varphi^{1,\downarrow}$ are, of course, defined in an analogous way. These continuous time processes again satisfy the self-financing conditions (157).

Finally, define the process $(\varphi^0_t, \varphi^1_t)_{0\leqslant t\leqslant T}$ as $(\varphi^{0,\uparrow}_t - \varphi^{0,\downarrow}_t, \varphi^{1,\uparrow}_t - \varphi^{1,\downarrow}_t)_{0\leqslant t\leqslant T}$. From Proposition 4.4 (*iii*) we obtain that this defines a self-financing trading strategy in the sense of Definition 4.2 with the desired terminal value $(\varphi^0_T, \varphi^1_T)$. The M-admissibility follows from Proposition 4.12.

We thus have shown that $\mathcal{A}^M$ is closed. The closedness of $\mathcal{C}^M$ is an immediate consequence. ■

In fact we have not only proved a *closedness* of $\mathcal{A}^M$ with respect to the topology of convergence in measure. Rather we have shown a *convex compactness* property (compare [64], [89]). Indeed, we have shown that, for any sequence $(\varphi^n_T)_{n=1}^\infty \in \mathcal{A}^M$, we can find a sequence of convex combinations which converges a.s., and therefore in measure, to an element $\varphi_T \in \mathcal{A}^M$.

Passage from L^0 to appropriate Banach spaces

The message of Theorem 4.13 is stated in terms of the topological vector space $L^0(\mathbb{R}^2)$ and with respect to convergence in measure. We now translate it into the setting of appropriately defined Banach spaces. This needs some preparation. For a fixed, positive number $S > 0$ we define the norm $|\cdot|_S$ on $\mathbb{R}^2$ by

$$|(x^0, x^1)|_S = \max\{|x^0 + x^1 S|, |x^0 + x^1(1-\lambda)S|\}. \tag{184}$$

Its unit ball is the convex hull of the four points $\{(1,0), (-1,0), (\frac{2-\lambda}{\lambda}, -\frac{2}{\lambda S}), (-\frac{2-\lambda}{\lambda}, \frac{2}{\lambda S})$.

To motivate this definition we consider for a fixed number $S > 0$, similarly as in (1), the solvency cone $K_S = \{(x^0, x^1) : x^0 \geqslant \max(-x^1 S, -x^1(1-\lambda)S)\}$. For $\xi \in \mathbb{R}$, let $K_S(\xi)$ be the shifted solvency cone $K_S(\xi) = K_S - \xi = \{(x^0, x^1) : (x^0 + \xi, x^1) \in K_S)\}$. With this notation, the unit ball of $(\mathbb{R}^2, |\cdot|_S)$ is the biggest set which is symmetric around 0 and contained in $K_S(1)$.

The dual norm $|\cdot|_S^*$ is given, for $(Z^0, Z^1) \in \mathbb{R}^2$, by

$$|(Z^0, Z^1)|_S^* = \max\{|Z^0|, |\tfrac{2-\lambda}{\lambda} Z^0 - \tfrac{2}{\lambda S} Z^1|\}, \tag{185}$$

as one readily verifies by looking at the extreme points of the unit ball of $(\mathbb{R}^2, |\cdot|_S)$. The unit ball of $(\mathbb{R}^2, |\cdot|_S^*)$ is the convex hull of the four points $\{(1, S), (-1, -S), (1, (1-\lambda)S), (-1, -(1-\lambda)S)\}$.

These norms on $\mathbb{R}^2$ are tailor-made to define Banach spaces L_S^1 and L_S^∞ in isometric duality where S will depend on $\omega \in \Omega$. Let $S = (S_t)_{0 \leqslant t \leqslant T}$ now denote an $\mathbb{R}_+$-valued process. We define the Banach space L_S^1 as

$$\begin{aligned} &L_S^1 = L_S^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) = \\ &\Big\{ Z_T = (Z_T^0, Z_T^1) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) : \|Z_T\|_{L_S^1} = \mathbb{E}\left[|(Z_T^0, Z_T^1)|_{S_T}^*\right] < \infty \Big\} \end{aligned} \tag{186}$$

Its dual L_S^∞ then is given by

$$\begin{aligned} &L_S^\infty = L_S^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) = \\ &\left\{ \varphi_T = (\varphi_T^0, \varphi_T^1) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) : \|\varphi_T\|_{L_S^\infty} = \operatorname{ess\,sup}\left[|(\varphi_T^0, \varphi_T^1)|_{S_T}\right] < \infty \right\}. \end{aligned} \tag{187}$$

These spaces are designed in such a way that $\mathcal{A} \cap L_S^\infty$ is "Fatou dense" in $\mathcal{A}$. We do not elaborate in detail on the notion of "Fatou closedness" which was introduced in [81] but only present the idea which is relevant in the present context.

For $\varphi_T = (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^M$ we have (154)

$$V_T^{\text{liq}} = \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \geqslant -M, \tag{188}$$

which may be written as

$$\min\left\{(\varphi_T^0 + \varphi_T^1(1-\lambda)S_T), (\varphi_T^0 + \varphi_T^1 S_T)\right\} \geqslant -M \tag{189}$$

or

$$\max\left\{-(\varphi_T^0 + \varphi_T^1(1-\lambda)S_T), -(\varphi_T^0 + \varphi_T^1 S_T)\right\} \leqslant M. \tag{190}$$

In order to obtain $|(\varphi_T^0, \varphi_T^1)|_{S_T} \leqslant M$ we still need the inequality

$$\max\left\{(\varphi_T^0 + \varphi_T^1(1-\lambda)S_T), (\varphi_T^0 + \varphi_T^1 S_T)\right\} \leqslant M. \tag{191}$$

In general, there is little reason why (191) should be satisfied, for an element $\varphi_T = (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^M$. Indeed, the agent may have become "very rich" which may cause (191) to fail to hold true. But there is an easy remedy: just "get rid of the superfluous assets"

More formally: fix $M > 0$, and $\varphi_T = (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^M$, as well as a number $C \geqslant M$. We shall define the C-truncation φ_T^C of φ_T in a pointwise way: if $|\varphi_T(\omega)|_{S_T(\omega)} \leqslant C$ we simply let

$$\varphi_T^C(\omega) = \varphi_T(\omega).$$

If $|\varphi_T(\omega)|_{S_T(\omega)} > C$ we define

$$\varphi_T^C = \mu(\varphi_T^0(\omega), \varphi_T^1(\omega)) + (1-\mu)(-M, 0) \tag{192}$$

which is a convex combination of $\varphi_T(\omega)$ and the lower left corner $(-M, 0)$ of the $M-$ ball of $(\mathbb{R}^2, |\cdot|_{S_T(\omega)})$; for $\mu \in [0,1]$ above we choose the biggest number in $[0,1]$ such that $|\varphi_T^C(\omega)|_{S_T(\omega)} \leqslant C$. Note that, for $C' \geqslant C \geqslant M$ we have $\varphi_T^{C'}(\omega) - \varphi_T^C(\omega) \in K_{S_T(\omega)}$, i.e. we can obtain φ_T^C from $\varphi_T^{C'}$ (as well as from φ_T) by a self-financing trade at time T.

By construction φ_T^C lies in the Banach space L_S^∞, its norm being bounded by C. Sending C to infinity the random variables $\varphi_T^{0,C}$ increase (with respect to the order induced by the cone K_T) a.s. to φ_T^0.

Summing up: the intersection $\mathcal{A} \cap L_S^\infty$ is dense in $\mathcal{A}$ in the sense that, for $\varphi_T \in \mathcal{A}$ there is an *increasing* sequence $(\varphi_T^k)_{k\geqslant 0}$ in $\mathcal{A} \cap L_S^\infty$ converging a.s. to φ_T. This is what we mean by "Fatou-dense".

Following a well-known line of argument (compare [26]), Theorem 4.13 thus translates into the following result.

Theorem 4.15. *Fix S and $\lambda > 0$, and suppose that $(CPS^{\lambda'})$ is satisfied, for each $0 < \lambda' < \lambda$. The convex cone $\mathcal{A} \cap L_S^\infty \subseteq L_S^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, as well as the convex cone $\mathcal{C} \cap L^\infty \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ are closed with respect to the weak-star topology induced by L_S^1 (resp. L^1).*

Proof: By the Krein-Smulian theorem [82] the cone $\mathcal{A} \cap L_S^\infty$ is σ^*-closed iff its intersection with the unit ball of L_S^∞ is σ^*-closed. Hence it suffices to show that $\mathcal{A} \cap$ (ball (L_S^∞)) is σ^*-compact. By a result of A. Grothendieck ([38], see also the version [26, Prop.5.2.4] which easily extends to the present 2-dimensional setting), the σ^*-compactness of a bounded, convex subset of L^∞ is equivalent to the following property: for every sequence $(\varphi_T^n)_{n=1}^\infty \in \mathcal{A} \cap$ (ball (L_S^∞)) converging a.s. to φ_T, we have that the limit again is in $\mathcal{A} \cap$ ball (L_S^∞). By the definition of the norm of L_S^∞ and using Proposition 4.12 we have that $\varphi_T^n \in \mathcal{A}^1$, for each n, so that Theorem 4.13 implies that the limit φ_T again is in $\mathcal{A}^1$. As the inequalities (190) and (191) clearly remain valid by passing from $(\varphi_T^n)_{n=1}^\infty$ to the limit φ_T we obtain that $\varphi_T \in \mathcal{A} \cap$ (ball L_S^∞). This shows the σ^*-closedness of $\mathcal{A} \cap L_S^\infty$.

The σ^*-closedness of $\mathcal{C}$ follows from the σ^*-closedness of $\mathcal{A}$ and the fact that L^∞ is a σ^*-closed subset of L_S^∞. ∎

Theorem 4.15 allows us to apply the duality theory to the dual pairs $\langle L^1_S, L^\infty_S\rangle$ and $\langle L^1, L^\infty\rangle$ respectively. Denoting as above by $(\mathcal{A} \cap L^\infty_S)^\circ$ (resp. $(\mathcal{C} \cap L^\infty)^\circ$) the polar of $\mathcal{A} \cap L^\infty$ in L^1_S (resp. of $\mathcal{C} \cap L^\infty$ in L^1), the bipolar theorem ([82]; see also Proposition A.1 in the appendix) as well as Theorem 4.15 imply that $(\mathcal{A} \cap L^\infty_S)^{\circ\circ} = \mathcal{A} \cap L^\infty_S$ and $(\mathcal{C} \cap L^\infty)^{\circ\circ} = \mathcal{C} \cap L^\infty$. In fact, we shall be able to characterize the polars $(\mathcal{A} \cap L^\infty_S)^\circ$ and $(\mathcal{C} \cap L^\infty_S)^\circ$ in terms of consistent price systems.

We remark that the distinction between $\mathcal{A}$ and $\mathcal{A} \cap L^\infty_S$ (resp. $\mathcal{C}$ and $\mathcal{C} \cap L^\infty$) is rather a formality; the passage to these intersections only serves to put us into the well-established framework of the duality theory of Banach spaces. For example, we shall consider the polar set

$$(\mathcal{C} \cap L^\infty)^\circ = \{Z^0_T \in L^1 : \langle \varphi^0_T, Z^0_T\rangle = \mathbb{E}[\varphi^0_T Z^0_T] \leqslant 0, \quad \text{for every } \varphi^0_T \in \mathcal{C} \cap L^\infty\} \tag{193}$$

and an analogous definition for $(\mathcal{A} \cap L^\infty_S)^\circ \subseteq L^1_S$. We note that we could equivalently define

$$\mathcal{C}^\circ = \{Z^0_T \in L^1 : \langle \varphi^0_T, Z^0_T\rangle = \mathbb{E}[\varphi^0_T Z^0_T] \leqslant 0 \quad \text{for every } \varphi^0_T \in \mathcal{C}\}$$

Indeed, as each $\varphi^0_T \in \mathcal{C}$ is uniformly bounded from below, the expectation appearing above is well-defined (possibly assuming the value infinity) and it follows from monotone convergence that

$$\mathbb{E}[\varphi^0_T Z^0_T] \leqslant 0 \quad \text{iff} \quad \mathbb{E}[(\varphi^0_T \wedge n) Z^0_T] \leqslant 0,$$

for every $n \geqslant 0$. A similar remark applies to $(\mathcal{A} \cap L^\infty_S)^\circ$. To alleviate notation we shall therefore write $\mathcal{C}^\circ$ and $\mathcal{A}^\circ$ instead of $(\mathcal{C} \cap L^\infty)^\circ$ and $(\mathcal{A} \cap L^\infty_S)^\circ$.

The dual variables

To characterize the polars of $\mathcal{A}$ and $\mathcal{C}$, let $(\tilde{S}, Q)$ be a consistent price system (Def. 4.1) for the process S under transaction costs λ. As usual, we denote by $(Z^0_t)_{0\leqslant t\leqslant T}$ the density process $Z^0_t = \mathbb{E}[\frac{dQ}{d\mathbb{P}}|\mathcal{F}_t]$ and by $(Z^1_t)_{0\leqslant t\leqslant T}$ the process $(Z^0_t \tilde{S}_t)_{0\leqslant t\leqslant T}$, so that Z^0 (resp. Z^1) is a martingale (resp. a local martingale) under $\mathbb{P}$.

Definition 4.16. *Given S and $\lambda > 0$ as above, we denote by $B(1)$ the convex, bounded set of non-negative random variables $\{Z_T = (Z^0_T, Z^1_T)\}$ such that Z_T is the terminal value of a consistent price process as above. Denote by $\mathcal{B}(1)$ the norm closure of $B(1)$ in L^1_S, and by $\mathcal{B}$ the cone generated by $\mathcal{B}(1)$, i.e.*

$$\mathcal{B} = \underset{y\geqslant 0}{\cup} \mathcal{B}(y),$$

where $\mathcal{B}(y) = y\mathcal{B}(1)$.

We denote by $D(1)$ the projection of $B(1)$ onto $L^1(\mathbb{R})$ (via the canonical projection of L^1_S onto its first coordinate), and by $\mathcal{D}(1)$ and $\mathcal{D}$ its norm closure and the cone generated by $\mathcal{D}(1)$, respectively.

Proposition 4.17. *Let S and $\lambda > 0$ be as in Theorem 4.13, and suppose again that $(CPS^{\lambda'})$ holds true, for all $0 < \lambda' < \lambda$.*

Then $\mathcal{B}$ (resp. $\mathcal{D}$) is a bounded set in L^1_S (resp. L^1) and $\mathcal{B}$ (resp. $\mathcal{D}$) equals the polar cone $\mathcal{A}^\circ$ of $\mathcal{A}$ (resp. $\mathcal{C}^\circ$ of $\mathcal{C}$) in L^1_S (resp. in L^1).

Proof: To obtain the inclusion $\mathcal{B} \subseteq \mathcal{A}^\circ$, we shall show that

$$\big\langle(\varphi^0_T, \varphi^1_T), (Z^0_T, Z^1_T)\big\rangle = \mathbb{E}[\varphi^0_T Z^0_T + \varphi^1_T Z^1_T] \leqslant 0, \tag{194}$$

for all $\varphi_T = (\varphi^0_T, \varphi^1_T) \in \mathcal{A}$ and for all $Z_T = (Z^0_T, Z^1_T) \in B(1)$.

Indeed, associate to φ_T an admissible trading strategy $(\varphi_t)_{0\leqslant t\leqslant T} = (\varphi^0_t, \varphi^1_t)_{0\leqslant t\leqslant T}$ and to Z_T a consistent price system $(\tilde{S}, Q) = ((\frac{Z^1_t}{Z^0_t})_{0\leqslant t\leqslant T}, Z^0_T)$. By Proposition 4.5 the process

$$\tilde{V}_t = \varphi^0_t + \varphi^1_t \tilde{S}_t, \qquad\qquad 0 \leqslant t \leqslant T,$$

is a Q-supermartingale, starting at $\tilde{V}_{0_-} = 0$, so that

$$\mathbb{E}_{\mathbb{P}}[\varphi^0_T Z^0_T + \varphi^1_T Z^1_T] = \mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T] \leqslant 0.$$

This shows (194) which, by continuity and positive homogeneity, also holds true, for all $Z_T = (Z^0_T, Z^1_T) \in \mathcal{B}$. We therefore have shown that $\mathcal{B}$ is contained in the polar $\mathcal{A}^\circ$ of $\mathcal{A}$.

As regards the reverse inclusion $\mathcal{A}^\circ \subseteq \mathcal{B}$, we have to show that, for $\varphi_T = (\varphi^0_T, \varphi^1_T) \in L^\infty_S$, such that (194) is satisfied, for all $Z_T = (Z^0_T, Z^1_T) \in B(1)$, we have that $\varphi_T \in \mathcal{A}$.

Fix $\bar{\varphi}_T = (\bar{\varphi}^0_T, \bar{\varphi}^1_T) \notin \mathcal{A}$. By Theorem 4.15 and the Hahn-Banach theorem we may find an element $\bar{Z}_T = (\bar{Z}^0_T, \bar{Z}^1_T) \in L^1_S$ such that (194) holds true, for $\bar{Z}_T$ and all $\varphi_T \in \mathcal{A}$ while

$$\big\langle(\bar{\varphi}^0_T, \bar{\varphi}^1_T), (\bar{Z}^0_T, \bar{Z}^1_T)\big\rangle > 0. \tag{195}$$

As $\mathcal{A}$ contains the non-positive functions, we have that $(\bar{Z}^0_T, \bar{Z}^1_T)$ takes values a.s. in $\mathbb{R}^2_+$. In fact, we may suppose that $\bar{Z}^0_T$ and $\bar{Z}^1_T$ are a.s. strictly positive. Indeed, by the assumption CPS^λ there is a λ-consistent price system $\hat{Z} = (\hat{Z}^0, \hat{Z}^1)$. For $\varepsilon > 0$, the convex combination $(1-\varepsilon)\bar{Z}_T + \varepsilon\hat{Z}_T$ still satisfies (194), for each $\varphi^1_T \in \mathcal{A}$. For $\varepsilon > 0$ sufficiently small, (195) is satisfied too.

Hence, by choosing $\varepsilon > 0$ sufficiently small, we may assume that $\bar{Z}_T^0$ and $\bar{Z}_T^1$ are strictly positive.

We also may assume that $\mathbb{E}[\bar{Z}_T^1] = 1$ so that $\frac{d\bar{Q}}{d\mathbb{P}} := \bar{Z}_T^0$ defines a probability measure $\bar{Q}$ which is equivalent to $\mathbb{P}$. We now have to work towards a contradiction.

To focus on the essence of the argument, let us assume for a moment that $S = (S_t)_{0\leqslant t\leqslant T}$ is uniformly bounded. We then may define the $\mathbb{R}_+^2$-valued martingale $\bar{Z} = (\bar{Z}^0, \bar{Z}^1)$ by

$$\bar{Z}_t = (\bar{Z}_t^0, \bar{Z}_t^1) = \mathbb{E}[(\bar{Z}_T^0, \bar{Z}_T^1)|\mathcal{F}_t], \qquad 0 \leqslant t \leqslant T. \tag{196}$$

Indeed by (185) and (186), we have $\bar{Z}_T^1 \leqslant C\bar{Z}_T^0 \leqslant C^*|\bar{Z}_T|_{S_T}^*$ almost surely, for some constants C, C^*, depending on the uniform bound of S. Hence $\bar{Z}_T$ is integrable so that $\bar{Z}_t$ in (196) is well-defined. We shall verify that $\bar{Z} = (\bar{Z}_t)_{0\leqslant t\leqslant T}$ indeed defines a consistent price system. To do so, we have to show that, for $0 \leqslant t \leqslant T$,

$$\tilde{S}_t := \frac{\bar{Z}_t^1}{\bar{Z}_t^0} \in [(1-\lambda)S_t, S_t], \qquad \text{a.s.} \tag{197}$$

Negating (197) we may find some $0 \leqslant u \leqslant T$ such that one of the following two sets has strictly positive measure

$$A_+ = \left\{\frac{\bar{Z}_u^1}{\bar{Z}_u^0} > S_u\right\}, \quad A_- = \left\{\frac{\bar{Z}_u^1}{\bar{Z}_u^0} < (1-\lambda)S_u\right\}.$$

In the former case, define the process $\varphi^1 = (\varphi^0, \varphi^1)$ as in (162) by

$$(\varphi_t^0, \varphi_t^1) = (-S_u, 1)\mathbb{1}_{A_+}\mathbb{1}_{[\![u,T]\!]}(t), \qquad 0 \leqslant t \leqslant T.$$

Using the boundedness of S, we conclude that $(\varphi_T^0, \varphi_T^1) = (\varphi_u^0, \varphi_u^1) = (-S_u, 1)\mathbb{1}_{A_+}$ is an element of $\mathcal{A}$ for which we get

$$\begin{aligned}\mathbb{E}\left[\varphi_T^0\bar{Z}_T^0 + \varphi_T^1\bar{Z}_T^1\right] =& \mathbb{E}\left[\mathbb{E}\left[\varphi_u^0\bar{Z}_T^0 + \varphi_u^1\bar{Z}_T^1|\mathcal{F}_u\right]\right] \\ =& \mathbb{E}\left[\varphi_u^0\bar{Z}_u^0 + \bar{\varphi}_u^1\bar{Z}_u^1\right] \\ =& \mathbb{E}\left[\bar{Z}_u^0\left(-S_u + \tfrac{\bar{Z}_u^1}{\bar{Z}_u^0}\right)\mathbb{1}_{A_+}\right] > 0,\end{aligned}$$

a contradiction to (194).

If $\mathbb{P}[A_-] > 0$ we apply a similar argument to (163).

Summing up: we have arrived at the desired contradiction proving the inclusion $\mathcal{A}^\circ \subseteq \mathcal{B}$, under the additional assumption that S is uniformly bounded.

Now we drop the boundedness assumption on S. By the continuity of S we may find a localizing sequence $(\tau_n)_{n=1}^{\infty}$ of $[0,T] \cup \{\infty\}$-valued stopping times, increasing a.s. to ∞, such that each stopped processes $S^{\tau_n} = (S_{t \wedge \tau_n})_{0 \leqslant t \leqslant T}$ is bounded. Indeed, it suffices to take $\tau_n = \inf\{t : S_t > n\}$.

Denote by $\mathcal{A}_{\tau_n} = \mathcal{A} \cap L_S^{\infty}(\Omega, \mathcal{F}_{\tau_n}, \mathbb{P})$ the subset of $\mathcal{A} \cap L_S^{\infty}$ formed by the elements $\varphi_T = (\varphi_T^0, \varphi_T^1)$ which are $\mathcal{F}_{\tau_n}$-measurable. We then have that $\mathcal{A}_{\tau_n}$ is the cone corresponding to the stopped process S^{τ_n} via Definition 4.7. By stopping, we also have that $\bigcup_{n=1}^{\infty} \mathcal{A}_{\tau_n} \cap L_S^{\infty}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ is weak-star dense in $\mathcal{A} \cap L_S^{\infty}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$.

Denote by $\bar{Z}_{\tau_n}$ the restriction of the functional $\bar{Z}_T = (\bar{Z}_T^0, \bar{Z}_T^1)$ to $L_S^{\infty}(\mathcal{F}_{\tau_n})$ which we may identify with a pair $(\bar{Z}_{\tau_n}^0, \bar{Z}_{\tau_n}^1)$ of $\mathcal{F}_{\tau_n}$-measurable functions.

By taking conditional expectations as in (196), we may associate to the random variables $(\bar{Z}_{\tau_n}^0, \bar{Z}_{\tau_n}^1)$ the corresponding martingales, denoted by $\bar{Z}^n = (\bar{Z}_t^{0,n}, \bar{Z}_t^{1,n})_{0 \leqslant t \leqslant \tau_n \wedge T}$.

Of course, this sequence of processes is consistent, i.e., for $n \leqslant m$, the process $\bar{Z}^m$, stopped at τ_n, equals the process $\bar{Z}^n$. As regards the first coordinate, it is clear that $(\bar{Z}_{\tau_n \wedge T}^0)_{n=1}^{\infty}$ converges in the norm of $L^1(\mathbb{P})$ to $\bar{Z}_T^0$, which is the density of the probability measure $\bar{Q}$. The associated density process is $\bar{Z}_t^0 = \mathbb{E}[\bar{Z}_T^0 | \mathcal{F}_t]$. The slightly delicate issue is the second coordinate of $\bar{Z}$. The sequence $(\bar{Z}_{\tau_n \wedge T}^1)$ only converges a.s. to $\bar{Z}_T^1$, but not necessarily with respect to the norm of $L^1(\mathbb{P})$. In other words, by pasting together the processes $(\bar{Z}_t^{1,n})_{0 \leqslant t \leqslant \tau_n \wedge T}$, and letting

$$\bar{Z}_t^1 = \lim_{n \to \infty} \bar{Z}_t^{1,n},$$

the limit holding true a.s., for each $0 \leqslant t \leqslant T$, we well-define a *local* $\mathbb{P}$-martingale $(\bar{Z}_t^1)_{0 \leqslant t \leqslant T}$. This process may fail to be a true $\mathbb{P}$-martingale. But this does not really do harm: the process $(\bar{Z}_t^0, \bar{Z}_t^1)_{0 \leqslant t \leqslant T}$ still is a consistent price system under transaction costs λ in the sense of Definition 4.1. Indeed, by the first part of the proof we have that, for $t \in [0,T]$ and $n \in \mathbb{N}$,

$$\frac{\bar{Z}_t^1}{\bar{Z}_t^0} \in [(1-\lambda)S_t, S_t], \qquad \text{a.s. on } \{t \leqslant \tau_n\}.$$

As $\bigcup_{n=1}^{\infty} \{t \leqslant \tau_n\} = \Omega$ a.s., for each fixed $0 \leqslant t \leqslant T$, we have obtained (197). We note in passing that Definition 4.1 was designed in a way that we allow for *local* martingales in the second coordinate $(\bar{Z}_t^1)_{0 \leqslant t \leqslant T}$.

Summing up: we have found a consistent price system $\bar{Z} = (\bar{Z}_t^0, \bar{Z}_t^1)_{0 \leqslant t \leqslant T}$ in the sense of Definition 4.1 such that the terminal value $(\bar{Z}_T^0, \bar{Z}_T^1)$ satisfies

(195). This contradiction shows that the cones $\mathcal{A}$ and $\mathcal{B}$ are in polar duality and finishes the proof of the first assertion of the theorem.

The corresponding assertion for the cones $\mathcal{C} \cap L^\infty$ and $\mathcal{D}$ now follows. For $\varphi_T^0 \in \mathcal{C}$ we have, by definition, that $(\varphi_T^0, 0) \in \mathcal{A}$ so that $\langle (\varphi_T^0, 0), (Z_T^0, Z_T^1) \rangle = \langle \varphi_T^0, Z_T^0 \rangle \leqslant 0$, for each consistent price system $Z = (Z^0, Z^1)$. This yields the inclusion $\mathcal{D} \subseteq (\mathcal{C} \cap L^\infty)^\circ$. Conversely, if $(\varphi_T^0, 0) \notin \mathcal{A}$ we may find by the above argument a consistent price system $\bar{Z}$ such that $\langle (\varphi_T^0, 0), (\bar{Z}_T^0, \bar{Z}_T^1) \rangle > 0$, which yields the inclusion $(\mathcal{C} \cap L^\infty)^\circ \subseteq \mathcal{D}$.

The proof of Proposition 4.17 now is complete. ■

We now are in a position to state and prove the central result of this section, the super-replication theorem (compare Corollary 1.11).

Theorem 4.18. *Suppose that the continuous, adapted process $S = (S_t)_{0 \leqslant t \leqslant T}$ satisfies $(CPS^{\lambda'})$, for each $0 < \lambda' < 1$, and fix $0 < \lambda < 1$.*

Suppose that the $\mathbb{R}^2$-valued random variable $\varphi_T = (\varphi_T^0, \varphi_T^1)$ satisfies

$$V_T^{liq}(\varphi_T^0, \varphi_T^1) = \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \geqslant -M. \tag{198}$$

For a constant $x^0 \in \mathbb{R}$ the following assertions then are equivalent:

(i) $\varphi_T = (\varphi_T^0, \varphi_T^1)$ is the terminal value of some self-financing, admissible trading strategy $(\varphi_t)_{0 \leqslant t \leqslant T} = (\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ under transaction costs λ, starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (x^0, 0)$.

(ii) $\mathbb{E}_Q[\varphi_T^0 + \varphi_T^1 \tilde{S}_T] \leqslant x^0$, for every λ-consistent price system $(\tilde{S}, Q)$.

Proof: First suppose that $\varphi_T = (\varphi_T^0, \varphi_T^1) \in L_S^\infty$. Then (i) is tantamount to $(\varphi_T^0 - x^0, \varphi_T^1)$ being an element of $\mathcal{A} \cap L_S^\infty$. By Proposition 4.17, Theorem 4.15, and the Bipolar Theorem (Proposition A.1 in the Appendix), this is equivalent to

$$\mathbb{E}_Q[\varphi_T^0 - x^0 + \varphi_T^1 \tilde{S}_T] \leqslant 0,$$

holding true for all λ-consistent price systems $(\tilde{S}, Q)$ which amounts to (ii).

Dropping the assumption $\varphi_T \in L_S^\infty$, we consider, for $C \geqslant M$, the C-truncations φ_T^C defined after (192) which are well-defined in view of (198). Recall that $\varphi_T^C \in L_S^\infty$ and $(\varphi_T^C)_{C \geqslant M}$ increases to φ_T, as $C \to \infty$. We may apply the first part of the argument to each φ_T^C and then send C to infinity: assume that (i) (and therefore, equivalently, (ii)) holds true, for each φ_T^C, where C is sufficiently large. Then (ii) also holds true for φ_T by monotone convergence, and (i) also holds true for φ_T by Theorem 4.13. ■

Corollary 4.19. *Under the assumptions of Theorem 4.18, let* $\varphi_T^0 \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ *be a random variable bounded from below, i.e.*

$$\varphi_T^0 \geqslant -M, \qquad\qquad a.s.$$

for some real number M*. For a real constant* x^0 *the following are equivalent.*

(i) $\varphi_T = (\varphi_T^0, 0)$ *is the terminal value of some self-financing, admissible trading strategy* $(\varphi_t)_{0 \leqslant t \leqslant T} = (\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ *under transaction costs* λ*, starting at* $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (x^0, 0)$*.*

(ii) $\mathbb{E}_Q[\varphi_T^0] \leqslant x^0$*, for every* λ*-consistent price system* $(\tilde{S}, Q)$*.*

Proof: Apply Theorem 4.18 to $(\varphi_T^0, 0)$. ■

Non-negative Claims

We shall need the following generalisation of the notion of λ-consistent price systems (compare Def. 5.1 below).

Definition 4.20. *Fix the continuous, adapted, strictly positive process* $S = (S_t)_{0 \leqslant t \leqslant T}$*, and* $\lambda > 0$*. The* λ-consistent equivalent super-martingale deflators *are defined as the set* $\mathcal{Z}^e = \mathcal{Z}^e(1)$ *of strictly positive processes* $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$*, starting at* $Z_0^0 = 1$*, such that, for every* x*-admissible,* λ*-self-financing process* $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$*, starting at* $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0, 0)$*, we have that the process*

$$(x + \varphi_t^0) Z_t^0 + \varphi_t^1 Z_t^1, \qquad\qquad 0 \leqslant t \leqslant T,$$

is a non-negative supermartingale. For $y > 0$ *we write* $\mathcal{Z}^e(y) = y\mathcal{Z}^e$*. By dropping the super-script* e *we define the sets* $\mathcal{Z}(y)$ *of* λ-consistent supermartingale deflators*, where we only impose the non-negativity of the elements* Z*.*

We note that Proposition 4.5 implies that $\mathcal{Z}^e$ contains the λ-consistent price systems, where we identify $(\tilde{S}, Q)$ with the process $(Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$ given by $Z_t^0 = \mathbb{E}[\frac{dQ}{d\mathbb{P}} | \mathcal{F}_t]$ and $Z_t^1 = \tilde{S}_t Z_t^0$.

For the applications in the next chapter, which concerns utility maximization, we shall deal with *positive* elements φ_T^0 only. For this setting we now develop a similar duality theory as in Theorem 4.18 and Corollary 4.19. We start with a definition relating the cones $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ defined in 4.8 and 4.16 above to bounded subsets of $L^0(\mathbb{R}^2)$ and $L_+^0(\mathbb{R})$ respectively.

Definition 4.21. *For $x > 0$, we define*

$$\begin{aligned}\mathfrak{A}(x) &= \{(\varphi_T^0, \varphi_T^1) : \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \geqslant 0, \\ &\qquad \text{and } (\varphi_T^0 - x, \varphi_T^1) \in \mathcal{A}\}, \\ \mathfrak{C}(x) &= \{\varphi_T^0 \geqslant 0 : \varphi_T^0 - x \in \mathcal{C}\} = \{\varphi_T^0 : (\varphi_T^0, 0) \in \mathfrak{A}(x)\}.\end{aligned}$$

For $y > 0$, we define

$$\begin{aligned}\mathfrak{B}(y) &= \{(Z_T^0, Z_T^1) : \text{there is } Z \in \mathcal{Z}(y) \text{ with terminal value } (Z_T^0, Z_T^1)\}, \\ \mathfrak{D}(y) &= \{Z_T^0 : \text{there is } Z \in \mathcal{Z}(y) \text{ with a terminal value } (Z_T^0, Z_T^1), \\ &\qquad \text{for some } Z_T^1\}.\end{aligned}$$

Theorem 4.22. *Suppose that the continuous, strictly positive process $S = (S_t)_{0 \leqslant t \leqslant T}$ satisfies condition $(CPS^{\lambda'})$, for each $0 < \lambda' < 1$. Fix $0 < \lambda < 1$.*

(i) The sets $\mathfrak{A}(x), \mathfrak{C}(x), \mathfrak{B}(y), \mathfrak{D}(y)$ defined in Definition 4.20 are convex, closed (w.r to convergence in measure) subsets of $L^0(\mathbb{R}^2)$ and $L^0_+(\mathbb{R})$ respectively. The sets $\mathfrak{C}(x)$ and $\mathfrak{D}(y)$ are also solid.

(ii) Fix $x > 0, y > 0$ and $\varphi_T^0 \in L^0_+(\mathbb{R})$. We have $\varphi_T^0 \in \mathfrak{C}(x)$ iff

$$\langle \varphi_T^0, Z_T^0 \rangle \leqslant xy, \tag{199}$$

for all $Z_T^0 \in \mathfrak{D}(y)$. In fact, we also have

$$\sup_{(\tilde{S},Q) \in \ CPS^{\lambda}} \mathbb{E}_Q[\varphi_T^0] = xy. \tag{200}$$

(ii′) We have $Z_T^0 \in \mathfrak{D}(y)$ iff

$$\langle \varphi_T^0, Z_T^0 \rangle \leqslant xy \tag{201}$$

for all $\varphi_T^0 \in \mathfrak{C}(x)$.

(iii) The sets $\mathfrak{A}(1)$ and $\mathfrak{C}(1)$ are bounded in $L^0(\mathbb{R}^2)$ and $L^0(\mathbb{R})$ respectively and contain the constant functions $(\mathbb{1}, 0)$ (resp. $\mathbb{1}$).

<ins>Proof:</ins> (i) The convexity of the four sets is obvious. As regards the solidity recall that a set $C \subseteq L^0_+(\mathbb{R})$ is solid if $0 \leqslant \psi_T^0 \leqslant \varphi_T^0 \in C$ implies that $\psi_T^0 \in C$. As regards $\mathfrak{C}(x)$, this property clearly holds true as one is allowed to "throw away bonds" at terminal time T. As regards the solidity of $\mathfrak{D}(y)$: if there is

$Z = (Z_t^0, Z_t^1)_{0\leqslant t\leqslant T} \in \mathcal{Z}(y)$, and Y_T^0 satisfies $0 \leqslant Y_T^0 \leqslant Z_T^0$, we may define an element $Y = (Y_t^0, Y_t^1)_{0\leqslant t\leqslant T} \in \mathcal{Z}(y)$ by letting

$$(Y_t^0, Y_t^1) = \begin{cases} (Z_t^0, Z_t^1), & 0 \leqslant t < T, \\ (Y_T^0, Z_T^1 \frac{Y_T^0}{Z_T^0}), & t = T, \end{cases}$$

which shows the solidity of $\mathfrak{D}(y)$.

The L^0-closedness of $\mathfrak{A}(x)$ and $\mathfrak{C}(x)$, follows from Theorem 4.13. Indeed $x > 0$ corresponds to the admissibility constant $M > 0$ in Theorem 4.13 and the operations of shifting these sets by the constant vector $(x\mathbb{1}, 0)$ and intersecting them with the positive orthant preserves the L^0-closedness.

Let us now pass to the closedness of $\mathfrak{B}(y)$ and $\mathfrak{D}(y)$. Fix a Cauchy sequence $Z_T^n = (Z_T^{0,n}, Z_T^{1,n})$ in $\mathfrak{B}(y)$ and associate to it the supermartingales $Z^n = (Z_t^{0,n}, Z_t^{1,n})_{0\leqslant t\leqslant T}$ as in Def 4.21. Applying Lemma 4.14 and passing to convex combinations similarly in the proof of Theorem 4.13 we may pass to a limiting càdlàg process $Z = (Z_t^0, Z_t^1)_{0\leqslant t\leqslant T}$ in the following way (the "Fatou Limit" construction from [20]).

First pass to pointwise limits of convex combinations of $(Z_r^{0,n}, Z_r^{1,n})_{n=1}^\infty$, where r ranges in the rational numbers in $[0, T]$ and then pass to the càdlàg versions, which exist as the limiting process $(Z_r^0, Z_r^1)_{r\in[0,T]\cap\mathbb{Q}}$ is a super-martingale (we suppose w.l.g. that T is rational). The fact that, for every 1-admissible λ-self-financing $\varphi = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ the process

$$V_t = (1 + \varphi_t^0) Z_t^0 + \varphi_t^1 Z_t^1, \qquad 0 \leqslant t \leqslant T,$$

is a super-martingale, now follows from Fatou's lemma. The argument for $\mathfrak{D}(y)$ is similar.

We thus have proved assertion (i).

(ii) Let $\varphi = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ be an admissible, self-financing process starting at $\varphi_{0_-} = (x, 0)$ and ending at $(\varphi_T^0, 0)$. Let $Z = (Z_t^0, Z_t^1)_{0\leqslant t\leqslant T}$ be a super-martingale deflator starting at $Z_0 = (y, Z_0^1)$, for some $Z_0^1 \in [(1-\lambda) y S_0, y S_0]$. By definition

$$\varphi_t^0 Z_t^0 + \varphi_t^1 Z_t^1, \qquad 0 \leqslant t \leqslant T,$$

is a super-martingale so that inequality (219) holds true.

Conversely, assertion (220) follows from Theorem 4.18 and Corollary 4.19.

(ii') If $Z_T^0 \in \mathfrak{D}(y)$ and $\varphi_T^0 \in \mathfrak{C}(x)$, we have already shown the inequality (221). As regards the "only if" assertion, condition (221) may be rephrased abstractly as the assertion that $\mathfrak{D}(1) = \frac{1}{y}\mathfrak{D}(y)$ equals the polar set of $\mathfrak{C}(1) =$

$\frac{1}{x}\mathfrak{C}(x)$ as defined in (202) below. On the other hand it follows from Proposition 4.17 and Corollary 4.19 that the polar of the set

$$D(1) = \{Z_T^0 \in L_+^0 : \tfrac{dQ}{d\mathbb{P}} = Z_T^0 \text{ for a consistent price } (\tilde{S}, Q)\}$$

equals $\mathfrak{C}(1)$. Hence by the subsequent version of the bipolar theorem we have that, if Z_T^0 satisfies (221), it is an element of the closed, convex, and solid hull of $D(1)$. As $D(1) \subseteq \mathfrak{D}(1)$ we conclude from (i) that this implies $Z_T^0 \in \mathfrak{D}(1)$.

(iii) By hypothesis (CPS^λ) there is a λ-consistent price system $(\tilde{S}, Q)$. We denote by $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$ the corresponding density process in $\mathcal{Z}^e$. For each $\varepsilon > 0$ there is $\delta > 0$ such that, for a subset $A \in \mathcal{F}$ with $\mathbb{P}[A] \geqslant \varepsilon$ we have $\mathbb{E}[\mathbb{1}_A Z_T^0] \geqslant \delta$ and $\mathbb{E}[\mathbb{1}_A Z_T^1] \geqslant \delta$. This shows that $\mathfrak{A}(1)$ is bounded in L^0. The L^0-boundedness of $\mathfrak{C}(1)$ follows and the final assertion is obvious. ■

We have used in the proof of (ii') above the subsequent version of the bipolar theorem pertaining to subsets *of the positive orthant* L_+^0 of L^0.

Proposition 4.23. *([10], compare also [89]) For a subset* $D \subseteq L_+^0(\Omega, \mathcal{F}, \mathbb{P})$ *we define its polar in* L_+^0 *as*

$$D^\circ = \{g \in L_+^0 : \mathbb{E}[gh] \leqslant 1, \text{ for all } h \in D\}. \tag{202}$$

Then the bipolar $D^{\circ\circ}$ *equals the closed, convex, solid hull of* D.

■