5 The local duality theory

In this section we extend the duality theory to the setting where the corresponding concepts such as no arbitrage, existence of consistent price systems etc. only hold *locally*. For example, this situation arises naturally in the stochastic portfolio theory as promoted by R. Fernholz and I. Karatzas. We refer to the paper [60] by I. Karatzas and C. Kardaras (compare also [63]) where the local duality theory is developed in the classical frictionless setting.

Recall that a property (P) of a stochastic process $S = (S_t)_{0 \leqslant t \leqslant T}$ holds locally if there is a sequence of stopping times $(\tau_n)_{n=1}^{\infty}$ increasing to infinity such that each of the stopped processes $S^{\tau_n} = (S_{t \wedge \tau_n})_{0 \leqslant t \leqslant T}$ has property (P).

We say that (P) is a local property if the fact that S has property (P) locally implies that S has property (P).

In the subsequent definition we formulate the notion of a super-martingale deflator in the frictionless setting. The tilde super-scripts indicate that we are in the frictionless setting.

Definition 5.1. *(see [60]) Let $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ be a semi-martingale based on and adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P}$.) The set of* equivalent super-martingale deflators $\tilde{\mathcal{Z}}^e$ *are defined as the* $]0, \infty[$*-valued processes $(\tilde{Z}_t)_{0 \leqslant t \leqslant T}$, starting at $\tilde{Z}_0 = 1$, such that, for every $\tilde{S}$-integrable predictable process $\tilde{H} = (\tilde{H}_t)_{0 \leqslant t \leqslant T}$ verifying*

$$1 + (\tilde{H} \cdot \tilde{S})_t \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{203}$$

the process

$$\tilde{Z}_t(1 + (\tilde{H} \cdot \tilde{S})_t), \qquad 0 \leqslant t \leqslant T \tag{204}$$

is a super-martingale under $\mathbb{P}$. Dropping the super-script e we obtain the corresponding class $\tilde{\mathcal{Z}}$ of $[0, \infty[$*-valued* super-martingale deflators.

We call $\tilde{Z} \in \tilde{\mathcal{Z}}$ a local martingale deflator *if, in addition, $\tilde{Z}$ is a local martingale.*

We say that $\tilde{S}$ satisfies the property (ESD) (resp. (ELD)) of existence of an equivalent super-martingale (resp. local martingale) deflator *if $\tilde{\mathcal{Z}}^e \neq \emptyset$ (resp. there is a local martingale $\tilde{Z}$ in $\tilde{\mathcal{Z}}^e$).*

We remark that, for a probability measure Q equivalent to $\mathbb{P}$ under which $\tilde{S}$ is a local martingale, we have that the density process $\tilde{Z}_t = \mathbb{E}[\frac{dQ}{d\mathbb{P}} | \mathcal{F}_t]$ defines a local martingale deflator.

We first give an easy example of a process $\tilde{S}$, for which (NFLVR) fails while there does exist a super-martingale deflator (see [60, Ex. 4.6] for a more sophisticated example, involving the three-dimensional Bessel process). In fact, we formulate this example in such a way that it also highlights the persistence of this phenomenon under transaction costs.

Proposition 5.2. *There is a continuous semi-martingale* $S = (S_t)_{0 \leqslant t \leqslant 1}$*, based on a Brownian filtration* $(\mathcal{F}_t)_{0 \leqslant t \leqslant 1}$*, such that there is an equivalent super-martingale deflator* $(Z_t)_{0 \leqslant t \leqslant 1}$ *for* S*. On the other hand, for* $0 \leqslant \lambda < \frac{1}{2}$*, there does not exist a* λ*-consistent price system* $(\tilde{S}, Q)$ *associated to* S*.*

Proof: Let $W = (W_t)_{t \geqslant 0}$ be an $(\mathcal{F}_t)_{t \geqslant 0}$-Brownian motion, where $(\mathcal{F}_t)_{t \geqslant 0}$ is the natural (right-continuous, saturated) filtration generated by W.

Define the martingale $Z = \mathcal{E}(-W)$

$$Z_t = \exp(-W_t - \tfrac{t}{2}), \qquad t \geqslant 0,$$

and let $N = Z^{-1}$, i.e.

$$N_t = \exp(W_t + \tfrac{t}{2}), \qquad t \geqslant 0,$$

so that N satisfies the SDE

$$\tfrac{dN_t}{N_t} = dW_t + dt.$$

Define the stopping time τ as

$$\tau = \inf\{t : \; Z_t = \tfrac{1}{2}\} = \inf\{t : \; N_t = 2\},$$

and note that τ is a.s. finite. Define the stock price process S as the time-changed restriction of N to the stochastic interval $[\![0, \tau]\!]$, i.e.

$$S_t = N_{\tan\left(\frac{\pi}{2}(t \wedge \tau)\right)}, \qquad 0 \leqslant t \leqslant 1.$$

By Girsanov there is only one candidate for the density process of an equivalent martingale measure, namely $\left(Z_{\tan\left(\frac{\pi}{2}(t \wedge \tau)\right)}\right)_{0 \leqslant t \leqslant 1}$. But the example is cooked up in such a way that $\left(Z_{\tan\left(\frac{\pi}{2}(t \wedge \tau)\right)}\right)_{0 \leqslant t \leqslant 1}$ only is a *local martingale*. Of course, $\left(Z_{\tan\left(\frac{\pi}{2}(t \wedge \tau)\right)}\right)_{0 \leqslant t \leqslant 1}$ is an equivalent local martingale deflator.

As regards the final assertion, fix $0 \leqslant \lambda < \frac{1}{2}$, and suppose that there is a λ-consistent price system $(\tilde{S}, Q)$. As $\tilde{S} \in [(1-\lambda)S, S]$ we have $\tilde{S}_0 \leqslant 1$ and $\tilde{S}_1 \geqslant 2(1-\lambda) > 1$, almost surely. On the other hand, assuming that $\tilde{S}$ is a Q-super-martingale implies that $\mathbb{E}_Q[\tilde{S}_1] \leqslant \mathbb{E}_Q[\tilde{S}_0]$, and we arrive at a contradiction. ■

Remark 5.3. *For later use we note that* $S_t = N_{\tan\left(\frac{\pi}{2}(t\wedge T)\right)}$ *is the so-called numéraire portfolio (see, e.g. [60]), i.e., the unique process of the form* $1+H\cdot S$ *verifying* $1 + (H \cdot S) \geqslant 0$*, and maximizing the logarithmic utility*

$$u(1) = \sup\{\mathbb{E}[\log(1 + (H \cdot S)_1)]\}.$$

The value function u *above has a finite value, namely* $u(1) = \log(2)$*, and, more generally,* $u(x) = \log(2) + \log(x)$*, although the process* S *does not admit an equivalent martingale measure. In other words, log-utility optimization does make sense although the process* S *obviously allows for an arbitrage as* $S_0 = 1$ *while* $S_1 = 2$*.*

We next resume two notions from [66]. The tilde indicates again that we are in the frictionless setting.

Definition 5.4. *Let* $\tilde{S} = (\tilde{S}_t)_{0\leqslant t\leqslant T}$ *be a semi-martingale.*
For $x > 0, y > 0$*, define the sets*

$$\tilde{\mathfrak{C}}(x) = \{\tilde{X}_T : 0 \leqslant \tilde{X}_T \leqslant x + (\tilde{H} \cdot \tilde{S})_T\}$$

where $\tilde{H}$ *runs through the predictable,* $\tilde{S}$*-integrable processes such that* $(\tilde{H} \cdot \tilde{S})_t \geqslant -x$*, for all* $0 \leqslant t \leqslant T$*, and let*

$$\tilde{\mathfrak{D}}(y) = \{y\tilde{Z}_T\}$$

where $\tilde{Z}_T$ *now runs through the terminal values of super-martingale deflators* $(\tilde{Z}_t)_{0\leqslant t\leqslant T} \in \tilde{\mathcal{Z}}$*.*

Let us comment on the issue of non-negativity versus strict positivity in the definition of $\tilde{\mathfrak{D}}(y)$. This corresponds to the difference between local martingale measures Q for the process $\tilde{S}$ which are either assumed to be equivalent or absolutely continuous with respect to $\mathbb{P}$. It is well-known in this more classical context that the *closure* of the set $\mathcal{M}^e(\tilde{S})$ of equivalent local martingale measures Q involves the passage to absolutely continuous martingale measures. Similarly, to obtain the *closedness* of $\tilde{\mathfrak{D}}(1)$ in the above theorem we have to allow for non-negative processes $(\tilde{Z}_t)_{0\leqslant t\leqslant T} \in \tilde{\mathcal{Z}}$ rather than strictly positive processes $(\tilde{Z}_t)_{0\leqslant t\leqslant T} \in \tilde{\mathcal{Z}}^e$.

We now formulate the analogue of the results of [60] in the context of transaction costs. To stay in line with the present setting we continue to suppose that $S = (S_t)_{0\leqslant t\leqslant T}$ is a continuous process based on and adapted to $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a Brownian filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$.

We present a local version of the fundamental theorem of asset pricing (Theorem 5.6 below) which pertains to the notion of equivalent super-martingale deflators. Here is the corresponding primal notion in terms of arbitrage in the frictionless setting.

Definition 5.5. *[60, Def. 4.1] Let $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ be a semi-martingale. We say that $\tilde{S}$ allows for an* unbounded profit with bounded risk *if there is $\alpha > 0$ such that, for every $C > 0$, there is a predictable, $\tilde{S}$-integrable process $\tilde{H}$ such that*

$$(\tilde{H} \cdot \tilde{S})_t \geqslant -1, \qquad 0 \leqslant t \leqslant T,$$

while

$$\mathbb{P}\Big[(\tilde{H} \cdot \tilde{S})_T \geqslant C\Big] \geqslant \alpha.$$

If $\tilde{S}$ does not allow for such profits, we say that $\tilde{S}$ satisfies the condition $(NUPBR)$ of no unbounded profit with bounded risk.

We now turn to the central result form the paper [60] of I. Karatzas and C. Kardaras. While these authors deal with the more complicated case of general semi-martingales (even allowing for convex constraints) we only deal with the case of continuous semi-martingales $\tilde{S}$. This simplifies things considerably as the problem then boils down to a careful inspection of Girsanov's formula.

Fix the continuous semi-martingale $\tilde{S}$. By the Bichteler-Dellacherie theorem (see, e.g., [75] or [3]), $\tilde{S}$ uniquely decomposes into

$$\tilde{S} = M + A$$

where M is a local martingale starting at $M_0 = \tilde{S}_0$, and A is predictable and of bounded variation starting at $A_0 = 0$. These processes M and A are continuous too and the quadratic variation process $\langle M \rangle_t$ is well-defined and a.s. finite. The so-called "structure condition" introduced by M. Schweizer [83] states that A is a.s. absolutely continuous with respect to $\langle M \rangle$, i.e.,

$$\tfrac{d\tilde{S}_t}{\tilde{S}_t} = dM_t + \varrho_t d\langle M \rangle_t \tag{205}$$

for some predictable process $(\varrho_t)_{0 \leqslant t \leqslant T}$.

If $\tilde{S}$ fails to be representable in the form (205), it is well-known and easy to prove that $\tilde{S}$ allows for arbitrage (in a very strong sense made precise, e.g., in [60, Def. 3.8]). The underlying idea goes as follows: if dA_t fails to be absolutely continuous with respect to $d\langle M \rangle_t$ then one can well-define a predictable trading strategy $H = (H_t)_{0 \leqslant t \leqslant T}$ which equals $+1$ where $dA_t > 0$ and $d\langle M \rangle_t = 0$ and equals -1 where $dA_t < 0$ and $d\langle M \rangle_t = 0$. The strategy H clearly yields an arbitrage.

We therefore may and shall assume the "structure condition"(205) in the sequel. The reader who is not so keen about the formalities of general continuous semi-martingales may very well think of the example of an SDE

$$\tfrac{d\tilde{S}_t}{\tilde{S}_t} = \sigma_t dW_t + \varrho_t dt, \tag{206}$$

where W is a Brownian motion σ and ϱ are predictable process such that $\sigma_t = 0$ implies that $\varrho_t = 0$ without missing anything essential in the subsequent arguments.

Assuming the integrability condition

$$\int_0^T \varrho_t^2 d\langle M\rangle_t < \infty, \quad \text{a.s.} \tag{207}$$

we may well-define the Girsanov density process

$$\tilde{Z}_t = \exp\left\{-\int_0^t \varrho_u dM_u - \tfrac{1}{2}\int_0^t \varrho_u^2 d\langle M\rangle_u\right\} \qquad 0 \leqslant t \leqslant T. \tag{208}$$

By Itô this is a strictly positive local martingale, such that $\tilde{Z}\tilde{S}$ is a local martingale too (compare, e.g., [67]). In particular (208) yields an *equivalent super-martingale deflator*.

The reciprocal $\tilde{N} = \tilde{Z}^{-1}$ is called the numéraire portfolio, i.e.

$$\tilde{N}_t = \exp\left\{\int_0^t \varrho_u dM_u + \tfrac{1}{2}\int_0^t \varrho_u^2 d\langle M\rangle_u\right\}. \tag{209}$$

By Itô's formula $\tilde{N}$ is a stochastic integral on $\tilde{S}$, namely $\frac{d\tilde{N}_t}{\tilde{N}_t} = \varrho_t \frac{d\tilde{S}_t}{\tilde{S}_t}$, and enjoys the property of being the optimal portfolio for the log-utility maximizer. For much more on this issue we refer, e.g., to [2].

Our aim is to characterize condition (207) in terms of the condition $(NUPBR)$ of Definition 5.5. Essentially (207) can fail in two different ways. We shall illustrate this with two proto-typical examples (compare [27]) of processes $\tilde{S}$, starting at $\tilde{S}_0 = 1$. First consider

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = dW_t + (1-t)^{-\frac{1}{2}} dt, \qquad 0 \leqslant t \leqslant 1, \tag{210}$$

so that $\int_0^{1-\varepsilon} \varrho_t^2 dt < \infty$, for all $\varepsilon > 0$, while $\int_0^1 \varrho_t^2 dt = \infty$ almost surely. In this case it is straightforward to check directly that the sequence $(\tilde{N}_{1-\frac{1}{n}})_{n=1}^\infty$, where $\tilde{N}$ is defined in (209), yields an unbounded profit with bounded risk, as $\tilde{N} > 0$ and $\lim_{t\to 1} \tilde{N}_t = \infty$, a.s.

The second example is

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = dW_t + t^{-\frac{1}{2}} dt, \qquad 0 \leqslant t \leqslant 1, \tag{211}$$

so that $\int_0^\varepsilon \varrho_t^2 dt = \infty$, for all $\varepsilon > 0$. This case is trickier as now the singularity is at the beginning of the interval $[0, 1]$, and not at the end. This leads to the concept of *immediate arbitrage* as anlayzed in [27]. Using the law of the iterated logarithm, it is shown there (Example 3.4) that in this case, one may find an $\tilde{S}$-integrand $\tilde{H}$ such that $\tilde{H} \cdot \tilde{S} \geqslant 0$ and $\mathbb{P}[(\tilde{H} \cdot \tilde{S})_t > 0] = 1$, for each $t > 0$. For the explicit construction of $\tilde{H}$ we refer to [27]. As one may multiply $\tilde{H}$ with an arbitrary constant $C > 0$ this again yields an unbounded profit with bounded risk.

Summing up, in both of the examples (210) and (211) we obtain an *unbounded profit with bounded risk.* These two examples essentially cover the general case.

We have thus motivated the following *local* version of the Fundamental Theorem of Asset Pricing (see [60, Th. 4.12] for a more general result).

Theorem 5.6. *Let* $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ *be a* continuous semi-martingale *of the form*

$$\tfrac{d\tilde{S}_t}{\tilde{S}_t} = dM_t + \varrho_t d\langle M \rangle_t,$$

where $(M_t)_{0 \leqslant t \leqslant T}$ *is a local martingale. The following assertions are equivalent.*

(i) *The condition* $(NUPBR)$ *of* no unbounded profit with bounded risk *holds true (Def. 5.5).*

(i') *Locally,* $\tilde{S}$ *satisfies the condition* $(NFLVR)$ *of* no free lunch with vanishing risk.

(i'') *The set* $\tilde{\mathfrak{C}}(1)$ *is bounded in* $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) *The process* ϱ *verifying* (205) *and* (207) *exists and satisfies* $\int_0^T \varrho_t^2 d\langle M \rangle_t < \infty$, *a.s.*

(ii') *The* Girsanov density process $\tilde{Z}$

$$\tilde{Z}_t = \exp\left\{ -\int_0^t \varrho_u dM_u - \tfrac{1}{2}\int_0^t \varrho_u^2 d\langle M \rangle_u \right\}, \qquad 0 \leqslant t \leqslant T,$$

is well-defined and therefore a strictly positive local martingale.

(ii'') *The* numéraire portfolio $\tilde{N} = \tilde{Z}^{-1}$

$$\tilde{N}_t = \exp\left\{ \int_0^t \varrho_u dM_u + \tfrac{1}{2}\int_0^t \varrho_u^2 d\langle M \rangle_u \right\}, \qquad 0 \leqslant t \leqslant T,$$

is well-defined (and therefore a.s. finite).

(iii) *The set of* equivalent super-martingale deflators $\tilde{\mathcal{Z}}^e$ *is non-empty (ESD).*

(iii') *The set of* equivalent local martingale deflators in $\tilde{\mathcal{Z}}^e$, *is non-empty* (ELD).

(iii'') *Locally, the set of equivalent martingale measures is non-empty.*

Proof: $(ii) \Leftrightarrow (ii'') \Leftrightarrow (ii') \Rightarrow (iii') \Leftrightarrow (iii'') \Rightarrow (iii)$ is obvious, and $(i'') \Leftrightarrow (i)$ holds true by Definition 5.5.

$(iii) \Rightarrow (i'')$: By definition, $\tilde{\mathfrak{C}}(1)$ fails to be bounded in L^0 if there is $\alpha > 0$ such that, for each $M > 0$, there is $\tilde{X}_T = 1 + (\tilde{H} \cdot \tilde{S})_T \in \tilde{\mathfrak{C}}(1)$ such that

$$\mathbb{P}[\tilde{X}_T \geqslant M] \geqslant \alpha. \tag{212}$$

Fix $\tilde{Z} \in \tilde{\mathcal{Z}}^e$. The strict positivity of $\tilde{Z}_T$, implies that

$$\beta := \inf\{\mathbb{E}[\tilde{Z}_T \mathbb{1}_A] : \mathbb{P}[A] \geqslant \alpha\}$$

is strictly positive. Letting $M > \frac{1}{\beta}$ in (212) we arrive at a contradiction to the super-martingale assumption

$$1 = \mathbb{E}[\tilde{Z}_0 \tilde{X}_0] \geqslant \mathbb{E}[\tilde{Z}_T \tilde{X}_T] \geqslant \beta M > 1.$$

$(i) \Rightarrow (ii)$ This is the non-trivial implication. It is straightforward to deduce from (i) that there is a predictable process ϱ satisfying (205) (compare [83] and the discussion preceding Theorem 5.6 .) We have to show that (207) is satisfied. The reader might keep the examples (210) and (211) in mind. Define the stopping time

$$\tau = \inf\left\{ t \in [0,T] : \int_0^t \varrho_u^2 d\langle M \rangle_u = \infty. \right\}.$$

Condition (ii) states that $\mathbb{P}[\tau < \infty] = 0$. Assuming the contrary, the set $\{\tau < \infty\}$ then splits into the two $\mathcal{F}_\tau$-measurable sets

$$A^c = \{\tau < \infty\} \cap \left\{ \lim_{t \nearrow \tau} \int_0^t \varrho_u^2 d\langle M \rangle_u = \infty \right\},$$
$$A^d = \{\tau < \infty\} \cap \left\{ \lim_{t \nearrow \tau} \int_0^t \varrho_u^2 d\langle M \rangle_u < \infty \right\},$$

where c refers to "continuous" and d to "discontinuous".

If $\mathbb{P}[A^c] > 0$ it suffices to define the stopping times

$$\tau_n = \inf\left\{t : \int_0^t \varrho_u^2 d\langle M\rangle_u \geqslant 2^n\right\}.$$

For each $n \in \mathbb{N}$, the numéraire portfolio $\tilde{N}_{\tau_n}$ at time τ_n is well-defined and given by

$$\tilde{N}_{\tau_n} = \exp\left\{\int_0^{\tau_n} \varrho_u dM_u + \tfrac{1}{2}\int_0^{\tau_n} \varrho_u^2 d\langle M\rangle_u\right\}.$$

It is straightforward to check that $\tilde{N}_{\tau_n}$ tends to $+\infty$ a.s. on A^c, which gives a contradiction to (i).

We still have to deal with the case $\mathbb{P}[A^c] = 0$ in which case we have $\mathbb{P}[A^d] > 0$. This is the situation of the "Immediate Arbitrage Theorem". We refer to [27, Th. 3.7] for a proof that in this case we may find an $\tilde{S}$-integrable, predictable process $\tilde{H}$ such that $(\tilde{H} \cdot \tilde{S})_t > 0$, for all $\tau < t \leqslant T$ almost surely on A^d. This contradicts assumption (i).

$(ii') \Rightarrow (i')$: Suppose that the Girsanov density process $\tilde{Z}$ is well-defined and strictly positive. We may define, for $\varepsilon > 0$, the stopping time

$$\tau_\varepsilon = \inf\left\{t : \ \tilde{Z}_t \geqslant \varepsilon^{-1}\right\}$$

so that $\mathbb{P}[\tau_\varepsilon < \infty] \leqslant \varepsilon$. The stopped process $\tilde{S}_\varepsilon^\tau$ then admits an equivalent martingale measure, namely $\frac{dQ}{d\mathbb{P}} = \tilde{Z}_{\tau_\varepsilon}$.

$(i') \Rightarrow (i)$ obvious as $(NUPBR)$ is a local property. ■

We now give a similar local version of the Fundamental Theorem of Asset Pricing in the context of transaction costs.

Definition 5.7. *Let $S = (S_t)_{0 \leqslant t \leqslant T}$ be a strictly positive, continuous process. We say that S allows for an* obvious arbitrage *if there are $\alpha > 0$ and $[0,T] \cup \{\infty\}$-valued stopping times $\sigma \leqslant \tau$ with $\mathbb{P}[\sigma < \infty] = \mathbb{P}[\tau < \infty] > 0$ such that either*

$$(a)\ S_\tau \geqslant (1+\alpha)S_\sigma, \qquad a.s.\ on\ \{\sigma < \infty\},$$

or

$$(b)\ S_\tau \leqslant \tfrac{1}{1+\alpha}S_\sigma, \qquad a.s.\ on\ \{\sigma < \infty\}.$$

We say that S allows for an obvious immediate arbitrage *if, in addition, we have*

$$(a)\ S_t \geqslant S_\sigma, \qquad for\ t \in [\![\sigma, \tau]\!],\ a.s.\ on\ \{\sigma < \infty\},$$

or

$$(b)\ S_t \leqslant S_\sigma, \qquad for\ t \in [\![\sigma, \tau]\!],\ a.s.\ on\ \{\sigma < \infty\}.$$

We say that S satisfies the condition (NOA) *(resp.* $(NOIA)$*) of* no obvious arbitrage *(resp.* no obvious immediate arbitrage*) if no such opportunity exists.*

It is indeed rather obvious how to make an arbitrage if (NOA) fails, provided the transaction costs $0 < \lambda < 1$ are smaller than α. Assuming e.g. condition (a), one goes long in the asset S at time σ and closes the position at time τ. In case of an *obvious immediate arbitrage* one is in addition assured that during such an operation the stock price will never fall under the initial value S_σ. In particular this gives an unbounded profit with bounded risk under transaction costs λ.

In the case of condition (b) one does a similar operation by going short in the asset S.

Next we formulate an analogue of Theorem 5.6 in the setting of transaction costs.

Theorem 5.8. *Let $S = (S_t)_{0 \leqslant t \leqslant T}$ be a strictly positive, continuous process. The following assertions are equivalent.*

(i) *Locally, there is no obvious immediate arbitrage* $(NOIA)$.

(i') *Locally, there is no obvious arbitrage* (NOA).

(i'') *Locally, for each $0 < \lambda < 1$, the process S does not allow for an arbitrage under transaction costs λ, i.e.*

$$\mathcal{C} \cap L^0_+ = \{0\}, \tag{213}$$

where $\mathcal{C}$ is the cone given by Definition 4.6 for the stopped process S^τ.

(i''') *Locally, for each $0 < \lambda < 1$, the process S does not allow for a free lunch with vanishing risk under transaction costs λ, i.e.*

$$\overline{\mathcal{C} \cap L^\infty} \cap L^\infty_+ = \{0\}, \tag{214}$$

where the bar denotes the closure with respect to the norm topology of L^∞.

(i'''') *Locally, for each $0 < \lambda < 1$, the process S does not allow for a free lunch under transaction costs λ, i.e.*

$$\overline{\mathcal{C} \cap L^\infty} \cap L^\infty_+ = \{0\}, \tag{215}$$

where now the bar denotes the closure with respect to the weak star topology of L^∞.

(ii) Locally, for each $0 < \lambda < 1$*, the condition* (CPS^λ) *of existence of a* λ-consistent price system *holds true.*

(ii′) For each $0 < \lambda < 1$ *the condition* (CLD^λ) *of existence of a* λ-consistent local martingale deflator *holds true.*

Proof: $(i'''') \Rightarrow (i''') \Rightarrow (i'') \Rightarrow (i') \Rightarrow (i)$ is straight-forward, as well as $(ii) \Leftrightarrow (ii')$.
$(i) \Rightarrow (ii)$: As assumption (ii) is a local property we may assume that S satisfies $(NOIA)$.

To prove (ii) we do a similar construction as in ([41], Proposition 2.1): we suppose in the sequel that the reader is familiar with the proof of [41], Proposition 2.1 and define the – preliminary – stopping time $\bar{\varrho}_1$ by

$$\bar{\varrho}_1 = \inf\left\{t > 0 : \tfrac{S_t}{S_0} \geqslant 1 + \lambda \text{ or } \tfrac{S_t}{S_0} \leqslant \tfrac{1}{1+\lambda}\right\}.$$

In fact, in [41] we wrote $\frac{\varepsilon}{3}$ instead of λ which does not matter as both quantities are arbitrary small.

Define the sets $\bar{A}_1^+, \bar{A}_1^-$, and $\bar{A}_0$ as

$$\bar{A}_1^+ = \{\bar{\varrho}_1 < \infty, S_{\bar{\varrho}_1} = (1+\lambda)S_0\}, \tag{216}$$

$$\bar{A}_1^- = \left\{\bar{\varrho}_1 < \infty, S_{\bar{\varrho}_1} = \tfrac{1}{1+\lambda}S_0\right\}, \tag{217}$$

$$\bar{A}_1^0 = \{\bar{\varrho}_1 = \infty\}. \tag{218}$$

It was observed in [41] that assumption (NOA) by definition rules out the cases $\mathbb{P}[\bar{A}_1^+] = 1$ and $\mathbb{P}[\bar{A}_1^-] = 1$. But under the present weaker assumption $(NOIA)$ we cannot a priori exclude the possibilities $\mathbb{P}[\bar{A}_1^+] = 1$ and $\mathbb{P}[\bar{A}_1^-] = 1$. To refine the argument from [41] in order to apply to the present setting, we distinguish two cases. Either we have $\mathbb{P}[\bar{A}_1^+] < 1$ and $\mathbb{P}[\bar{A}_1^-] < 1$; in this case we let $\varrho_1 = \bar{\varrho}_1$ and proceed exactly as in the proof of ([41], Proposition 2.1) to complete the first inductive step.

The second case is that one of the probabilities $\mathbb{P}[\bar{A}_1^+]$ or $\mathbb{P}[\bar{A}_1^-]$ equals one. We assume w.l.g. $\mathbb{P}[\bar{A}_1^+] = 1$, the other case being similar.

Define the real number $\beta \leqslant 1$ as the essential infimum of the random variable $\min_{0 \leqslant t \leqslant \bar{\varrho}_1} \frac{S_t}{S_0}$. We must have $\beta < 1$, otherwise the pair $(0, \bar{\varrho}_1)$ would define an *immediate obvious arbitrage.* We also have the obvious inequality $\beta \geqslant \frac{1}{1+\lambda}$.

We define, for $1 > \gamma \geqslant \beta$ the stopping time

$$\bar{\varrho}_1^\gamma = \inf\left\{t > 0 \ : \ \tfrac{S_t}{S_0} \geqslant 1 + \lambda \text{ or } \tfrac{S_t}{S_0} \leqslant \gamma\right\}.$$

Defining $\bar{A}_1^{\gamma,+} = \{S_{\bar{\varrho}_1^\gamma} = (1+\lambda)S_0\}$ and $\bar{A}_1^{\gamma,-} = \{S_{\bar{\varrho}_1^\gamma} = \gamma S_0\}$ we find an a.s. partition of $\bar{A}_1^+$ into the sets $\bar{A}_1^{\gamma,+}$ and $\bar{A}_1^{\gamma,-}$. Clearly $\mathbb{P}[\bar{A}_1^{\gamma,-}] > 0$, for $1 > \gamma > \beta$. We claim that $\lim\limits_{\gamma\searrow\beta} \mathbb{P}[\bar{A}_1^{\gamma,-}] = 0$. Indeed, supposing that this limit were positive, we again could find an *obvious immediate arbitrage* as in this case we have that $\mathbb{P}[\bar{A}_1^{\beta,-}] > 0$. Hence the pair of stopping times

$$\sigma = \bar{\varrho}_1^\beta . \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = \beta S_0\}} + \infty \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = (1+\lambda)S_0\}}$$

and

$$\tau = \bar{\varrho}_1 . \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = \beta S_0\}} + \infty \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = (1+\lambda)S_0\}}$$

would define an *obvious immediate arbitrage*.

We thus may find $1 > \gamma > \beta$ such that $\mathbb{P}[\bar{A}_1^{\gamma,-}] < \frac{1}{2}$. After having found this value of γ we can define the stopping time ϱ_1 in its final form as

$$\varrho_1 := \bar{\varrho}_1^\gamma.$$

Next we define, similarly as in (216) and (217) the sets

$$A_1^+ = \{\varrho_1 < \infty, S_{\varrho_1} = (1+\lambda)S_0\}$$

$$A_1^- = \{\varrho_1 < \infty, S_{\varrho_1} = \gamma S_0\}$$

to obtain a partition of Ω into two sets of positive measure.

As in [41] we define a probability measure Q_1 on $\mathcal{F}_{\varrho_1}$ by letting $\frac{dQ_1}{d\mathbb{P}}$ to be constant on these two sets, where the constants are chosen such that $Q_1[A_1^+] = \frac{1-\beta}{1+\lambda-\beta}$ and $Q_1[A_1^-] = \frac{\lambda}{1+\lambda-\beta}$. We then may define the Q_1-martingale $(\tilde{S}_t)_{0\leqslant t\leqslant \varrho_1}$ by

$$\tilde{S}_t = \mathbb{E}_{Q_1}[S_{\varrho_1}|\mathcal{F}_t], \qquad 0 \leqslant t \leqslant \varrho_1,$$

to obtain a process remaining in the interval $[\gamma S_0, (1+\lambda)S_0]$.

The above weights for Q_1 were chosen in such a way to obtain

$$\tilde{S}_0 = \mathbb{E}_{Q_1}[S_{\varrho_1}] = S_0.$$

This completes the first inductive step similarly as in [41]. Summing up, we obtained ϱ_1, Q_1 and $(\tilde{S}_t)_{0\leqslant t\leqslant\varrho_1}$ precisely as in the proof of ([41], Proposition 2.1) with the following additional possibility: it may happen that ϱ_1 does not stop when S_t first hits $(1+\lambda)S_0$ or $\frac{S_0}{1+\lambda}$, but rather when S_t first hits $(1+\lambda)S_0$ or βS_0, for some $\frac{1}{1+\lambda} < \beta < 1$. In this case we have $\mathbb{P}[A_1^0] = 0$ and we made sure that $\mathbb{P}[A_1^-] < \frac{1}{2}$, i.e., we have a control on the probability of $\{S_{\varrho_1} = \beta S_0\}$.

We now proceed as in [41] with the inductive construction of ϱ_n, Q_n and $(\tilde{S}_t)_{0\leqslant t\leqslant \varrho_n}$. The new ingredient is that again we have to take care (conditionally on $\mathcal{F}_{\varrho_{n-1}}$) of the additional possibility $\mathbb{P}[A_n^+] = 1$ or $\mathbb{P}[A_n^-] = 1$. Supposing again w.lg. that we have the first case, we deal with this possibility precisely as for $n = 1$ above, but now we make sure that $\mathbb{P}[A_n^-] < 2^{-n}$.

This completes the inductive step and we obtain, for each $n \in \mathbb{N}$, an equivalent probability measure Q_n on $\mathcal{F}_{\varrho_n}$ and a Q_n-martingale $(\tilde{S}_t)_{0\leqslant t\leqslant \varrho_n}$ taking values in the bid ask spread $([\frac{1}{1+\lambda}S_t, (1+\lambda)S_t])_{0\leqslant t\leqslant \varrho_n}$. We note in passing that there is no loss of generality in having chosen this normalization of the bid ask spread instead of the usual normalization $[(1-\lambda')S', S']$ by passing from S to $S' = (1-\frac{\lambda}{2})S$ and from λ to $\lambda' = \frac{\lambda}{2}$.

There is one more thing to check to complete the proof of (ii) : we have to show that the stopping times $(\varrho_n)_{n=1}^{\infty}$ increase almost surely to infinity. This is verified in the following way: suppose that $(\varrho_n)_{n=1}^{\infty}$ remains bounded on a set of positive probability. On this set we must have that $\frac{S_{\varrho_{n+1}}}{S_{\varrho_n}}$ equals $(1+\lambda)$ or $\frac{1}{1+\lambda}$, except for possibly finitely many $n's$. Indeed, the above requirement $\mathbb{P}[A_n^-] < 2^{-n}$ makes sure that a.s. the novel possibility of moving by a value different from $(1+\lambda)$ or $\frac{1}{1+\lambda}$ can only happen finitely many times. Therefore we may, as in [41], conclude from the continuity and strict positivity of the trajectories of S that ϱ_n increases a.s. to infinity which completes the proof of (ii).

$(ii) \Rightarrow (i'''')$ As (ii) as well as (i'''') are local properties holding true for each $0 < \lambda < 1$, it will suffice to show that (CPS^{λ}) implies (215), for fixed $0 < \lambda < 1$.

Let $(\tilde{S}, Q)$ be a λ-consistent price system and define the half-space H of $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$

$$H = \left\{\varphi_T^0 \in L^{\infty} : \mathbb{E}_Q[\varphi_T^0] \leqslant 0\right\},$$

which is σ^*-closed and satisfies $H \cap L_+^{\infty} = \{0\}$. It follows from Proposition 4.5 that, for all self-financing, admissible trading strategies $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ we have that $(\varphi_t^0 Z_t^0 + \varphi_t^1 Z_t^1)_{0\leqslant t\leqslant T}$ is a super-martingale under Q, which implies that $\mathcal{C} \cap L^{\infty}$. Hence (215) holds true.

■

Recall Theorem 4.22 from the previous section. It states the polarity between the sets $\mathfrak{C}(x)$ and $\mathfrak{D}(y)$ in L_+^0. This result which will turn out to be the basis of the duality theory of portfolio optimization in the next.

The crucial hypothesis in Theorem 4.22 is the assumption of $(CPS^{\lambda'})$, for each $0 < \lambda' < 1$. It turns out that it is sufficient to impose this hypothesis only *locally* i.e. under one of the conditions listed in Theorem 5.8. The proof is rather standard but somewhat lengthy and was carried out in detail in

[18] and [21]. Here we content ourselves to simply stating this result without going through the proof.

Theorem 5.9. *Suppose that the continuous, strictly positive process* $S = (S_t)_{0 \leqslant t \leqslant T}$ *satisfies condition* $(CPS^{\lambda'})$ **locally**, *for each* $0 < \lambda' < 1$. *Fix* $0 < \lambda < 1$.

(i) The sets $\mathfrak{A}(x), \mathfrak{C}(x), \mathfrak{B}(y), \mathfrak{D}(y)$ *defined in Definition 4.21 are convex, closed (w.r to convergence in measure) subsets of* $L^0(\mathbb{R}^2)$ *and* $L^0_+(\mathbb{R})$ *respectively. The sets* $\mathfrak{C}(x)$ *and* $\mathfrak{D}(y)$ *are also solid.*

(ii) Fix $x > 0, y > 0$ *and* $\varphi^0_T \in L^0_+(\mathbb{R})$. *We have* $\varphi^0_T \in \mathfrak{C}(x)$ *iff*

$$\langle \varphi^0_T, Z^0_T \rangle \leqslant xy, \tag{219}$$

for all $Z^0_T \in \mathfrak{D}(y)$. *In fact, we also have*

$$\sup_{(\tilde{S},Q) \in \, CPS^{\lambda}} \mathbb{E}_Q[\varphi^0_T] = xy. \tag{220}$$

(ii′) We have $Z^0_T \in \mathfrak{D}(y)$ *iff*

$$\langle \varphi^0_T, Z^0_T \rangle \leqslant xy \tag{221}$$

for all $\varphi^0_T \in \mathfrak{C}(x)$.

(iii) The sets $\mathfrak{A}(1)$ *and* $\mathfrak{C}(1)$ *are bounded in* $L^0(\mathbb{R}^2)$ *and* $L^0(\mathbb{R})$ *respectively and contain the constant functions* $(\mathbb{1}, 0)$ *(resp.* $\mathbb{1}$*).*