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The Story in a Nutshell

1.1 Arbitrage

The notion of arbitrage is crucial to the modern theory of Finance. It is the corner-stone of the option pricing theory due to F. Black, R. Merton and M. Scholes [BS 73], [M 73] (published in 1973, honoured by the Nobel prize in Economics 1997).

The idea of arbitrage is best explained by telling a little joke: a professor working in Mathematical Finance and a normal person go on a walk and the normal person sees a 100 € bill lying on the street. When the normal person wants to pick it up, the professor says: don't try to do that. It is absolutely impossible that there is a 100 € bill lying on the street. Indeed, if it were lying on the street, somebody else would have picked it up before you. (end of joke)

How about financial markets? There it is already much more reasonable to assume that there are no arbitrage possibilities, i.e., that there are no 100 € bills lying around and waiting to be picked up. Let us illustrate this with an easy example.

Consider the trading of \$ versus € that takes place simultaneously at two exchanges, say in New York and Frankfurt. Assume for simplicity that in New York the \$/€ rate is 1 : 1. Then it is quite obvious that in Frankfurt the exchange rate (at the same moment of time) also is 1 : 1. Let us have a closer look why this is the case. Suppose to the contrary that you can buy in Frankfurt a \$ for 0.999 €. Then, indeed, the so-called "arbitrageurs" (these are people with two telephones in their hands and three screens in front of them) would quickly act to buy \$ in Frankfurt and simultaneously sell the same amount of \$ in New York, keeping the margin in their (or their bank's) pocket. Note that there is no normalising factor in front of the exchanged amount and the arbitrageur would try to do this on a scale as large as possible.

It is rather obvious that in the situation described above the market cannot be in equilibrium. A moment's reflection reveals that the market forces triggered by the arbitrageurs will make the \$ rise in Frankfurt and fall in

New York. The arbitrage possibility will disappear when the two prices become equal. Of course, "equality" here is to be understood as an approximate identity where — even for arbitrageurs with very low transaction costs — the above scheme is not profitable any more.

This brings us to a first — informal and intuitive — definition of arbitrage: an arbitrage opportunity is the possibility to make a profit in a financial market *without risk* and *without net investment of capital*. The *principle of no-arbitrage* states that a mathematical model of a financial market should not allow for arbitrage possibilities.

1.2 An Easy Model of a Financial Market

To apply this principle to less trivial cases than the Euro/Dollar example above, we consider a still extremely simple mathematical model of a financial market: there are two assets, called the bond and the stock. The bond is riskless, hence by definition we know what it is worth tomorrow. For (mainly notational) simplicity we neglect interest rates and assume that the price of a bond equals 1 € today as well as tomorrow, i.e.,

$$B_0 = B_1 = 1$$

The more interesting feature of the model is the stock which is risky: we know its value today, say (w.l.o.g.)

$$S_0 = 1,$$

but we don't know its value tomorrow. We model this uncertainty stochastically by defining S_1 to be a random variable depending on the random element $\omega \in \Omega$. To keep things as simple as possible, we let Ω consist of two elements only, g for "good" and b for "bad", with probability $\mathbf{P}[g] = \mathbf{P}[b] = \frac{1}{2}$. We define $S_1(\omega)$ by

$$S_1(\omega) = \begin{cases} 2 & \text{for } \omega = g \\ \frac{1}{2} & \text{for } \omega = b. \end{cases}$$

Now we introduce a third financial instrument in our model, an *option on the stock* with strike price K: the buyer of the option has the right — but not the obligation — to buy one stock at time $t = 1$ at a predefined price K. To fix ideas let $K = 1$. A moment's reflexion reveals that the price C_1 of the option at time $t = 1$ (where C stands for "call") equals

$$C_1 = (S_1 - K)_+,$$

i.e., in our simple example

$$C_1(\omega) = \begin{cases} 1 & \text{for } \omega = g \\ 0 & \text{for } \omega = b. \end{cases}$$

Hence we know the value of the option at time $t = 1$, *contingent on the value of the stock*. But what is the price of the option today?

The classical approach, used by actuaries for centuries, is to price contingent claims by taking expectations. In our example this gives the value $C_0 := \mathbf{E}[C_1] = \frac{1}{2}$. Although this simple approach is very successful in many actuarial applications, it is not at all satisfactory in the present context. Indeed, the rationale behind taking the expected value is the following argument based on the law of large numbers: in the long run the buyer of an option will neither gain nor lose in the average. We rephrase this fact in a more financial lingo: the performance of an investment into the option would in average equal the performance of the bond (for which we have assumed an interest rate equal to zero). However, a basic feature of finance is that an investment into a risky asset should in average yield a better performance than an investment into the bond (for the sceptical reader: at least, these two values should not necessarily coincide). In our "toy example" we have chosen the numbers such that $\mathbf{E}[S_1] = 1.25 > 1 = S_0$, so that in average the stock performs better than the bond. This indicates that the option (which clearly is a risky investment) should not necessarily have the same performance (in average) as the bond. It also shows that the old method of calculating prices via expectation is not directly applicable. It already fails for the stock and hence there is no reason why the price of the option should be given by its expectation $\mathbf{E}[C_1]$.

1.3 Pricing by No-Arbitrage

A different approach to the pricing of the option goes like this: we can buy at time $t = 0$ a *portfolio* $\varPi$ consisting of $\frac{2}{3}$ of stock and $-\frac{1}{3}$ of bond. The reader might be puzzled about the negative sign: investing a negative amount into a bond — "going short" in the financial lingo — means borrowing money.

Note that — although normal people like most of us may not be able to do so — the "big players" can go "long" as well as "short". In fact they can do so not only with respect to the bond (i.e. to invest or borrow money at a fixed rate of interest) but can also go "long" as well as "short" in other assets like shares. In addition, they can do so at (relatively) low transaction costs, which is reflected by completely neglecting transaction costs in our present basic modelling.

Turning back to our portfolio $\varPi$ one verifies that the value $\varPi_1$ of the portfolio at time $t = 1$ equals

$$\varPi_1(\omega) = \begin{cases} 1 & \text{for } \omega = g \\ 0 & \text{for } \omega = b. \end{cases}$$

The portfolio "replicates" the option, i.e.,

$$C_1 \equiv \varPi_1, \tag{1.1}$$

or, written more explicitly,

$$C_1(g) = \Pi_1(g), \tag{1.2}$$

$$C_1(b) = \Pi_1(b). \tag{1.3}$$

We are confident that the reader now sees why we have chosen the above weights $\frac{2}{3}$ and $-\frac{1}{3}$: the mathematical complexity of determining these weights such that (1.2) and (1.3) hold true, amounts to solving two linear equations in two variables.

The portfolio Π has a well-defined price at time $t = 0$, namely $\Pi_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0 = \frac{1}{3}$. Now comes the "pricing by no-arbitrage" argument: equality (1.1) implies that we also must have

$$C_0 = \Pi_0 \tag{1.4}$$

whence $C_0 = \frac{1}{3}$. Indeed, suppose that (1.4) does not hold true; to fix ideas, suppose we have $C_0 = \frac{1}{2}$ as we had proposed above. This would allow an arbitrage by buying ("going long in") the portfolio Π and simultaneously selling ("going short in") the option C. The difference $C_0 - \Pi_0 = \frac{1}{6}$ remains as arbitrage profit at time $t = 0$, while at time $t = 1$ the two positions cancel out *independently of whether the random element ω equals g or b.*

Of course, the above considered size of the arbitrage profit by applying the above scheme to one option was only chosen for expository reasons: it is important to note that you may multiply the size of the above portfolios with your favourite power of ten, thus multiplying also your arbitrage profit.

At this stage we see that the story with the 100 € bill at the beginning of this chapter did not fully describe the idea of an arbitrage: The correct analogue would be to find instead of a single 100 € bill a "money pump", i.e., something like a box from which you can take one 100 € bill after another. While it might have happened to some of us, to occasionally find a 100 € bill lying around, we are confident that nobody ever found such a "money pump".

Another aspect where the little story at the beginning of this chapter did not fully describe the idea of arbitrage is the question of information. We shall assume throughout this book that all agents have the same information (there are no "insiders"). The theory changes completely when different agents have different information (which would correspond to the situation in the above joke). We will not address these extensions.

These arguments should convince the reader that the *"no-arbitrage principle"* is economically very appealing: in a liquid financial market there should be no arbitrage opportunities. Hence a mathematical model of a financial market should be designed in such a way that it does not permit arbitrage.

It is remarkable that this rather obvious principle yielded a unique price for the option considered in the above model.

1.4 Variations of the Example

Although the preceding "toy example" is extremely simple and, of course, far from reality, it contains the heart of the matter: the possibility of replicating a contingent claim, e.g. an option, by trading on the existing assets and to apply the no-arbitrage principle.

It is straightforward to generalise the example by passing from the time index set $\{0,1\}$ to an arbitrary finite discrete time set $\{0,\ldots,T\}$, and by considering T independent Bernoulli random variables. This binomial model is called the Cox-Ross-Rubinstein model in finance (see Chap. 3 below).

It is also relatively simple — at least with the technology of stochastic calculus, which is available today — to pass to the (properly normalised) limit as T tends to infinity, thus ending up with a stochastic process driven by Brownian motion (see Chap. 4 below). The so-called geometric Brownian motion, i.e., Brownian motion on an exponential scale, is the celebrated *Black-Scholes model* which was proposed in 1965 by P. Samuelson, see [S 65]. In fact, already in 1900 L. Bachelier [B 00] used Brownian motion to price options in his remarkable thesis "Théorie de la spéculation" (a member of the jury and rapporteur was H. Poincaré).

In order to apply the above no-arbitrage arguments to more complex models we still need one additional, crucial concept.

1.5 Martingale Measures

To explain this notion let us turn back to our "toy example", where we have seen that the unique arbitrage free price of our option equals $C_0 = \frac{1}{3}$. We also have seen that, by taking expectations, we obtained $\mathbf{E}[C_1] = \frac{1}{2}$ as the price of the option, which was a "wrong price" as it allowed for arbitrage opportunities. The economic rationale for this discrepancy was that the expected return of the stock was higher than that of the bond.

Now make the following mind experiment: suppose that the world were governed by a different probability than $\mathbf{P}$ which assigns different weights to g and b, such that under this new probability, let's call it $\mathbf{Q}$, the expected return of the stock equals that of the bond. An elementary calculation reveals that the probability measure defined by $\mathbf{Q}[g] = \frac{1}{3}$ and $\mathbf{Q}[b] = \frac{2}{3}$ is the unique solution satisfying $\mathbf{E}_{\mathbf{Q}}[S_1] = S_0 = 1$. Mathematically speaking, the process S is a *martingale* under $\mathbf{Q}$, and $\mathbf{Q}$ is a *martingale measure* for S.

Speaking again economically, it is not unreasonable to expect that in a world governed by $\mathbf{Q}$, the recipe of taking expected values should indeed give a price for the option which is compatible with the no-arbitrage principle. After all, our original objection, that the average performance of the stock and the bond differ, now has disappeared. A direct calculation reveals that in our "toy example" these two prices for the option indeed coincide as

$$\mathbf{E}_{\mathbf{Q}}[C_1] = \tfrac{1}{3}.$$

Clearly we suspect that this numerical match is not just a coincidence. At this stage it is, of course, the reflex of every mathematician to ask: what is precisely going on behind this phenomenon? A preliminary answer is that the expectation under the new measure $\mathbf{Q}$ defines a linear function of the span of B_1 and S_1. The price of an element in this span should therefore be the corresponding linear combination of the prices at time 0. Thus, using simple linear algebra, we get $C_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0$ and moreover we identify this as $\mathbf{E}_{\mathbf{Q}}[C_1]$.

1.6 The Fundamental Theorem of Asset Pricing

To make a long story very short: for a general stochastic process $(S_t)_{0\leq t\leq T}$, modelled on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$, the following statement *essentially* holds true. For any "contingent claim" C_T, i.e. an $\mathcal{F}_T$-measurable random variable, the formula

$$C_0 := \mathbf{E}_{\mathbf{Q}}[C_T] \tag{1.5}$$

yields precisely the arbitrage-free prices for C_T, when $\mathbf{Q}$ runs through the probability measures on $\mathcal{F}_T$, which are equivalent to $\mathbf{P}$ and under which the process S is a martingale (*"equivalent martingale measures"*). In particular, when there is precisely one equivalent martingale measure (as it is the case in the Cox-Ross-Rubinstein, the Black-Scholes and the Bachelier model), formula (1.5) gives the unique arbitrage free price C_0 for C_T. In this case we may "replicate" the contingent claim C_T as

$$C_T = C_0 + \int_0^T H_t dS_t, \tag{1.6}$$

where $(H_t)_{0\leq t\leq T}$ is a predictable process (a *"trading strategy"*) and where H_t models the holding in the stock S during the infinitesimal interval $[t, t+dt]$.

Of course, the stochastic integral appearing in (1.6) needs some care; fortunately people like K. Itô and P.A. Meyer's school of probability in Strasbourg told us very precisely how to interpret such an integral.

The mathematical challenge of the above story consists of getting rid of the word "essentially" and to turn this program into precise theorems.

The central piece of the theory relating the no-arbitrage arguments with martingale theory is the so-called Fundamental Theorem of Asset Pricing. We quote a general version of this theorem, which is proved in Chap. 14.

Theorem 1.6.1 (Fundamental Theorem of Asset Pricing). *For an $\mathbf{R}^d$-valued semi-martingale $S = (S_t)_{0\leq t\leq T}$ t.f.a.e.:*

(i) *There exists a probability measure* $\mathbf{Q}$ *equivalent to* $\mathbf{P}$ *under which* S *is a* sigma-martingale*.*

(ii) S *does not permit a* free lunch with vanishing risk*.*

This theorem was proved for the case of a probability space Ω consisting of finitely many elements by Harrison and Pliska [HP 81]. In this case one may equivalently write *no-arbitrage* instead of *no free lunch with vanishing risk* and *martingale* instead of *sigma-martingale*.

In the general case it is unavoidable to speak about more technical concepts, such as *sigma-martingales* (which is a generalisation of the notion of a local martingale) and *free lunches*. A *free lunch* (a notion introduced by D. Kreps [K 81]) is something like an arbitrage, where — roughly speaking — agents are allowed to form integrals as in (1.6), to subsequently "throw away money" (if they want do so), and finally to pass to the limit in an appropriate topology. It was the — somewhat surprising — insight of [DS 94] (reprinted in Chap. 9) that one may take the topology of *uniform convergence* (which allows for an economic interpretation to which the term "with vanishing risk" alludes) and still get a valid theorem.

The remainder of this book is devoted to the development of this theme, as well as to its remarkable scope of applications in Finance.