The Asymptotic Theory of Transaction Costs

Lecture Notes by
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Introduction

The present lecture notes are based on several advanced courses which I gave at the University of Vienna between 2011 and 2013. In 2015 I gave a similar course ("Nachdiplom-Vorlesung") at ETH Zürich. The purpose of these lectures was to present and organize the recent progress on portfolio optimization under proportional transaction costs $\lambda > 0$. Special emphasis is given to the asymptotic behaviour when λ tends to zero.

The theme of portfolio optimization is a classical topic of Mathematical Finance, going back to the seminal work of Robert Merton in the early seventies (considering the frictionless case without transaction costs). Mathematically speaking, this question leads to a concave optimization problem under linear constraints. A technical challenge arises from the fact that – except for the case of finite probability spaces Ω – the optimization takes place over infinite–dimensional sets.

There are essentially two ways of attacking such an optimization problem.

The *primal* method consists in directly addressing the problem at hand. Following the path initiated by Robert Merton, this leads to a partial differential equation of *Hamilton–Jacobi–Bellman* type. This PDE method can also be successfully extended to the case of proportional transaction costs. Important work on this line was done by G. Constantinides [40], B. Dumas and E. Luciano [75], M. Taksar, M. J. Klass, D. Assaf [234], M. Davis and A. Norman [57], St. Shreve and M. Soner [224], just to name some of the early work on this topic.

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An alternative method consists in passing to the *dual version* of the problem. This dual method is also sometimes called the *martingale method* as one now optimizes over the constraint variables, which in the present context turn out to be martingales (or their generalizations such as super-martingales). In the course of the analysis an important role is played by the *Legendre transform* or *conjugate function* V of the utility function U appearing in the primal version of the problem.

In these notes we focus on the dual method as well as the interplay between the dual and the primal problem. This approach yields to a central concept of our approach, namely the concept of a *shadow price process*. Mathematically speaking, this is an infinite-dimensional generalization of the fundamental concept of a Lagrange multiplier. It has a clear economic interpretation as a price process which – without transaction costs – yields the same optimal portfolio as the original price process under transaction costs. This gives a direct link of the present optimization theory under transaction costs with the more classical frictionless theory.

This brings us to a major open challenge for Mathematical Finance which constitutes much of the motivation for the present notes and the underlying research. The question is how to design an economically as well as mathematically meaningful framework to deal with financial models which are based on *fractional Brownian motion*.