Asymptotic Arbitrage
in Non-Complete Large Financial Markets

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ABSTRACT. Kabanov and Kramkov introduced the notion of "large financial markets". Instead of considering–as usual in mathematical finance–a stochastic stock price process S based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ one considers a sequence $(S^n)_{n \geq 1}$ of such processes based on a sequence $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t \in I^n}, \mathbb{P}^n)_{n \geq 1}$ of filtered probability spaces. The interpretation is that an investor can invest not only in one stock exchange but in several (in the model countably many) stock exchanges.

The usual notion of arbitrage then may be interpreted by "asymptotic" arbitrage concepts, where it is essential to distinguish between two different kinds introduced by Kabanov and Kramkov. If for each $n \in \mathbb{N}$ the market is complete i.e., there is exactly one local martingale measure Q^n for the process S^n on $\mathcal{F}^n$ which is equivalent to $\mathbb{P}^n$, then Kabanov and Kramkov showed that contiguity of $(\mathbb{P}^n)_{n \geq 1}$ with respect to $(Q^n)_{n \geq 1}$ (respectively vice versa) is equivalent to the absence of asymptotic arbitrage of first (respectively second) kind.

In the present paper we extend this result to the non-complete case i.e., where for each $n \in \mathbb{N}$ the set of equivalent local martingale measures for the process S^n is non-empty but not necessarily a singleton. The question arises whether we can extend the theorem of Kabanov and Kramkov to this situation by selecting a proper sequence $(Q^n)_{n \geq 1}$ of equivalent local martingale measures.

It turns out that the theorem characterising asymptotic arbitrage of first kind may be directly extended to this setting while for the theorem characterising asymptotic arbitrage of second kind some modifications are needed. We also provide an example showing that these modifications cannot be avoided.

Introduction

In this paper we deal with arbitrage possibilities in a large financial market, a concept originally introduced by Kabanov and Kramkov, see [10]. Following these authors we define a large financial market to be a sequence of filtered probability spaces $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t \in \mathbb{R}_+}, \mathbb{P}^n)$. On each of these "small" spaces we can trade in $d(n)$ securities, whose price processes are denoted by an $\mathbb{R}^{d(n)}$-valued $(\mathcal{F}_t^n)$-adapted semimartingale.

Key words and phrases. arbitrage, asymptotic arbitrage, contiguity of measures, equivalent martingale measure, free lunch, free lunch with vanishing risk, large financial market.

We suppose that for each "small market" there exists a probability measure Q^n on $\mathcal{F}^n$ that is equivalent to the original measure $\mathbb{P}^n$, such that S^n is a local Q^n-martingale. So we obviously do not have any arbitrage opportunities on the small markets, a fact that is very wellknown (see [8,9]). Nevertheless by choosing a reasonably large portfolio (i.e. by trading on a large number of small markets) we may be able to make some kind of approximation of an arbitrage profit. Kabanov and Kramkov called this form of arbitrage an asymptotic arbitrage and distinguished two kinds. Asymptotic arbitrage of first kind (see definition 1.1 below) can be interpreted as an opportunity of getting arbitrarily rich with positive probability by risking arbitrarily small losses, i.e. taking a "vanishing risk" (compare the notion of "no free lunch with vanishing risk", introduced in [4]). Asymptotic arbitrage of second kind (see definition 1.2 below) describes the possibility of gaining something with probability arbitrarily close to 1, but by taking the risk of loosing a uniformly bounded amount of money (say for example one ECU), i.e. one has the chance of a very likely profit but only with bounded (not vanishing) risk. So the difference between the two kinds of asymptotic arbitrage is that on one hand by risking nearly nothing we can become very rich but only on a set of positive probability, on the other hand we can win something, maybe very little, with very high probability, but there is the possibility to loose one ECU on a set of probability nearly zero.

In the "classical" case where we do not have a sequence of financial markets but only one fixed market there has been done a lot of work relating the absence of arbitrage or similar concepts (such as "no free lunch"[11], "no free lunch with bounded risk"[2,12], "no free lunch with vanishing risk" [4]) to the existence of an equivalent local martingale measure for the price process of the available securities, e.g. [2,4,6,8,9,11,12,13]. In a large financial market there is a similar situation. We will specify conditions on the local martingale measures of the small spaces, respectively on sequences of such measures, that are necessary and sufficient for the absence of asymptotic arbitrage of first or second kind.

Kabanov and Kramkov already presented such conditions but only for the special case when the set of equivalent local martingale measures for each small space (denoted by $\mathcal{M}^e(\mathbb{P}^n)$) consists of a single Q^n (for each n). The notion of contiguity of sequences of probability measures, a concept often used in mathematical statistics, plays an important role. Kabanov and Kramkov showed that $(\mathbb{P}^n)_{n\geq 1}$ is contiguous with respect to $(Q^n)_{n\geq 1}$, respectively $(Q^n)_{n\geq 1}$ contiguous with respect to $(\mathbb{P}^n)_{n\geq 1}$, if and only if there is no asymptotic arbitrage of first kind, respectively second kind. For the general case ($\mathcal{M}^e(\mathbb{P}^n)$ not a singleton) Kabanov and Kramkov proved the sufficiency, i.e. the existence of some sequence $(Q^n)_{n\geq 1}, Q^n \in \mathcal{M}^e(\mathbb{P}^n)$, with the respective contiguity property implies the absence of asymptotic arbitrage of the respective kind.

We generalize these results to the case where $\mathcal{M}^e(\mathbb{P}^n)$ is not necessarily a singleton and observe an interesting asymmetry. For the case of asymptotic arbitrage of first kind we show that the result of Kabanov and Kramkov remains valid (i.e. we prove the necessity of the above condition). In the other case we have to modify the condition on the equivalent local martingale measures and thus establish a condition that is necessary and sufficient for the absence of asymptotic arbitrage of second kind. For the case where $\mathcal{M}^e(\mathbb{P}^n)$ is a singleton this condition equals the contiguity condition of Kabanov and Kramkov. But we present a counterexample where $\mathcal{M}^e(\mathbb{P}^n)$ is not a singleton showing that in this case the necessity of the contiguity condition fails.

1.Definitions and Notations

Let $(B^n)_{n=1}^{\infty} = (\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\in\mathbb{R}_+}, \mathbb{P}^n)_{n=1}^{\infty}$ be a sequence of filtered probability spaces where the filtration satisfies the usual conditions. For $n \in \mathbb{N}$ let $S^n = (S_t^n)_{t\in\mathbb{R}_+}$ be a locally bounded $(\mathcal{F}_t^n)$-adapted semimartingale on B^n with values in $\mathbb{R}^{d(n)}$, describing the (discounted) price of $d(n)$ available securities. We choose the time set to be $\mathbb{R}_+$ to cover the most general case.
We define $\mathcal{M}(\mathbb{P}^n) = \{Q^n \ll \mathbb{P}^n, S^n \text{ local } Q^n\text{-martingale}\}$ to be the set of all absolutely continuous local martingale measures for the process S^n and $\mathcal{M}^e(\mathbb{P}^n) = \{Q^n \in \mathcal{M}(\mathbb{P}^n) | Q^n \sim \mathbb{P}^n\}$ the set of the equivalent ones. As in Kabanov-Kramkov [10] we assume that for any n: $\mathcal{M}^e(\mathbb{P}^n) \neq \emptyset$; in particular it is no restricion at all to require S^n to be a cadlag semimartingale. Moreover on each fixed probability space B^n any condition of no-arbitrage-type is satisfied.

A trading strategy on B^n will be an $\mathbb{R}^{d(n)}$- valued predictable S^n-integrable stochastic process $H^n = (H_t^n)_{t\in\mathbb{R}_+}$. The predictability of H^n describes mathematically the obvious assumption that one should not be able to foresee the future. We only admit general admissible integrands as trading strategies, i.e. we require that there is $a \in \mathbb{R}_+$ such that for all n and t we have that $(H^n \cdot S^n)_t \geq -a$ almost surely, describing the natural idea that the losses of a portfolio should stay bounded (compare [4]).

As we always require $\mathcal{M}^e(\mathbb{P}^n) \neq \emptyset$, whence the admissibility condition on the integrands will imply that, for all n, $(H^n \cdot S^n)_\infty = \lim_{t\to\infty}(H^n \cdot S^n)_t$ exists and is finite a.e. ($(H^n \cdot S^n)$ is an L^1-bounded supermartingale for each $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$). Note that $(H^n \cdot S^n)_t$ denotes the stochastic integral of H^n with respect to S^n at time t and describes the cumulated gains or losses according to the strategy H^n until time t.

We reformulate the notions of Asymptotic Arbitrage of first and second kind (AA1, AA2), see [10]:

Definition 1.1. *A sequence $(H^n)_{n=1}^{\infty}$ of admissible trading strategies realizes asymptotic arbitrage of first kind (AA1), iff*

1. $(H^n \cdot S^n)_t \geq -c_n$ *for all* $t \in \mathbb{R}_+$, *i.e.,* H^n *is* c_n*-admissible,*
2. $\limsup_{n\to\infty} \mathbb{P}^n((H^n \cdot S^n)_\infty \geq C_n) > 0$,

where $c_n > 0$ tend to zero and $C_n > 0$ to infinity.

AA1 describes the possibility of getting arbitrarily rich with positive probability by taking an arbitrarily small (vanishing) risk.

Definition 1.2. *A sequence of admissible trading strategies realizes asymptotic arbitrage of second kind (AA2), iff*

1. $(H^n \cdot S^n)_t \geq -1$, *for all* $t \in \mathbb{R}_+$, *i.e.* H^n *is 1-admissible.*
2. $\exists\, c > 0$, *such that* $\limsup_{n\to\infty} \mathbb{P}^n((H^n \cdot S^n)_\infty \geq c) = 1$.

AA2 can be interpreted as an opportunity of gaining something with probability arbitrarily close to one, but by taking a uniformly bounded risk.

A large financial market satisfies no asymptotic arbitrage of first, respectively second kind (NAA1, NAA2) iff it does not allow the respective arbitrage opportunities.

A sequence of measures $(Q^n)_{n\in\mathbb{N}}$ is called contiguous with respect to $(\mathbb{P}^n)_{n\in\mathbb{N}}$, denoted by $(Q^n)_{n\geq 1} \triangleleft (\mathbb{P}^n)_{n\geq 1}$, iff for all sequences $(A^n)_{n\in\mathbb{N}}$, $A^n \in \mathcal{F}^n$, $\mathbb{P}^n(A^n) \to 0$ implies that $Q^n(A^n) \to 0$.

Kabanov and Kramkov proved the following theorem [10].

Theorem 1.3. *Suppose that, for all n, $\mathcal{M}^e(\mathbb{P}^n)$ consists of a single measure Q^n.*

(1) $(\mathbb{P}^n)_{n\geq 1} \triangleleft (Q^n)_{n\geq 1}$ *if and only if NAA1 is satisfied.*
(2) $(Q^n)_{n\geq 1} \triangleleft (\mathbb{P}^n)_{n\geq 1}$ *if and only if NAA2 is satisfied.*

In order to extend this result to the case where $\mathcal{M}^e(\mathbb{P}^n)$ is not necessarily a singleton we suppose for the rest of the paper that $\mathcal{M}^e(\mathbb{P}^n) \neq \emptyset$, for each n, and gather some notations and results. For each n define

$$\mathcal{K}^n = \{(H^n \cdot S^n)_\infty | H^n \text{ admissible}\}$$

to be the set of all random variables which are outcomes of admissible trading strategies considered as a subset of $L^0(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. Let

$$\mathcal{C}_0^n = \mathcal{K}^n - L_+^0(\Omega^n, \mathcal{F}^n, \mathbb{P}^n),\ \mathcal{C}^n = \mathcal{C}_0^n \cap L^\infty(\Omega^n, \mathcal{F}^n, \mathbb{P}^n).$$

The next result is in the spirit of a theorem of Yor [14] and is a direct consequence of theorem 4.2 of Delbaen, Schachermayer [4].

Theorem 1.4. *$\mathcal{C}^n$ is a weak*- (i.e. $\sigma(L^\infty, L^1)$-) closed cone in $L^\infty(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$.*

The following duality relation between $\mathcal{C}^n$ and $\mathcal{M}^e(\mathbb{P}^n)$ (see, e.g.,[5], theorem 6) will be crucial in our treatment as the proof of theorem 2.1 and 2.2 below will be based on Hahn-Banach arguments.

Lemma 1.5. *An element $g \in L^1(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, normalized by $\mathbb{E}_{\mathbb{P}^n}(g) = 1$, is the density of a measure $Q^n \in \mathcal{M}(\mathbb{P}^n)$ if and only if $\mathbb{E}_{\mathbb{P}^n}(gh) \leq 0$ for all $h \in \mathcal{C}^n$.*
Similarly an element $h \in L^\infty(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ is in $\mathcal{C}^n$ iff $\mathbb{E}_{Q^n}(h) \leq 0$ for all $Q^n \in \mathcal{M}(\mathbb{P}^n)$ (or equivalently for all $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$).

Finally we need the following result of Ansel, Stricker [1], which essentially goes back to Émery [7].

Theorem 1.6. *If M is a d-dimensional local martingale, H a d-dimensional admissible integrand for M, then $H \cdot M$ is a local martingale and therefore being bounded from below a supermartingale.*

2. Results

We suppose that $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\in\mathbb{R}_+}, \mathbb{P}^n)$, S^n is as stated in section 1, in particular that $\mathcal{M}^e(\mathbb{P}^n) \neq \emptyset$ for each $n \in \mathbb{N}$.

Theorem 2.1. *NAA1 is satisfied if and only if there exists a sequence $(Q^n)_{n\geq 1}$, $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ for all n, such that $(P^n)_{n\geq 1} \lhd (Q^n)_{n\geq 1}$.*

Theorem 2.2. *NAA2 is satisfied if and only if for each $\epsilon > 0$ there exist measures $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ and $\delta > 0$, such that for any set $A^n \in \mathcal{F}^n$ with $\mathbb{P}^n(A^n) < \delta$ we have that $Q^n(A^n) < \epsilon$.*

While theorem 2.1 is a straight generalization of the Kabanov-Kramkov theorem, the necessary and sufficient condition stated in theorem 2.2 is of a somewhat technical nature and reflects an interchange of quantifiers. We do *not* assert that we may choose a sequence $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ such that for each $\epsilon > 0$ there is $\delta > 0$ etc. (which would precisely mean that $(Q^n)_{n\geq 1}$ is contiguous with respect to $(\mathbb{P}^n)_{n\geq 1}$). We only assert that for each $\epsilon > 0$ there is a sequence $(Q^n)_{n\geq 1}, Q^n \in \mathcal{M}^e(\mathbb{P}^n)$, (depending on ϵ) such that there is $\delta > 0$ etc. Clearly if $\mathcal{M}^e(\mathbb{P}^n)$ is reduced to a singleton then the choice of $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ cannot depend on ϵ and therefore theorem 2.2 contains the Kabanov-Kramkov theorem as a special case. But the subsequent example shows that this interchange of quantifiers is indeed necessary for theorem 2.2 to be valid in the present generality.

Example 2.3. *There is a sequence of probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ and $\mathbb{R}^n$-valued processes $(S_t^n)_{t\in\{0,1\}}$, such that:*

1. *NAA2 is satisfied, while*
2. *for any sequence $(Q^n)_{n\geq 1}$, $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ for all n, we have that $(Q^n)_{n\geq 1} \not\lhd (\mathbb{P}^n)_{n\geq 1}$.*

3. Proofs

Proof of Theorem 2.1. We just prove the "only if"-part as the "if"-part is proved in Kabanov-Kramkov [10].
So assume that NAA1 is satisfied. We shall construct a sequence $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ such that $(\mathbb{P}^n)_{n\geq 1} \lhd (Q^n)_{n\geq 1}$. Let $M \in \mathbb{R}_+$ and define, for $n \in \mathbb{N}$,

$$D_M^n = \{h^n \in L_+^\infty(\Omega^n, \mathcal{F}^n, \mathbb{P}^n) | \mathbb{E}_{\mathbb{P}^n}(h^n) = 1, \|h^n\|_\infty \leq M\}.$$

<u>Claim :</u> For any $M \geq 1$ exists $\gamma_M > 0$, such that, for all n,

$$(*) \qquad \operatorname{dist}_{\|\cdot\|_\infty}(\mathcal{C}^n, D_M^n) \geq 2\gamma_M,$$

where $\mathcal{C}^n$ is the set defined in section 1.
<u>Proof of the claim :</u> If not, there exists M_0 and a sequence $(n_k)_{k=1}^\infty$ of natural numbers, such that

$$\operatorname{dist}_{\|\cdot\|_\infty}(\mathcal{C}^{n_k}, D_{M_0}^{n_k}) < \frac{1}{4k^2}.$$

That means there exist an admissible integrands H^{n_k} and $f^{n_k} \in L^0_+$, $h^{n_k} \in D^{n_k}_{M_0}$, such that $\|(H^{n_k} \cdot S^{n_k})_\infty - f_{n_k} - h_{n_k}\|_\infty < \frac{1}{4k^2}$, i.e.

$$-\frac{1}{4k^2} + f_{n_k} + h_{n_k} < (H^{n_k} \cdot S^{n_k})_\infty < \frac{1}{4k^2} + f_{n_k} + h_{n_k}, \text{ a.s.}$$

Since f_{n_k}, h_{n_k} are nonnegative we have that $(H^{n_k} \cdot S^{n_k})_\infty \geq -\frac{1}{4k^2}$ a.e.
Besides that $h^{n_k} \in D^{n_k}_{M_0}$ and therefore $\mathbb{P}^{n_k}(\{h^{n_k} \geq \frac{1}{2}\}) \geq \frac{1}{2M_0}$, and this implies that for all k:

$$\mathbb{P}^{n_k}((H^{n_k} \cdot S^{n_k})_\infty \geq \frac{1}{4}) \geq \frac{1}{2M_0}.$$

Now define a new integrand $\widetilde{H}^{n_k} := 4kH^{n_k}$. Then we have that

$$(\widetilde{H}^{n_k} \cdot S^{n_k})_\infty \geq -\frac{1}{k}, \text{a.e.}$$

As we assume that $\mathcal{M}^e(\mathbb{P}^{n_k}) \neq \emptyset$ we have that $\widetilde{H}^{n_k}$ is $\frac{1}{k}$-admissible (see [4], Proposition 3.5). On the other hand $\mathbb{P}^{n_k}((\widetilde{H}^{n_k} \cdot S^{n_k})_\infty \geq k) \geq \frac{1}{2M_0}$ a.e., so we get an asymptotic arbitrage of first kind, thus proving the claim (*).

Take now $M \geq 1$ fixed. By theorem 1.4 we have that $\mathcal{C}^n$ is weak*-closed and D^n_M is clearly weak*-compact. Hence by Hahn-Banach we can find $g^{n,M} \in L^1(\mathbb{P}^n)$, $\|g^{n,M}\|_1 = 1$, such that,

(1) $$\sup_{f^n \in \mathcal{C}^n} \mathbb{E}_{\mathbb{P}^n}(g^{n,M} f^n) \leq \inf_{h^n \in D^n_M} \mathbb{E}_{\mathbb{P}^n}(g^{n,M} h^n) - \gamma_M.$$

As $\mathcal{C}^n$ is a cone the left hand side equals zero and $g^{n,M} \geq 0$, because $-L^\infty_+ \subseteq \mathcal{C}^n$. By lemma 1.5 we see that $g^{n,M}$ is the density of a measure in $\mathcal{M}(\mathbb{P}^n)$.
Moreover we have that for any $h^n \in D^n_M$:

(2) $$\mathbb{E}_{\mathbb{P}^n}(g^{n,M} h^n) \geq \gamma_M.$$

We claim that this implies that

(3) $$\mathbb{P}^n(\{g^{n,M} < \gamma_M\}) < \frac{1}{M}.$$

Indeed, suppose $\mathbb{P}^n(\{g^{n,M} < \gamma_M\}) = p \geq \frac{1}{M}$. Let $h^n := \frac{1}{p}\chi_{\{g^{n,M} < \gamma_M\}}$ which is in D^n_M. Because of (2) $\gamma_M \leq \mathbb{E}_{\mathbb{P}^n}(g^{n,M} h^n) = \frac{1}{p}\mathbb{E}_{\mathbb{P}^n}(g^{n,M}\chi_{\{g^{n,M} < \gamma_M\}}) < \gamma_M$, a contradiction, proving (3).

Let now $M = 1, 2, \dots$ and find for each $M \in \mathbb{N}$ the $\gamma_M > 0$ as above and, for each n, the density $g^{n,M}$ of a measure in $\mathcal{M}(\mathbb{P}^n)$ as above. Define

$$G^n := \sum_{M=1}^{\infty} 2^{-M} g^{n,M},$$

so that G^n is the density of a measure Q^n in $\mathcal{M}(\mathbb{P}^n)$, since–by the local-boundedness assumption on S^n–$\mathcal{M}(\mathbb{P}^n)$ is convex and closed in $L^1(\mathbb{P}^n)$. We even have that Q^n is in $\mathcal{M}^e(\mathbb{P}^n)$, because $\{G^n = 0\} \subseteq \bigcap_{M=1}^{\infty}\{g^{n,M} < \gamma_M\}$.
Finally we claim that $(\mathbb{P}^n)_{n\geq 1} \lhd (Q^n)_{n\geq 1}$.
We have to show that, for $\epsilon > 0$, we can find $\delta > 0$, such that for $A^n \in \mathcal{F}^n$ with $Q^n(A^n) < \delta$ we have that $\mathbb{P}^n(A^n) < \epsilon$.
Let $\epsilon > 0$. Choose M so that $2M^{-1} < \epsilon$ and fix this M. Let now $\delta < 2^{-M}\gamma_M M^{-1}$ and $A^n \in \mathcal{F}^n$ with $Q^n(A^n) < \delta$. We see that

$$\begin{aligned}\mathbb{P}^n(A^n) &= \mathbb{P}^n(A^n \cap \{g^{n,M} < \gamma_M\}) + \mathbb{P}^n(A^n \cap \{g^{n,M} \geq \gamma_M\})\\ &\leq M^{-1} + \mathbb{P}^n(A^n \cap \{G_n \geq 2^{-M}\gamma_M\})\\ &\leq M^{-1} + \frac{2^M}{\gamma_M}Q^n(A^n) < 2M^{-1} < \epsilon.\end{aligned}$$

So $(Q^n)_{n\geq 1} \lhd (\mathbb{P}^n)_{n\geq 1}$ and we are finished. $\square$

PROOF OF THEOREM 2.2. Sufficiency of the ϵ-δ-condition:
Suppose that there is an arbitrage opportunity of second kind, i.e. there is a sequence $(H^n)_{n=1}^{\infty}$ of 1-admissible integrands and $c > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}^n((H^n \cdot S^n)_\infty < c) = 0.$$

Let $\epsilon > 0$ be small enough such that $-\epsilon + c(1-\epsilon) > 0$ and apply the assumption to find $\delta > 0$ and a sequence $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ such that, for $A^n \in \mathcal{F}^n$, $\mathbb{P}^n(A^n) < \delta$, we have that $Q^n(A^n) < \epsilon$. Choose n big enough such that $\mathbb{P}^n((H^n \cdot S^n)_\infty < c) < \delta$ so that $Q^n((H^n \cdot S^n)_\infty < c) < \epsilon$ which implies that

$$\mathbb{E}_{Q^n}((H^n \cdot S^n)_\infty) \geq -\epsilon + c(1-\epsilon) > 0.$$

On the other hand, as $(H^n \cdot S^n)$ is a Q^n-supermartingale (see theorem 1.6)

$$\mathbb{E}_{Q^n}((H^n \cdot S^n)_\infty) \leq 0,$$

a contradicion.
Necessity of the ϵ-δ-condition:
Suppose to the contrary that there exists $1 > \epsilon > 0$ such that for any $\delta > 0$ there is $n \in \mathbb{N}$ such that for each $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ there exists a set $A_{Q^n} \in \mathcal{F}^n$ with

$$\mathbb{P}^n(A_{Q^n}) < \delta, \text{but } Q^n(A_{Q^n}) \geq \epsilon.$$

Choose a sequence $\delta_k \to 0$ and the corresponding n_k as above. Now we construct a sequence of admissible integrands H^{n_k} admitting AA2. We do this by constructing H^{n_k} for each fixed k, so we omit superscripts. Let $\widehat{\mathcal{M}(\mathbb{P})}$ be the cone in $L^1(\mathbb{P})$ generated by the densities $\frac{dQ}{d\mathbb{P}}$, where $Q \in \mathcal{M}(\mathbb{P})$,and let

$$\Gamma := \{g \in L^1_+(\mathbb{P}), \|g\|_1 = 1, g \leq \frac{\epsilon}{4\delta} \text{ a.s.}\}.$$

We claim that

$$\operatorname{dist}_{\|\cdot\|_1}(\Gamma, \widehat{\mathcal{M}(\mathbb{P})}) \geq \frac{\epsilon}{4}. \tag{1}$$

We have to show that for all $\lambda \in \mathbb{R}_+$

$$\operatorname{dist}_{\|\cdot\|_1}(\Gamma, \lambda\mathcal{M}(\mathbb{P})) \geq \frac{\epsilon}{4}.$$

This is obvious for $\lambda \notin]1-\frac{\epsilon}{4}, 1+\frac{\epsilon}{4}[$. So suppose $\lambda \in]1-\frac{\epsilon}{4}, 1+\frac{\epsilon}{4}[$. Let Q be a measure in $\mathcal{M}(\mathbb{P})$, A_Q such that $\mathbb{P}(A_Q) < \delta$ and $Q(A_Q) \geq \epsilon$ and g a function in Γ. Then we have

$$\begin{aligned}\|\lambda\frac{dQ}{d\mathbb{P}} - g\|_1 &\geq |\mathbb{E}(\lambda\frac{dQ}{d\mathbb{P}}\chi_{A_Q}) - \mathbb{E}(g\chi_{A_Q})| \\ &\geq (1-\frac{\epsilon}{4})\epsilon - \frac{\epsilon}{4\delta}\delta > \frac{\epsilon}{4},\end{aligned}$$

whence (1) is proved.
By Hahn-Banach there exists $f \in L^\infty(\mathbb{P})$ such that $\|f\|_\infty = 1$ and

$$\sup_{h\in\widehat{\mathcal{M}(\mathbb{P})}} \mathbb{E}_{\mathbb{P}}(hf) \leq \inf_{g\in\Gamma} \mathbb{E}_{\mathbb{P}}(gf) - \frac{\epsilon}{4}. \tag{2}$$

As $\widehat{\mathcal{M}(\mathbb{P})}$ is a cone we may again conclude that the left hand side equals zero. We deduce from lemma 1.5 that f is in $\mathcal{C}$, i.e. there is an admissible integrand H, such that

$$f \leq (H\cdot S)_\infty. \tag{3}$$

Clearly $(H\cdot S)_\infty \geq -1$ whence H is 1-admissible (compare again Proposition 3.5 of [4].) (2) implies that for any $g \in \Gamma$:

$$\mathbb{E}_{\mathbb{P}}(gf) \geq \frac{\epsilon}{4}. \tag{4}$$

We claim that

$$\mathbb{P}(f < \frac{\epsilon}{4}) < \frac{4\delta}{\epsilon}. \tag{5}$$

Indeed, similarly as in the proof of theorem 2.1 above, suppose $\mathbb{P}(f < \frac{\epsilon}{4}) = p \geq \frac{4\delta}{\epsilon}$. Let $g := \frac{1}{p}\chi_{\{f<\frac{\epsilon}{4}\}}$ which is in Γ. Because of (4) we have that $\frac{\epsilon}{4} \leq \mathbb{E}_{\mathbb{P}}(fg) = \frac{1}{p}\mathbb{E}_{\mathbb{P}}(f\chi_{\{f<\frac{\epsilon}{4}\}}) < \frac{\epsilon}{4}$, a contradicion, proving (5).

Using superscripts again we see that we found a sequence of 1-admissible integrands (H^{n_k}) such that

$$\lim_{k\to\infty} \mathbb{P}^{n_k}(\{(H^{n_k}\cdot S^{n_k})_\infty \geq \frac{\epsilon}{4}\}) \geq \lim_{k\to\infty} \mathbb{P}^{n_k}(\{f^{n_k} \geq \frac{\epsilon}{4}\}) = 1,$$

since $\delta_k \to 0$. Hence the sequence $(H^{n_k})_{k\geq 1}$ admits AA2, a contradiction. $\square$

Construction of Example 2.3. For n and $1 \leq j \leq n$ define the random variable $f^{n,j}$ on a suitable base $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ by

$$f^{n,j} = \begin{cases} 1 & \text{on } A^{n,j}, \mathbb{P}^n(A^{n,j}) = 1 - 2^{-(j+2)} - 2^{-(n+1)} \\ -2^{-j} & \text{on } B^{n,j}, \mathbb{P}^n(B^{n,j}) = 2^{-(j+2)} \\ -2^{j} & \text{on } C^{n,j}, \mathbb{P}^n(C^{n,j}) = 2^{-(n+1)}, \end{cases}$$

such that $(f^{n,j})_{j=1}^n$ are independent. The process S^n is then defined as $S_0^n = 0$, $S_1^n = (f^{n,1},\ldots,f^{n,n})$. Moreover we define the filtration: $\mathcal{F}_0^n = \{\emptyset, \Omega^n\}, \mathcal{F}_1^n = \sigma(f^{n,1},\ldots,f^{n,n})$.

NAA2 is satisfied:
Indeed, otherwise there is some $c > 0$ such that for each $\epsilon > 0$ there is $n \in \mathbb{N}$ and a 1-admissible integrand such that

$$\mathbb{P}^n((H^n \cdot S^n)_1 \geq c) \geq 1 - \epsilon.$$

We shall show that for ϵ small enough this leads to a contradiction. Indeed, in view of the triviality of $\mathcal{F}_0^n$, we may write

$$(H^n \cdot S^n)_1 = \sum_{j=1}^n h^{n,j} f^{n,j}$$

for some real numbers $(h^{n,j})_{j=1}^n$. We will break the random variable $(H^n \cdot S^n)_1$ into three pieces $g_1^n + g_2^n + g_3^n$:

$$g_1^n = \sum_{\{j\leq n : h^{n,j} \leq 0\}} h^{n,j} f^{n,j}.$$

We remark that $\mathbb{P}^n(g_1^n \leq 0) \geq 1 - \sum_{j=1}^n 2^{-(j+2)} - \frac{n}{2^{n+1}} \geq \frac{1}{2}$. For $j_0 \in \mathbb{N}$ to be specified below, let

$$g_2^n = \sum_{\{j\leq j_0 : h^{n,j} > 0\}} h^{n,j} f^{n,j}$$

and remark that $\mathbb{P}^n(g_2^n \leq 0) \geq \prod_{j=1}^{j_0} 2^{-(j+2)} \geq 2^{-j_0^2 - 2j_0}$. As regards the third part

$$g_3^n = \sum_{\{j_0 < j \leq n : h^{n,j} > 0\}} h^{n,j} f^{n,j},$$

we note that in view of the 1-admissibility of H^n and the independence of the $(f^{n,j})_{j=1}^n$ we have $h^{n,j} \leq 2^{-j}$ (for $h^{n,j} > 0$) so that

$$g_3^n \leq \sum_{j=j_0+1}^{n} 2^{-j} \leq 2^{-j_0}.$$

Now choose $j_0 \in \mathbb{N}$, $j_0 \geq 3$, such that $2^{-j_0} < \frac{c}{2}$. If $\mathbb{P}^n((H^n \cdot S^n)_1 \geq c) \geq 1 - \epsilon$ we have that

$$\mathbb{P}^n(g_1^n + g_2^n \geq \frac{c}{2}) \geq 1 - \epsilon.$$

On the other hand

$$\mathbb{P}^n(g_1^n + g_2^n \leq 0) \geq \mathbb{P}^n(g_1^n \leq 0)\mathbb{P}^n(g_2^n \leq 0) \geq 2^{-j_0^2-2j_0-1} \geq 4^{-j_0^2},$$

a contradiction for ϵ small enough, finishing the proof.

We prove now the second part of the assertion in example 2.3. Suppose that $(Q^n)_{n\geq 1}$ is a sequence with $Q^n \in \mathcal{M}^e(\mathbb{P}^n)$ for all n and $(Q^n)_{n\geq 1} \triangleleft (\mathbb{P}^n)_{n\geq 1}$. Since Q^n is a martingale measure we must have, for $j = 1, \ldots, n$,

$$0 = \mathbb{E}_{Q^n}(f^{n,j}) = Q^n(A^{n,j}) - 2^{-(j+1)}Q^n(B^{n,j}) - 2^{j+1}Q^n(C^{n,j}).$$

Since $Q^n(A^{n,j}) + Q^n(B^{n,j}) + Q^n(C^{n,j}) = 1$ (for each j), we see that

$$Q^n(B^{n,j}) = \frac{1 - (2^{(j+1)} + 1)Q^n(C^{n,j})}{1 + 2^{-(j+1)}}.$$

Since $(Q^n)_{n\geq 1} \triangleleft (\mathbb{P}^n)_{n\geq 1}$ we have that $Q^n(C^{n,j}) \to 0$ for each fixed j. Hence, for each fixed j, $Q^n(B^{n,j})$ tends to $\frac{1}{1+2^{-(j+1)}} = 1 - \frac{2^{-(j+1)}}{1+2^{-(j+1)}}$. Choosing an increasing sequence $(n_j)_{j=1}^{\infty}$ such that $Q^{n_j}(B^{n_j,j}) \geq 1 - \frac{2^{-j}}{1+2^{-(j+1)}}$ we arrive at a contradiction as $\mathbb{P}^{n_j}(B^{n_j,j})$ tends to zero. $\square$

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