Illinois Journal of Mathematics
Volume 54, Number 4, Winter 2010, Pages 1463–1480
S 0019-2082

HIDING A CONSTANT DRIFT—A STRONG SOLUTION

VILMOS PROKAJ AND WALTER SCHACHERMAYER

Abstract. Let B be a Brownian motion. We show that there is a process H predictable in the natural filtration of B, such that $H \cdot S$ is a Brownian motion in its own filtration, where $S_t = B_t + t$. In other words, H hides the constant drift. This gives a positive answer to a question posed by Marc Yor.

1. Introduction

In this paper, we continue the work initiated in [4] and [2]. These papers deal with the following question of Marc Yor: let B be a Brownian motion in its natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Is it possible to define an $(\mathcal{F}_t)_{t\geq 0}$ predictable sign process H such that the stochastic integral $H \cdot S$ with $S_t = B_t + t$ gives a Brownian motion in its own filtration?

In [4], a process $(\mu_t)_{t\geq 0}$ uniformly close to a given constant μ was constructed such that, with a suitable choice of H, predictable in the filtration of B, the integral $\beta_t = \int_0^t H_u(dB_u + \mu_u\,du)$ is a Brownian motion in its own filtration. Next, in [2], we found a weak solution to the problem for constant μ, namely we have proved that one can define two predictable process (B, H) in the filtration $(\mathcal{F}_t^W)_{t\geq 0}$ of a Brownian motion W, such that B is a Brownian motion in $(\mathcal{F}_t^W)_{t\geq 0}$ and $\beta = H \cdot S$ is a Brownian motion in its own filtration, where $S_t = B_t + t$. This was a weak solution as H was not adapted to the filtration generated by B.

Received June 15, 2010; received in final form October 3, 2011.

The research of the first author was supported by the European Union and the European Social Fund under the Grant Agreement TÁMOP 4.2.1./B-09/1/KMR-2010-0003.

The second author gratefully acknowledges financial support from the Austrian Science Fund (FWF) under Grant P19456, and from the European Research Council (ERC) under Grant FA506041. Furthermore this work was financially supported by the Christian Doppler Research Association (CDG).

2010 *Mathematics Subject Classification.* Primary 60H05, 60G44, 60J65. Secondary 60G05, 60H10.

In this note, we show that there exists a strong solution, that is H can be defined from B in a predictable way, solving the problem posed by Marc Yor in its original form. Our main theorem is the following.

THEOREM 1. *Let B be a Brownian motion and denote by $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. Fix $\mu\in\mathbb{R}$. There is an $(\mathcal{F}_t)_{t\geq 0}$-predictable process H such that $H\cdot S$ is a Brownian motion in its own filtration, where $S_t=B_t+\mu t$.*

In the next section, we describe the strategy of our solution. Then we motivate our construction in the discrete time setting. Section 4 contains the proof of a simplified version of Theorem 1, which is followed by the proof of Theorem 1 and the proof of the auxiliary results.

2. Heuristic description

In what follows B is a Brownian motion in the filtration $(\mathcal{F}_t)_{t\geq 0}$. The case $\mu=0$ in Theorem 1 is of no interest. If $\mu\neq 0$, then without loss of generality we may assume that $\mu=1$ by the scaling invariance of the Brownian motion. Hence, in most of the paper we may and do assume that $\mu=1$.

Assume, that we have found a predictable process H such that $\beta=H\cdot S$ is a Brownian motion in its own filtration, where $S_t=B_t+t$. Then considering the quadratic variation of β we obtain that H_t takes its value in $\{-1,+1\}$ for almost all t with probability one. It is also easy to see that $\beta=H\cdot S$ is a Brownian motion in its own filtration if and only if H_t is independent of $\mathcal{F}^{\beta}_t$, for almost all $t\geq 0$. Indeed, let γ be any measurable $\mathcal{F}^{\beta}$-adapted bounded process then

$$0=\mathbf{E}\left(\int_0^t \gamma_s\,d\beta_s\right)=\mathbf{E}\left(\int_0^t \gamma_s H_s\,ds\right)=\int_0^t \mathbf{E}\bigl(\gamma_s\mathbf{E}\bigl(H_s|\mathcal{F}^{\beta}_s\bigr)\bigr)\,ds.$$

First, we used that β is Brownian motion in its own filtration, then we substituted its definition and used that B is a Brownian motion. To finish the argument, let γ be a measurable version of the process $\gamma_s=\mathbf{E}(H_s|\mathcal{F}^{\beta}_s)$. This gives that $\mathbf{E}(H_s|\mathcal{F}^{\beta}_s)=0$ for almost all $s\geq 0$ and since $H_s\in\{-1,1\}$ we also obtain that H_s is independent of $\mathcal{F}^{\beta}_s$.

So, we can reformulate our problem as follows:

Is it possible to define a $\{-1,1\}$-valued process H which is predictable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that for almost all $t\geq 0$ the random variable H_t is independent of $\mathcal{F}^{\beta}_t$, where $\beta=H\cdot S$ and $S_t=B_t+t$?

We temporarily relax the requirement that $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by B. Instead, we assume that besides B there is a random variable U independent of B and uniformly distributed on $[0,1]$. Then, $(\mathcal{F}_t)_{t\geq 0}$ is the smallest filtration such that U is $\mathcal{F}_0$ measurable and B is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

This was the starting point of [2], where we defined H such that H_t is one if the value of the random variable U is below its conditional median, given $(\beta_s)_{0\leq s\leq t}$, and minus one otherwise.

Next, we give a heuristic explanation, why we could not obtain a strong solution in [2]. In the next paragraph we assume that the reader is familiar with [2] and freely use its notation. The random sign H is based on $\hat{D}_t(x)$, the conditional distribution function of U given $\mathcal{F}_t^\beta$. By standard arguments, see Lemma 10 of [2], $D_t = \hat{D}_t(U)$ is independent of $\mathcal{F}_t^\beta$ and uniformly distributed on $[0,1]$, hence $H_t = \operatorname{sign}(1/2 - D_t)$ defined by the median rule takes ± 1 with equal probability independently from $\mathcal{F}_t^\beta$. The problem occurs when the process D_t reaches $1/2$, say at the stopping time τ. Then H starts bouncing between plus and minus one. Roughly speaking, this flickering is driven by the fluctuation of $\beta_{\tau+t} - \beta_\tau$ around zero. Hence, S is like the Lévy-transform of β and this prevents β from being adapted to $(\mathcal{F}_t^{S,U})_{t\geq 0}$. This argument will be made precise in Remark 11 below.

A strong solution requires an additional randomness. Instead of the "median rule" a somewhat more complicated rule will be used in the present paper. With the median rule, we divide the range of U into two subsets having equal conditional probability. However, this can be done in many ways, not only with the median rule. Let $h:\mathbb{R}\to\mathbb{R}$ be the function $h(x) = (-1)^{[2x]}$, where $[\cdot]$ denotes the integer part. The random sign $H_t = h(D_t)$ defines the median rule, used in [2]. However, with $H_t = h(D_t + a)$ we also get a random sign independent from $\mathcal{F}_t^\beta$ for any $a \in \mathbb{R}$. Moreover a can be time-varying, for example, the value of an independent Brownian motion W at t. To be precise, we modify the definitions $\hat{D}(x) = \mathbf{P}(U < x | \mathcal{F}_t^{\beta,W})$, $H_t = h(\hat{D}_t(U) + W_t)$ and $\beta = H \cdot S$, where U, W, B are independent, (B, W) is a two dimensional Brownian motion, U uniformly distributed on $[0,1]$ and $S_t = B_t + t$. With this modification, when $\hat{D}_t(U) + W_t$ reaches a point in $\frac{1}{2}\mathbb{Z}$, then the flickering of H comes from the fluctuation of $\hat{D}_t(U) + W_t$. By oversimplifying the problem, that is, replacing $\hat{D}_t(U)$ by β we arrive at the following question:

Is there a strong solution of the perturbed Tanaka equation

$$dX_t = \operatorname{sign}(X_t)\, dB_t + dW_t, \tag{1}$$

where B and W are independent Brownian motions?

We call (1) the perturbed Tanaka equation. It was investigated in [1]. It turns out that if the additional noise W is strong enough then the answer is affirmative. We recall the result of this paper with the following theorem.

THEOREM 2 (Theorem 2 of [1]). *Let (M, N) be a two dimensional continuous local martingale on a filtered probability space. Assume that $\langle M, N\rangle = 0$ and $d\langle M\rangle \leq c\, d\langle N\rangle$ for some constant $c > 0$. Then, the solution of*

$$dX_t = \operatorname{sign}(X_t)\, dM_t + dN_t$$

is pathwise unique.

The heuristic reasoning above, although somewhat over-simplifying, captures an important feature of the problem and in Section 6 below Theorem 2 will be the key tool.

3. The discrete case

To motivate equation (9) below, we illustrate things in a discrete time setting, similarly as in [2]. Fix the time increment Δt and a time grid $t_j = j\Delta t$, $j \ge 0$. Let $\mu \in \mathbb{R}$ be such that $|\mu(\Delta t)^{\frac{1}{2}}| < 1$ and define the process $(S_{t_j})_{j\ge 0}$ by $S_0 = 0$ and requiring that the increments $\Delta S_{t_j} = S_{t_{j+1}} - S_{t_j}$, $j \ge 0$ form an i.i.d. sequence with

$$\mathbf{P}\big(\Delta S_{t_j} = +(\Delta t)^{\frac{1}{2}}\big) = \frac{1+\mu(\Delta t)^{\frac{1}{2}}}{2},$$
$$\mathbf{P}\big(\Delta S_{t_j} = -(\Delta t)^{\frac{1}{2}}\big) = \frac{1-\mu(\Delta t)^{\frac{1}{2}}}{2},$$

so that S is a discrete analog of Brownian motion with drift μ.

Similarly as in [2] let U be a random variable uniformly distributed on $[0,1]$ and independent of S. The filtration $(\mathcal{F}_{t_j})_{j\ge 0}$ is assumed to be such that U is $\mathcal{F}_0$ measurable and $(S_{t_j})_{j\ge 0}$ is $(\mathcal{F}_{t_j})_{j\ge 0}$-adapted. As a new ingredient, in comparison to [2] we suppose that there also is given a sequence of real numbers $(w_{t_j})_{j\ge 0}$. This will play the role of a trajectory of the process W in the above heuristic discussion pertaining to the continuous limit. Since the Brownian motion W is supposed to be independent of all other variables, we can first condition on W and work under the conditional probability. In other words, we can use a typical trajectory of W. In analogy, we assume in the present discrete setting that $(w_{t_j})_{j=0}^{\infty}$ simply is any given sequence of real numbers.

As in [2], we shall construct inductively a predictable, $\{-1,1\}$-valued process $(H_{t_j})_{j\ge 0}$ such that $((H\cdot S)_{t_j})_{j\ge 0}$ is an unbiased random walk, that is, a martingale, in its own filtration.

To do so, we again construct inductively the $[0,1]$-valued process $D_{t_j} = \hat{D}_{t_j}(U)$ adapted to $(\mathcal{F}_{t_j})_{j\ge 0}$. Denoting by $(\mathcal{G}_{t_j})_{j\ge 0}$ the filtration generated by $H\cdot S$ we want to have that the law of the random variable D_{t_j} conditionally on $\mathcal{G}_{t_j}$, equals the law of the random variable U, that is, is uniform on $[0,1]$. This is achieved by evaluating the conditional distribution function of U given $\mathcal{G}_{t_j}$, denoted by $(\hat{D}_{t_j}(x))_{x\in[0,1]}$, at the random point U. We use here that $\hat{D}_{t_j}(x)$ is continuous in x. As we shall see, it even has a density.

We start by letting $D_{t_0} = \hat{D}_{t_0}(U) = U$. In [2], we then defined the $\mathcal{F}_{t_0}$-measurable $\{-1,1\}$-valued random variable H_{t_1} by the "median rule"

$$H_{t_1} = \operatorname{sign}\left(\frac{1}{2} - \hat{D}_{t_0}(U)\right) = \operatorname{sign}\left(\frac{1}{2} - U\right). \tag{2}$$

In the present paper, we replace this rule by the "w_t-shifted median rule," that is,

$$H_{t_{j+1}} = \begin{cases} +1 & \text{if } \hat{D}_{t_j}(U) \dotplus w_{t_j} \in]0, \frac{1}{2}[, \\ -1 & \text{if } \hat{D}_{t_j}(U) \dotplus w_{t_j} \in]\frac{1}{2}, 1[, \end{cases} \tag{3}$$

where $\dotplus$ denotes addition modulo one.

For example, if $w_{t_0} = 0$ or, more generally, if $w_{t_j} \in \mathbb{Z}$, we again find the "median rule" (2) which was used in [2].

If $w_{t_0} \in]0, \frac{1}{2}[$ or, more generally, $w_{t_0} \in]0, \frac{1}{2}[+\mathbb{Z}$, we have that H_{t_1} equals 1, if $U \in]0, \frac{1}{2} - w_{t_0}[\cup]1 - w_{t_0}, 1[$, and H_{t_1} equals -1, if $U \in]\frac{1}{2} - w_{t_0}, 1 - w_{t_0}[$. The case $w_{t_0} \in]\frac{1}{2}, 1[+\mathbb{Z}$ is similar.

Clearly $\mathbf{P}(H_{t_1} = 1) = \mathbf{P}(H_{t_1} = -1) = \frac{1}{2}$, independently of the choice of w_{t_0}. What is slightly different now in comparison to [2] is the Bayesian updating rule to obtain $\hat{D}_{t_1}(x)$ which is the conditional law of U given $H_{t_1} S_{t_1}$.

To fix ideas let us assume that $w_{t_0} \in]0, \frac{1}{2}[$. Conditionally on the event $\{H_{t_1} S_{t_1} = (\Delta t)^{1/2}\}$, we obtain for the conditional distribution function $\hat{D}_{t_1}(x)$

$$\hat{D}_{t_1}(x) = \begin{cases} \bigl(1 + \mu(\Delta t)^{\frac{1}{2}}\bigr)x \quad \text{for } 0 \le x \le \frac{1}{2} - w_{t_0}, \\ \bigl(1 + \mu(\Delta t)^{\frac{1}{2}}\bigr)\bigl(\frac{1}{2} - w_{t_0}\bigr) \\ \quad + \bigl(1 - \mu(\Delta t)^{\frac{1}{2}}\bigr)\bigl(x - \bigl(\frac{1}{2} - w_{t_0}\bigr)\bigr) \\ \qquad\qquad\qquad \text{for } \frac{1}{2} - w_{t_0} \le x \le 1 - w_{t_0}, \\ \bigl(1 + \mu(\Delta t)^{\frac{1}{2}}\bigr)\bigl(\frac{1}{2} - w_{t_0}\bigr) + \bigl(1 - \mu(\Delta t)^{\frac{1}{2}}\bigr)\frac{1}{2} \\ \quad + \bigl(1 + \mu(\Delta t)^{\frac{1}{2}}\bigr)\bigl(x - (1 - w_{t_0})\bigr) \\ \qquad\qquad\qquad \text{for } 1 - w_{t_0} \le x \le 1. \end{cases}$$

The random variable $D_{t_1} = \hat{D}_{t_1}(U)$ is defined as the conditional distribution function $\hat{D}_{t_1}(\cdot)$ at the random point U.

Using the function $f(x) = \operatorname{dist}(x, \mathbb{Z})$ and remembering that we condition on the event $\{H_{t_1} S_{t_1} = (\Delta t)^{1/2}\}$, this may be compactly written as

$$\hat{D}_{t_1}(U) - \hat{D}_{t_0}(U) = \mu\bigl(f\bigl(w_{t_0} + \hat{D}_{t_0}(U)\bigr) - f(w_{t_0})\bigr) H_{t_1} S_{t_1}. \tag{4}$$

This is the special case, for $j = 0$, of the formula

$$\Delta D_{t_j} = \mu\bigl(f(w_{t_j} + D_{t_j}) - f(w_{t_j})\bigr)\Delta(H \cdot S)_{t_j}. \tag{5}$$

In fact, repeating the above argument one verifies that formula (5) gives the random variable D_{t_j}, which is nothing else then the conditional distribution function $\hat{D}_{t_j}(\cdot)$ of U given $\mathcal{G}_{t_j}$ evaluated at the random point U, hence

$$\operatorname{law}\bigl[D_{t_j} | \bigl((H \cdot S)_{t_i}\bigr)_{i=0}^{j}\bigr] = \operatorname{law}[U]. \tag{6}$$

Using the above defined function $f(x) = \operatorname{dist}(x, \mathbb{Z})$ formula (3) may in fact be written more compactly as

$$H_{t_{j+1}} = f'(w_{t_j} + D_{t_j}). \tag{7}$$

We recall that, as in [2] formulas (2), (3) and (7) are only defined almost surely. In the present discrete time setting, this does not cause a problem as only countably many time steps are involved.

Summing up our new findings as compared to [2] in the discrete case: for any real sequence $(w_{t_j})_{j\geq 0}$, formula (7) inductively defines a predictable process $(H_{t_j})_{j\geq 0}$, for which we get, for $j \geq 1$,

$$\mathbf{P}\big(H_{t_{j+1}} = 1 | \big((H \cdot S)_{t_i}\big)_{i=0}^{j}\big) = \mathbf{P}\big(H_{t_{j+1}} = -1 | \big((H \cdot S)_{t_i}\big)_{i=0}^{j}\big) = \frac{1}{2},$$

as follows from the fact that the Bayesian updating rule (5) inductively defines the random variable D_{t_j} verifying (6).

We did these extra miles involving an additional trajectory w of a process W in order to prepare for the continuous time limit of the above construction, in particular for the SDE

$$dD_t = \mu\big(f(W_t + D_t) - f(W_t)\big) H_t \, dS_t \tag{8}$$

which is the formal limit of the difference equation (5). Here the continuous time process $S_t = B_t + \mu t$ is a standard Brownian motion $(B_t)_{t\geq 0}$ with drift μ and $H_t = f'(W_t + D_t)$. While in [2] we considered the situation $W = 0$, we now let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion independent of the Brownian motion B.

The crucial issue is the question whether the SDE has a solution $D = (D_t)_{t\geq 0}$, with initial value $D_0 = U$, where the uniformly distributed $[0,1]$-valued random variable U is independent of B and W.

In [2], it was shown that, letting $W = 0$, there is a unique *weak solution* $D = (D_t)_{t\geq 0}$ of (8) but *no strong solution*. In the present paper, we shall show that, if we choose W to be a standard Brownian motion independent of U and B, we obtain *a unique strong solution* to (8). Why the presence of a nontrivial W changes the situation drastically was motivated in the previous section and is explained in more detail in [1].

We finish this section with two remarks. The first is on the drift μ. If we replace the constant drift $\mu \in \mathbb{R}$ above by a bounded, real-valued process $(\mu_t)_{t\geq 0}$, which is *adapted to the filtration of* W, nothing essential changes in the above discrete setting as well as in the continuous setting analyzed below, so that the results formulated for constant μ carry over to this setting in a straightforward way. If, however, the process $(\mu_t)_{t\geq 0}$ also depends on B and/or U, things are not so evident and we do not know whether there is a positive solution to Yor's question in this general framework.

For the second remark, let $\mu \neq 0$ be constant and H a $\{-1, 1\}$-valued process obtained from Theorem 1. It defines a transformation T on the Wiener space $C([0,1])$ which takes the Brownian motion B into another Brownian motion $\beta = H \cdot S$, where $S_t = B_t + \mu t$. Of course, the transformation T is not unique in any sense, there are many possible choices even in the construction presented below. Nonetheless, T is a measure preserving transformation on the Wiener-space, similarly to the Lévy-transform. We do not know whether it is possible to construct H in such a way that the corresponding transformation T is ergodic.

4. Simplified form of Theorem 1

We shall show in Section 5 below how to deduce Theorem 1 from the following more technical version.

PROPOSITION 3. *Let B be a Brownian motion, and U a random variable independent of B, uniformly distributed on $[0,1]$. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the smallest filtration satisfying the usual hypotheses, such that U is $\mathcal{F}_0$ measurable, and B is $(\mathcal{F}_t)_{t\geq 0}$ adapted.*

Fix $\mu \in \mathbb{R}$. Then, there is an $(\mathcal{F}_t)_{t\geq 0}$-predictable process H such that

(i) *$\beta = H \cdot S$ is a Brownian motion in its own filtration, where $S_t = B_t + \mu t$,*
(ii) *$\tau = \inf\{t > 0 : \mathcal{L}(\beta)_t \neq \mathcal{L}(S)_t\}$ is almost surely positive, where $\mathcal{L}$ is the "Lévy" transform $\mathcal{L}(X) = \operatorname{sign}(X) \cdot X$.*

For the proof of Proposition 3, we need some preparation. Denote by f the function $f(x) = \operatorname{dist}(x, \mathbb{Z})$, where dist denotes the distance of its arguments. It is a periodic function with period one. On the interval $[0,1]$ it equals $x \wedge (1-x)$. Let us consider the formal analogue to (5), that is,

$$dD_t = \mu\big(f(D_t + w_t) - f(w_t)\big) f'(D_t + w_t)\, dS_t, \quad D_0 = U, \tag{9}$$

where $S_t = B_t + \mu t$, f' is the derivative of f and w is a deterministic continuous function. At the points of $\frac{1}{2}\mathbb{Z}$, where f' is undefined, we set the value of f' to 1.

We prove in Section 6 below the following statement.

PROPOSITION 4. *There is a continuous function $w : [0,\infty) \to \mathbb{R}$ such that the stochastic differential equation (9) admits a strong solution, that is we have a process D adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and satisfying (9).*

In particular, a typical trajectory of a Brownian motion can serve as w.

Letting $H_t = f'(D_t + w_t)$ we want to prove that $\beta = H \cdot S$ is a Brownian motion in its own filtration, following the ideas used in [2]. To show this, it is enough to prove that $(\beta_t)_{t\in[0,T]}$ has the correct law for each $T > 0$.

On $\mathcal{F}_T$ we can define a new measure $\mathbf{Q}$ by the Cameron–Martin formula $d\mathbf{Q} = \exp\{-\mu B_T - T\mu^2/2\}\, d\mathbf{P}$. Under $\mathbf{Q}$ the process $(S_t)_{t\in[0,T]}$ is a Brownian motion and therefore $(\beta_t)_{t\in[0,T]}$ is so too, as H takes values in $\{-1,+1\}$.

Hence, it is enough to prove that on $\mathcal{F}_T^\beta$ the measures $\mathbf{P}$ and $\mathbf{Q}$ coincide. In other words, it is enough to show that

$$\mathbf{E}_{\mathbf{Q}}\left(\frac{d\mathbf{P}}{d\mathbf{Q}}\middle|\mathcal{F}_T^\beta\right)=1. \tag{10}$$

Here $d\mathbf{P}/d\mathbf{Q}=\exp\{\mu B_T+T\mu^2/2\}=\exp\{\mu S_T-T\mu^2/2\}$.

Now, let us consider for $x\in[0,1]$ the parametric SDE:

$$d\hat{D}_t(x)=\mu\big(f\big(\hat{D}_t(x)+w_t\big)-f(w_t)\big)\,d\beta_t,\quad \hat{D}_0(x)=x. \tag{11}$$

We use the notation $\hat{D}$ rather than D to distinguish between the family $\hat{D}_t(x)$, $x\in[0,1]$ of processes and the single process D_t which is the solution of (9).

We use the next result, whose proof is also deferred to Section 6. It is somewhat more general than we actually need. Roughly speaking it states that under suitable conditions the solution of a one dimensional SDE as a function of the initial value is absolutely continuous.

PROPOSITION 5. *Let β be a continuous semimartingale on the filtered probability space $(\Omega,(\mathcal{F}_t)_{t\geq0},\mathbf{P})$ and $F:\mathbb{R}\times\Omega\times[0,\infty)\to\mathbb{R}$ be a bounded mapping. Assume that*

(i) *F is $\mathcal{B}(\mathbb{R})\times\mathcal{P}$ measurable where $\mathcal{P}$ is the predictable sigma-field of $\Omega\times[0,\infty)$.*
(ii) *There is a constant $L>0$, such that for each fixed (ω,t) the real function $x\mapsto F(x,\omega,t)$ is Lipschitz continuous with constant L. The derivative in x, also $\mathcal{B}(\mathbb{R})\times\mathcal{P}$ measurable, is denoted by ∂_xF.*

Then, the parametric equation

$$d\hat{D}_t(x)=F\big(\hat{D}_t(x),\cdot,t\big)\,d\beta_t,\quad \hat{D}_0(x)=x, \tag{12}$$

has a solution which is continuous in (x,t) almost surely. This solution is absolutely continuous in x. If

$$\bar{S}_t(x)=\int_0^t\partial_xF\big(\hat{D}_s(x),\cdot,s\big)\,d\beta_s$$

is the $\mathcal{B}(\mathbb{R})\times\mathcal{P}$ measurable version of the parametric integral on the right, then $Y_t(x)=\exp\{\bar{S}_t(x)-\frac{1}{2}\langle\bar{S}(x)\rangle_t\}$ is a possible choice for the Radon–Nikodym derivative of $x\mapsto\hat{D}_t(x)$.

In our case, $F(x,\omega,t)=\mu(f(x+w_t)-f(w_t))$ clearly fulfills the assumptions of Proposition 5. The process β is a continuous semimartingale in the filtration $(\mathcal{F}_t)_{t\geq0}$ hence the same holds in its natural filtration $(\mathcal{F}_t^\beta)_{t\geq0}$. Since $\hat{D}_T(0)=0$ and $\hat{D}_T(1)=1$, we get by Proposition 5

$$1=\hat{D}_T(1)-\hat{D}_T(0)=\int_0^1Y_t(x)\,dx=\int_0^1e^{\bar{S}_T(x)-\langle\bar{S}(x)\rangle_T/2}\,dx, \tag{13}$$

where

$$\bar{S}_T(x) = \mu \int_0^T f'\big(\hat{D}_s(x) + w_s\big)\, d\beta_s \tag{14}$$

is clearly $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_T^\beta$ measurable.

We still need a simple observation on parametric processes, whose proof is left to the reader.

PROPOSITION 6. *Let β be a continuous semimartingale on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and $\xi_t(x,\omega)$ be a parametric $\mathcal{B}(\mathbb{R}) \times \mathcal{P}$ measurable process, where $\mathcal{P}$ is the predictable σ-field. Assume that the stochastic integral $X(x) = \xi(x) \cdot \beta$ exists for all x and $X_t(x,\omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{P}$ measurable.*

If U is $\mathcal{F}_0$-measurable, then $X_t(U) = \int_0^t \bar{\xi}_s\, d\beta_s$, where $\bar{\xi}_s = \xi_s(U)$.

We apply Proposition 6 to obtain that the process $\hat{D} := \hat{D}(U)$ solves the equation

$$d\hat{D}_t = \mu\big(f(\hat{D}_t + w_t) - f(w_t)\big)\, d\beta_t, \quad \hat{D}_0 = U.$$

The solution of this equation is pathwise unique and since by the definition of β, the process D also solves this equation, we have $D = \hat{D}(U)$. Then, $H_t = f'(D_t + w_t) = f'(\hat{D}_t(U) + w_t)$ and the combination of (14) and Proposition 6 gives that

$$\bar{S}_t(U) = \int_0^t \mu f'\big(\hat{D}_s(x) + w_s\big)\, d\beta_s\bigg|_{x=U} = \mu \int_0^t f'\big(\hat{D}_s(U) + w_s\big)\, d\beta_s = \mu S_t.$$

Proof of Proposition 3. We use (13) and apply the formula $\mathbf{E}(g(U,X)|X) = \mathbf{E}(g(U,x))|_{x=X}$, which holds provided that U, X are independent. We apply this formula under $\mathbf{Q}$ with $X = (\beta_t)_{t\in[0,T]}$ and

$$g\big(x, (\beta_t)_{t\in[0,T]}\big) = \exp\big\{\bar{S}_T(x) - \big\langle \bar{S}_T(x)\big\rangle_T/2\big\} = \exp\big\{\bar{S}_T(x) - \mu^2 T/2\big\}.$$

Recall that $(\beta_t)_{t\in[0,T]}$ is a Brownian motion under $\mathbf{Q}$ in the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the variable U is $\mathcal{F}_0$-measurable, whence the independence. Thus,

$$\begin{aligned}\mathbf{E}_\mathbf{Q}\big(e^{\mu S_T - T\mu^2/2} | \mathcal{F}_T^\beta\big) &= \mathbf{E}_\mathbf{Q}\big(e^{\bar{S}_T(U) - T\mu^2/2} | \mathcal{F}_T^\beta\big)\\ &= \int_0^1 e^{\bar{S}_T(x) - \langle \bar{S}(x)\rangle_T/2}\, dx = \hat{D}_T(1) - \hat{D}_T(0) = 1.\end{aligned} \tag{15}$$

Thus, (10) holds and Property (i) of Proposition 3 follows.

To finish the proof put $\tau = \inf\{t > 0 : D_t + w_t \in \frac{1}{2}\mathbb{Z}\}$. Then τ is strictly positive almost surely, and up to the stopping time τ the sign process H is identical to $\operatorname{sign}(\frac{1}{2} - U)$. This implies that $(\mathcal{L}(\beta))_{t\wedge\tau} = (\mathcal{L}(S))_{t\wedge\tau}$, hence Property (ii) of Proposition 3 also holds. □

Next we formulate some peculiar properties of the construction. Throughout, we use the notation introduced in Proposition 3 and in its proof.

Integrating over $[0, x]$ in (15) instead of $[0, 1]$, we obtain the next statement.

COROLLARY 7. *The parametric process $\hat{D}_t(x)$ defined by* (11) *satisfies*

$$\hat{D}_t(x) = \mathbf{P}\big(U < x | \mathcal{F}_t^\beta\big).$$

Similarly, as in [2], U is encoded in the sample path of β, provided w in (9) is e.g. a typical trajectory of a Brownian motion. More precisely, we assume that w has *divergent occupation density* (local time) in the following sense

(i) the function $s(t,x) = \int_0^t \mathbb{1}_{(w_s<x)}\, ds$. is absolutely continuous in the variable x for all t and
(ii) $\lim_{t\to\infty} \partial_x s(t,x) = \infty$ for Lebesgue almost all x.

These properties clearly hold for a typical trajectory of a Brownian motion.

COROLLARY 8. *Assume that $\mu \neq 0$ and the shift w used in* (9) *has divergent occupation density. Then the random variable U is $\sigma(\beta)$ measurable.*

Proof. Note that by Corollary 7 the process $\hat{D}_t(x)$ defined by (11) is a closed martingale under $\mathbf{P}$ and

$$\hat{D}_\infty(x) = \lim_{t\to\infty} \hat{D}_t(x) = \mathbf{P}\big(U < x | \sigma(\beta)\big) \quad \text{for each } x \in [0,1].$$

Since, for a fixed x the process $\hat{D}(x)$ is a convergent martingale its quadratic variation remains bounded almost surely. This can only happen if the limit $\hat{D}_\infty(x)$ takes values in $\{0,1\}$. To see this, fix a typical ω and $\varepsilon > 0$. Let T so large that $|\hat{D}_t(x) - \hat{D}_\infty(x)| < \varepsilon$ if $t > T$. Then

$$\infty > \big\langle \hat{D}(x)\big\rangle_\infty \geq \int_T^\infty \mu^2 \big(f\big(\hat{D}_t(x) + w_t\big) - f(w_t)\big)^2\, dt.$$

The integrand has a lower estimate in the form

$$\mu^2 \big(f\big(\hat{D}_t(x) + w_t\big) - f(w_t)\big)^2 \geq g\big(\hat{D}_\infty(x), w_t\big) \quad \text{for } t > T,$$

where

$$g(x,y) = \mu^2 \big(\big(\big|f(x+y) - f(y)\big| - \varepsilon\big) \vee 0\big)^2.$$

Now using that w has an occupation density $l_t(x) = \partial_x s(t,x)$ we can write

$$\infty > \big\langle \hat{D}(x)\big\rangle_\infty \geq \int_{\mathbb{R}} \lim_{K\to\infty} g\big(\hat{D}_\infty(x), y\big)\big(l_K(y) - l_T(y)\big)\, dy.$$

Since $(l_K(y) - l_T(y)) \to \infty$ for Lebesgue almost all y, the boundedness of the integral implies that $g(\hat{D}_\infty(x), y) = 0$ for almost all y. In other words

$$\big|f\big(\hat{D}_\infty(x) + y\big) - f(y)\big| \leq \frac{\varepsilon}{\mu^2} \qquad \text{for Lebesgue almost all } y.$$

Letting $\varepsilon \to 0$ we obtain

$$f\big(\hat{D}_\infty(x) + y\big) = f(y) \quad \text{for Lebesgue almost all } y.$$

This clearly gives that $\hat{D}_\infty(x) \in \mathbb{Z} \cap [0,1] = \{0,1\}$ for a typical $\omega \in \Omega$.

We obtained that for a fixed $x \in [0,1]$

$$\lim_{t\to\infty} \hat{D}_t(x) = \mathbf{P}\big(U < x | \sigma(\beta)\big) \in \{0,1\} \quad \text{almost surely.}$$

This holds almost surely for all $x \in [0,1] \cap \mathbb{Q}$ simultaneously and we conclude that U is measurable with respect to the complete σ-algebra $\mathcal{F}^{\beta}_{\infty}$. □

COROLLARY 9. *Assume that $\mu \neq 0$ and the shift w used in (9) has divergent occupation density. Then $\sigma(\beta) = \sigma(B,U)$.*

Proof. By construction $\sigma(\beta) \subset \sigma(B,U)$. The other direction follows from the previous corollary using that $\hat{D}_t(U) = D_t$ and $S = H \cdot \beta$ with $H_t = f'(D_t + w_t)$. □

5. Proof of Theorem 1 from Proposition 3

We use the same idea as in [4] and [2], that is, we take a partition of $(0,\infty)$ into non overlapping subintervals $I_k = [t_k, t_{k+1}]$, where t_k is an increasing sequence indexed by integers, accumulating at zero and at infinity. On each subinterval I_k, we apply Proposition 3 using the extra randomness obtained from the past of the process B.

In what follows, we again let $\mu = 1$ as this does not restrict the generality and use the notation

$$\begin{aligned} B^{(k)}_t &= B_{t+t_k} - B_{t_k}, \quad t \in [0, t_{k+1} - t_k], \\ S^{(k)}_t &= B^{(k)}_t + t, \quad t \in [0, t_{k+1} - t_k], \end{aligned}$$

and similarly for other processes. To carry out the above program, we have to define random variable U_k for each $k \in \mathbb{Z}$, that can be used on the next subinterval I_{k+1}.

Assume that we have a candidate for U_{k-1}. Applying Proposition 3 with U_{k-1} and $B^{(k)}$ yields a Brownian motion $\bar{\beta}^{(k)} = \bar{\beta}^{(k)}(U_{k-1}, B^{(k)})$. Taking the Lévy transform $\mathcal{L}(\bar{\beta}^{(k)})$ gives another Brownian motion which agrees with high probability with $\mathcal{L}(S^{(k)})$ provided that the length of I_k is small.

Now, we can define U_k from the random signs of the excursions of $\mathcal{L}(\bar{\beta}^{(k)})$. Formally, we have a mapping ϕ_k that gives U_k from the data U_{k-1} and $B^{(k)}$, that is,

$$\phi_k : [0,1] \times C[0, t_{k+1} - t_k] \to [0,1]$$

such that $\phi_k(U, B^{(k)})$ is independent of $\mathcal{L}^2(\bar{\beta}^{(k)}(U, B^{(k)}))$ and uniformly distributed on $[0,1]$ provided that U is uniform and independent of $B^{(k)}$.

We show below how to define V_k from the random signs of the excursions of $\mathcal{L}(S^{(k)})$ such that

$$\mathbf{P}\big(\phi_k\big(U, B^{(k)}\big) \neq V_k\big) \leq F_\tau\big(|I_k|\big) \tag{16}$$

for any U which is independent of $B^{(k)}$ and uniformly distributed in $[0,1]$. In (16) F_τ denotes the distribution function of the strictly positive random variable $\tau=\inf\{t>0:\mathcal{L}(\beta)_t\neq\mathcal{L}(S)_t\}$.

Now, let us start at the interval I_n, define $U^{(n,n)}=V_n$ and continue recursively with $U^{(n,k)}=\phi_k(U^{(n,k-1)},B^{(k)})$ for $k>n$. Then, $U^{(n,k)}$ is measurable with respect to the σ–algebra $\mathcal{F}_{t_{k+1}}$ and uniformly distributed on $[0,1]$ for each $k\geq n$.

By definition we have

$$\mathbf{P}\bigl(\exists k\geq n, U^{(n,k)}\neq U^{(n-1,k)}\bigr)\leq\mathbf{P}\bigl(\phi_n\bigl(V_{n-1},B^{(n)}\bigr)\neq V_n\bigr)\leq F_\tau\bigl(|I_n|\bigr).$$

If t_n tends to zero sufficiently fast as n goes to $-\infty$, then

$$\sum_{n\leq 0}F_\tau\bigl(|I_n|\bigr)<\infty.$$

With such a choice $\lim_{n\to-\infty}U^{(n,k)}$ exists by the Borel–Cantelli lemma. Moreover, this convergence is quasi-constant, that is for almost all ω, there is $n_0(\omega)$ such that for $n<n_0(\omega)$, even $U^{(n,k)}(\omega)=U_k(\omega)$ holds for $k\geq n$. This implies that $\phi_k(U_{k-1},B^{(k)})=U_k$ almost surely, U_k is $\mathcal{F}_{t_{k+1}}$-measurable and uniformly distributed on $[0,1]$.

Now, we are done. On each interval I_k we can use the random variable U_{k-1} and $B^{(k)}$ to obtain $\bar\beta^{(k)}$ as above by applying Proposition 3, that is

$$\bar\beta^{(k)}_t=\int_0^t\bar H^{(k)}_s\,dS^{(k)}_s\quad\text{for } t\leq t_{k+1}-t_k,$$

where $S^{(k)}_t=B^{(k)}_t+t$. Put

$$H^{(k)}_t=\operatorname{sign}\bigl(\mathcal{L}\bigl(\bar\beta^{(k)}\bigr)_t\bigr)\operatorname{sign}\bigl(\bar\beta^{(k)}_t\bigr)\bar H^{(k)}_t\quad\text{for } k\in\mathbb{Z},\ t\in[0,t_{k+1}-t_k].$$

Then

$$\beta^{(k)}_t=\int_0^t H^{(k)}_s dS^{(k)}_s=\mathcal{L}^2\bigl(\bar\beta^{(k)}\bigr)_t$$

is independent of U_k and we obtain that $\beta^{(k)}$, $k\in\mathbb{Z}$ is a sequence of independent processes. For each $k\in\mathbb{Z}$ the process $\beta^{(k)}$ is a Brownian motion in its own filtration on the finite time horizon $[0,t_{k+1}-t_k]$.

The last step is the joining of these pieces. Let

$$H_t=H^{(k)}_{t-t_k}\quad\text{for } t\in[t_k,t_{k+1})$$

and observe that

$$\beta_t=\int_0^t H_s\,dS_s=\sum_k\beta^{(k)}_{0\vee((t\wedge t_{k+1})-t_k)}$$

is a Brownian motion in its own filtration.

We used relation (16) in the above proof. Next we show, how to define V_k and ϕ_k such that (16) holds. As it is indicated above, ϕ_k yields a random variable from the signs of the excursions of $\mathcal{L}(\bar{\beta}^{(k)})$ and V_k is a uniform variable from the random signs of the excursions of $\mathcal{L}(S^{(k)})$. To obtain $\bar{\beta}^{(k)}$ we applied Proposition 3: it provides us with a mapping $\psi : [0,1] \times C[0,\infty) \to C[0,\infty)$ such that if U and B are independent, U is uniform on $[0,1]$ and B is a Brownian motion then $\psi(U,B)$ is Brownian motion. The restriction to $C[0,t_{k+1}-t_k]$ is denoted by ψ_k which is then a mapping $\psi_k : [0,1] \times C[0,t_{k+1}-t_k] \to C[0,t_{k+1}-t_k]$. Proposition 15 of [2] gives an example of a function $u_k : C[0,t_{k+1}-t_k] \to [0,1]$ with the property that $u_k(B)$ is a random variable uniformly distributed on $[0,1]$ provided that B is a Brownian motion with time horizon $[0,t_{k+1}-t_k]$, moreover $u_k(B)$ is independent of $\mathcal{L}(B)$.

Now $u_k(\mathcal{L}(\bar{\beta}^{(k)}))$ and $u_k(\mathcal{L}(S^{(k)}))$ coincide if $\mathcal{L}(\bar{\beta}^{(k)}) = \mathcal{L}(S^{(k)})$, that is with probability $F_\tau(t_{k+1}-t_k)$. So relation (16) holds if U_k and V_k is defined as $g(u_k(\mathcal{L}(\bar{\beta}^{(k)})))$ and $g(u_k(\mathcal{L}(S^{(k)})))$ respectively, with the same measurable function g. This is the content of the next lemma. For the application of the lemma, we only need that both $u_k(\mathcal{L}(\bar{\beta}^{(k)}))$ and $u_k(\mathcal{L}(S^{(k)}))$ have diffuse laws. It is obvious for $u_k(\mathcal{L}(\bar{\beta}^{(k)}))$. For the other variable it follows from the fact that the law of $S^{(k)}$ is absolutely continuous with respect to the law of the Brownian motion, that is, that of $\bar{\beta}^{(k)}$, and then the same holds for their functions, for example, to $u_k \circ \mathcal{L}$.

LEMMA 10. *Let U and V be random variables with diffuse law. Then, there exists a Borel measurable function $g : \mathbb{R} \to [0,1]$ such that both $g(U)$ and $g(V)$ are uniformly distributed on $[0,1]$.*

Proof. We can and do assume that U,V takes values in $[0,1]$. Denote by μ and ν the law of U and V, respectively. We prove that there is a sequence (α_n) of partitions of $[0,1]$, such that α_{n+1} is finer than α_n and $\mu(I) = \nu(I) = 2^{-n}$ for each $I \in \alpha_n$. Then each $I \in \alpha_n$ is the union of two elements of α_{n+1}, denote one of them by I^1, and the other with I^0. Once we have done this, we can define $\varepsilon_n(x) = \mathbb{1}_{(x \in \cup_{I \in \alpha_n} I^1)}$ and $g(x) = \sum_{n=0}^{\infty} 2^{-(n+1)} \varepsilon_n(x)$. It is a routine exercise to check that for $t \in (0,1)$ we have $\mu(g < t) = \nu(g < t) = t$, where $(g < t) = \{x : g(x) < t\}$.

We define the partition sequence $(\alpha_n)_{n \geq 0}$ recursively. Let $\alpha_0 = [0,1]$. Assume that α_n is already defined and let $I \in \alpha_n$. For the recursion step, it is enough to show that if $\mu(I) = \nu(I)$ then there is a decomposition $I = I^0 \cup I^1$ into disjoint sets such that $\mu(I^0) = \mu(I^1) = \nu(I^0) = \nu(I^1)$.

Let $h(x) = (-1)^{[2x]}$ where $[x]$ is the integer part of x and define

$$L(t) = \int_I h\left(t + \frac{\nu(I \cap [0,x])}{\nu(I)}\right) \mu(dx).$$

The integrand in this formula depends on the conditional distribution function $\nu(I \cap [0,x])/\nu(I)$. By definition ± 1 level sets of h split I into two equal part with respect to the measure ν for any t.

The function L is continuous, since μ is diffuse, and $L(0) = -L(1/2)$, whence $L(t_0) = 0$ for some $t_0 \in [0, 1/2]$. Then $I^1 = \{x \in I : h(t_0 + \nu(I \cap [0, x])/\nu(I)) = 1\}$ and $I^0 = I \setminus I$ divides I into two subset such that $\mu(I^1) = \mu(I^0) = \nu(I^1) = \nu(I^0)$. □

6. Proof of the auxiliary results

6.1. Proof of Proposition 4. We prove here that the following equation has a pathwise unique solution

$$dD_t = \mu\big(f(D_t + W_t) - f(W_t)\big) f'(D_t + W_t)\, d(B_t + \mu t) \tag{17}$$

for any initial value. The driving process of this equation is (B, W) a two dimensional Brownian motion. Application of a classical theorem of Yamada and Watanabe, see, for example, [5, Theorem 1.7 of Chapter IX] gives that the pathwise uniqueness of the solution implies that every solution is strong, that is, adapted to the filtration generated by the initial value and the driving Brownian motion (B, W). Conditioning on W we obtain that almost all sample paths of W can serve as w in Proposition 4.

A weak solution to (17) can be easily given. Indeed, let (β, W) be a two dimensional Brownian motion and U independent of (β, W) be the initial value. Then there is a solution of the equation

$$dD_t = \mu\big(f(D_t + W_t) - f(W_t)\big)\, d\beta_t, \quad D_0 = U,$$

since f is a Lipschitz continuous function. Define $S_t = \int_0^t f'(D_u + W_u)\, d\beta_u$. Then (S, W) is also a two-dimensional Brownian motion.

Then we have a solution to (9) by the usual measure change argument. Formally, (9) is a two dimensional equation driven by the two dimensional Brownian motion (B, W). The first, is the displayed equation (9) and the second hidden equation is simply $W = W$. The diffusion coefficient is the diagonal matrix $\sigma(D_t, W_t) = \operatorname{diag}(\mu(f(D_t + W_t) - f(W_t)), 1)$, the drift is $b(D_t, W_t) = (\mu(f(D_t + W_t) - f(W_t)), 0)^T$. The previous argument shows that $e(\sigma, 0)$ has a solution, hence application of Theorem 1.11 in Chapter IX of [5] gives that $e(\sigma, b)$ also has a solution, since $b = \sigma c$ with $c = (1, 0)^T$.

So let us assume that (D, S, W) is a solution on some filtered probability space. We prove that if (D', S, W) is another solution on the same filtered space with $D'_0 = D_0$ then D' and D are indistinguishable. This proves the pathwise uniqueness, and hence the statement.

Let $(\tau_k)_{k=0}^\infty$ be the following sequence of stopping times:

$$\tau_0 = 0, \qquad \tau_1 = \inf\left\{t \geq 0 : D_t + W_t \in \frac{1}{2}\mathbb{Z}\right\},$$

$$\tau_{k+1} = \inf\left\{t > \tau_k : \big|D_t + W_t - (D_{\tau_k} + W_{\tau_k})\big| > \frac{1}{2}\right\}, \quad k \geq 1$$

and similarly $(\tau'_k)_{k=0}^{\infty}$ with D' instead of D. By induction on k we show that $\tau_k = \tau'_k$ and $D_{t\wedge\tau_k} = D'_{t\wedge\tau_k}$ for all $t \geq 0$ almost surely. This is clearly proved if we show that between τ_k and τ_{k+1} the triple (D, S, W) satisfies an equation with pathwise unique solution.

Up to the time τ_1 any solution (D, S, W) of (9) satisfies the equation

$$dD_t = \mu\big(f(W_t + D_t) - f(W_t)\big) f'(W_0 + D_0)\, dS_t$$

which has pathwise unique solution, hence $\tau_1 = \tau'_1$ and $D_{t\wedge\tau_1} = D'_{t\wedge\tau_1}$ for all $t \geq 0$.

Assume that we already know that $\tau_k = \tau'_k$ and up to this stopping time $D = D'$. At the time point τ_k, the value of the process $D + W \in \frac{1}{2}\mathbb{Z}$. Fixing $k \geq 1$ denote

$$\begin{aligned} X_t &= D_{t+\tau_k} + W_{t+\tau_k} - (D_{\tau_k} + W_{\tau_k}),\\ \bar{S}_t &= S_{t+\tau_k} - S_{\tau_k},\\ \bar{W}_t &= W_{t+\tau_k},\\ \bar{B}_t &= \bar{S}_t - t. \end{aligned}$$

If $D_{\tau_k} + W_{\tau_k} \in \mathbb{Z}$, then on $[0, \tau_{k+1} - \tau_k]$ the process X solves the equation

$$\begin{aligned} dX_t &= \mu\big(f(X_t) - f(\bar{W}_t)\big) f'(X_t)\, d\bar{S}_t + d\bar{W}_t\\ &= \mu f(X_t) f'(X_t)\, d\bar{S}_t - \mu f'(X_t) f(\bar{W}_t)\, d\bar{S}_t + d\bar{W}_t. \end{aligned}$$

For $|x| < \frac{1}{2}$, we have $f(x) = |x|$ and $f(x)f'(x) = x$, hence to prove the induction step for this case we have to show, that the solution of the next equation is pathwise unique:

$$dX_t = \mu X_t\, d\bar{S}_t - \mu \operatorname{sign}(X_t) f(\bar{W}_t)\, d\bar{S}_t + d\bar{W}_t. \tag{18}$$

An application of Itô's formula gives that X satisfies this equation if and only $Y_t = e^{-\mu\bar{B}_t - \mu^2 t/2} X_t$ solves

$$\begin{aligned} dY_t &= -\operatorname{sign}(Y_t)\mu f(\bar{W}_t) e^{-\mu\bar{B}_t - \mu^2 t/2}\, d\bar{B}_t + e^{-\mu\bar{B}_t - \mu^2 t/2}\, d\bar{W}_t\\ &= \operatorname{sign}(Y_t)\, dM_t + dN_t, \end{aligned}$$

where

$$\begin{aligned} M_t &= -\mu \int_0^t f(\bar{W}_s) e^{-\mu\bar{B}_s - \mu^2 s/2}\, d\bar{B}_s,\\ N_t &= \int_0^t e^{-\mu\bar{B}_s - \mu^2 s/2}\, d\bar{W}_s. \end{aligned}$$

Now, (M, N) is a given two dimensional martingale (defined from $\bar{S}$ and $\bar{W}$). They are strongly orthogonal, i.e. $\langle M, N\rangle = 0$ and N dominates M, that is, $d\langle M\rangle \leq \mu^2 d\langle N\rangle$, hence by Theorem 2 of [1] the solution of this equation is pathwise unique.

The other case that can be treated similarly. If $D_{\tau_k} + W_{\tau_k} \in \frac{1}{2} + \mathbb{Z}$ then for $t \in [\tau_{k+1} - \tau_k]$ we have that $f(D_{t+\tau_k} + W_{t+\tau_k}) = f(X_t + \frac{1}{2})$ and similarly for f'. So we have to deal with the equation

$$dX_t = \mu\left(f\left(X_t + \frac{1}{2}\right) - f(\bar{W}_t)\right) f'\left(X_t + \frac{1}{2}\right) d\bar{S}_t + d\bar{W}_t.$$

Now, we can use that $f(x + \frac{1}{2}) = \frac{1}{2} - f(x)$. So in this case, we have to show that the solution of the next equation is pathwise unique:

$$dX_t = \mu\left(f(X_t) - f\left(\bar{W}_t + \frac{1}{2}\right)\right) f'(X_t)\, d\bar{S}_t + d\bar{W}_t.$$

This is the same type of equation as (18) was in the other case. So, we have proved the induction step and the proposition as well.

REMARK 11. The above reasoning shows that the fundamental equation used in [2], that is,

$$dD_t = -\mu\big(D_t \wedge (1 - D_t)\big) \operatorname{sign}\left(D_t - \frac{1}{2}\right) dS_t, \quad D_0 = U \tag{19}$$

is a transformed version of the Tanaka equation. Indeed, application of the Itô formula gives that D is the solution of (19) if and only if $Y_t = (D_t - \frac{1}{2}) \exp\{-\mu B_t - \mu^2 t/2\}$ is the solution of

$$dY_t = \operatorname{sign}(Y_t)\, dM_t, \quad Y_0 = U, \tag{20}$$

where $M_t = \int_0^t e^{-\mu B_t - \mu^2 t/2}\, dB_t$. The driving local martingale M in (20) is a time-changed Brownian motion, where the time change is obtained from B and Y can be viewed as the time-changed solution of the Tanaka equation.

6.2. Proof of Proposition 5. By assumption F is random Lipschitz, hence functional Lipschitz in the terminology of Protter, see [3, pp. 250] and [3, Chapter V, Theorem 37] gives the existence and continuity of the solution.

The proof of Lemma 12 in [2] applies without any serious changes to the process $\hat{D}_t(x)$, since it only uses the fact that $F(\hat{D}_t(x), \cdot, t) - F(\hat{D}_t(y), \cdot, t) = q_t(\hat{D}_t(x) - \hat{D}_t(y))$ with a uniformly bounded predictable process q. So we have that there is a parametric process $\bar{Y}_t(x)$, such that $\sup_{s \le t} \int_x^y \bar{Y}_s^2(u)\, du < \infty$ almost surely for all x, y, t and on a common almost sure event Ω'

$$\int_x^y \bar{Y}_t(u)\, du = \hat{D}_t(y) - \hat{D}_t(x) \quad \text{for all } (x, y, t).$$

Then, if $\omega \in \Omega'$

$$\begin{aligned}
&\int_x^y \partial_x F\big(\hat{D}_t(u), \omega, t\big) \bar{Y}_t(u)\, du \\
&\quad = F\big(\hat{D}_t(y), \omega, t\big) - F\big(\hat{D}_t(x), \omega, t\big) \quad \text{for all } (x, y, t).
\end{aligned}$$

The Fubini theorem for stochastic integrals [3, Chapter V, Theorem 65] can be applied since $\int_x^y \bar{Y}_t^2(u)\,du$ is locally bounded for all (x,y), i.e.,

$$\begin{aligned}\int_x^y \bar{Y}_t(u)\,du &= \hat{D}_t(y) - \hat{D}_t(x) \\ &= y - x + \int_0^t F\big(\hat{D}_s(y),\cdot,s\big) - F\big(\hat{D}_s(x),\cdot,s\big)\,d\beta_s \\ &= \int_x^y \left(1 + \int_0^t \partial_x F\big(\hat{D}_s(u),\cdot,s\big)\bar{Y}_s(u)\,d\beta_s\right) du \\ &= \int_x^y Y_t(u)\,du,\end{aligned} \tag{21}$$

where $Y_t(u)$ is the $\mathcal{B}(\mathbb{R}) \times \mathcal{P}$ measurable version of the parametric integral

$$Y(u) = 1 + \int \partial_x F\big(\hat{D}_s(u),\cdot,s\big)\bar{Y}_s(u)\,d\beta_s. \tag{22}$$

Identity (21) holds for all (x,y,t) on an almost sure event $\Omega'' \subset \Omega'$. For $\omega \in \Omega''$ and $s \geq 0$, the equality $\bar{Y}_s(u) = Y_s(u)$ holds for Lebesgue almost all u. Thus, we can replace $\bar{Y}$ by Y on right hand side of (22), obtaining that

$$Y(u) = 1 + \int \partial_x F\big(\hat{D}_s(u),\cdot,s\big)Y_s(u)\,d\beta_s$$

for Lebesgue almost all u. Hence, for these u values $Y_t(u) = \exp\{\bar{S}_t(u) - \frac{1}{2}\langle \bar{S}(u)\rangle_t\}$ and the stochastic exponential of $\bar{S}$ gives the Radon–Nikodym derivative.

References

[1] V. Prokaj, *The solution of the perturbed Tanaka-equation is pathwise unique*, to appear in Ann. Probab.

[2] V. Prokaj, M. Rásonyi and W. Schachermayer, *Hiding a constant drift*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), 498–514. MR 2814420

[3] P. E. Protter, *Stochastic integration and differential equations*, 2nd ed., Applications of mathematics (New York), Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2004. MR 2020294

[4] M. Rásonyi, W. Schachermayer and R. Warnung, *Hiding a drift*, Ann. Probab. **37** (2009), 2459–2479. MR 2573564

[5] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR 1725357

Vilmos Prokaj, Eötvös Loránd University, Department of Probability Theory and Statistics, 1117 Budapest, Pázmány P. sétány 1/C, and Computer and Automation Institute of the Hungarian Academy of Sciences, 1111 Budapest, Kende utca 13-17, Hungary

E-mail address: prokaj@cs.elte.hu

Walter Schachermayer, University of Vienna, Faculty of Mathematics, Nordbergstrasse 15, 1090 Vienna, Austria

E-mail address: walter.schachermayer@univie.ac.at