Admissible Trading Strategies under Transaction Costs

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Abstract

A well known result in stochastic analysis reads as follows: for an $\mathbb{R}$-valued super-martingale $X = (X_t)_{0 \le t \le T}$ such that the terminal value X_T is non-negative, we have that the entire process X is non-negative. An analogous result holds true in the no arbitrage theory of mathematical finance: under the assumption of no arbitrage, a portfolio process $x + (H \cdot S)$ verifying $x + (H \cdot S)_T \ge 0$ also satisfies $x + (H \cdot S)_t \ge 0$, for all $0 \le t \le T$.

In the present paper we derive an analogous result in the presence of transaction costs. A counter-example reveals that the consideration of transaction costs makes things more delicate than in the frictionless setting.

1 A Theorem on Admissibility

We consider a stock price process $S = (S_t)_{0 \le t \le T}$ in continuous time with a fixed horizon T. This stochastic process is assumed to be based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, satisfying the usual conditions of completeness and right continuity. We assume that S is adapted and has *càdlàg* (right continuous, left limits), and strictly positive trajectories, i.e. the function $t \to S_t(\omega)$ is càdlàg and strictly positive, for almost each $\omega \in \Omega$.

In mathematical finance a key assumption is that the process S is *free of arbitrage*. The Fundamental Theorem of Asset Pricing states that this

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property is *essentially* equivalent to the property that S admits an equivalent local martingale measure (see, [9], [4], or the books [5],[13]).

Definition 1.1. *The process S admits an equivalent local martingale measure, if there is a probability measure $Q \sim \mathbb{P}$ such that S is a local martingale under Q.*

Fix a process S satisfying the above assumption and note that Def.1.1 implies in particular that S is a semi-martingale as this property is invariant under equivalent changes of measure. Turning to the theme of the paper, we now consider *trading strategies*, i.e. S-integrable predictable processes $H = (H_t)_{0 \leq t \leq T}$. We call H *admissible* if there is $M > 0$ such that

$$(H \cdot S)_t \geq -M, \qquad \mathbb{P} - a.s. \quad \text{for} \qquad 0 \leq t \leq T. \tag{1}$$

The stochastic integral

$$(H \cdot S)_t = \int_0^t H_u dS_u, \qquad\qquad 0 \leq t \leq T, \tag{2}$$

then is a local Q-martingale by a result of Ansel-Stricker (see [1] and [15]). Assumption (1) also implies that the local martingale $H \cdot S$ is a *super-martingale* (see [5], Prop.7.2.7) under each equivalent local martingale measure Q. We thus infer from the well known result mentioned in the abstract that $(H \cdot S)_T \geq -x$ almost surely implies that $(H \cdot S)_t \geq -x$ almost surely under Q (and therefore also under $\mathbb{P}$), for all $0 \leq t \leq T$.

We resume our findings in the subsequent well-known Proposition (compare [14], Prop.4.2).

Proposition 1.2. *Let S admit an equivalent local martingale measure, let H be admissible, and suppose that there is $x \in \mathbb{R}_+$ such that*

$$x + (H \cdot S)_T \geq 0, \qquad\qquad \mathbb{P} - a.s. \tag{3}$$

Then

$$x + (H \cdot S)_\tau \geq 0, \qquad\qquad \mathbb{P} - a.s. \tag{4}$$

for every $[0, T]$-valued stopping time τ.

We now introduce transaction costs: fix $0 \leq \lambda < 1$. We define the *bid-ask spread* as the interval $[(1-\lambda)S, S]$. The interpretation is that an agent can buy the stock at price S, but sell it only at price $(1-\lambda)S$. Of course, the case $\lambda = 0$ corresponds to the usual frictionless theory.

In the setting of transaction costs the notion of *consistent price systems*, which goes back to [10] and [3], plays a role analogous to the notion of equivalent martingale measures in the frictionless theory.

Definition 1.3. *Fix $1 > \lambda \geq 0$. A process $S = (S_t)_{0 \leq t \leq T}$ satisfies the condition (CPS^λ) of* having a consistent price system under transaction costs λ *if there is a process $\widetilde{S} = (\widetilde{S}_t)_{0 \leq t \leq T}$, such that*

$$(1-\lambda)S_t \leq \widetilde{S}_t \leq S_t, \qquad 0 \leq t \leq T,$$

as well as a probability measure Q on $\mathcal{F}$, equivalent to $\mathbb{P}$, such that $(\widetilde{S}_t)_{0 \leq t \leq T}$ is a local martingale under Q.

We say that S admits consistent price systems for arbitrarily small transaction costs *if (CPS^λ) is satisfied, for all $\lambda > 0$.*

In [8] the condition of *admitting consistent price systems* for arbitrarily small transaction costs has been related to the condition of *no arbitrage* under arbitrarily small transaction costs, thus proving a version of the Fundamental Theorem of Asset Pricing under small transaction costs (compare [12] for a large amount of related material).

It is important to note that we *do not assume* that S is a semi-martingale as one is forced to do in the frictionless theory [4, Theorem 7.2]. Only the process $\widetilde{S}$ appearing in Definition 1.3 always has to be a semi-martingale, as it becomes a local martingale after passing to an equivalent measure Q.

To formulate a result analogous to Proposition 1.2 in the setting of transaction costs we have to define the notion of $\mathbb{R}^2$-valued *self-financing trading strategies.*

Definition 1.4. *Fix a strictly positive stock price process $S = (S_t)_{0 \leq t \leq T}$ with càdlàg paths, as well as transaction costs $1 > \lambda > 0$.*

A self-financing trading strategy starting with zero endowment *is a pair of predictable, finite variation processes $(\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that*

(i) $\varphi_0^0 = \varphi_0^1 = 0$,

(ii) denoting by $\varphi_t^0 = \varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}$ and $\varphi_t^1 = \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow}$, the canonical decompositions of φ^0 and φ^1 into the difference of two increasing processes, starting at $\varphi_0^{0,\uparrow} = \varphi_0^{0,\downarrow} = \varphi_0^{1,\uparrow} = \varphi_0^{1,\downarrow} = 0$, these processes satisfy

$$d\varphi_t^{0,\uparrow} \leq (1-\lambda)S_t d\varphi_t^{1,\downarrow}, \quad d\varphi_t^{0,\downarrow} \geq S_t d\varphi_t^{1,\uparrow}, \quad 0 \leq t \leq T. \tag{5}$$

The trading strategy $\varphi = (\varphi^0, \varphi^1)$ *is called* admissible *if there is* $M > 0$ *such that the liquidation value* V_t^{liq} *satisfies*

$$V_t^{liq}(\varphi^0, \varphi^1) := \varphi_t^0 + (\varphi_t^1)^+(1-\lambda)S_t - (\varphi_t^1)^- S_t \geq -M, \tag{6}$$

a.s., for $0 \leq t \leq T$.

The processes φ_t^0 and φ_t^1 model the holdings at time t in units of bond and stock respectively. We normalize the bond price by $B_t \equiv 1$. The differential notation in (5) needs some detailed explanation. If φ is continuous, then (5) has to be understood as the integral requirement.

$$\int_\sigma^\tau ((1-\lambda)S_t d\varphi_t^{1,\downarrow} - d\varphi_t^{0,\uparrow}) \geq 0, \qquad a.s. \tag{7}$$

for all stopping times $0 \leq \sigma \leq \tau \leq T$, and analogously for the second differential in (5). The above integral makes pathwise sense as Riemann-Stieltjes intregral, as φ is continuous and of finite variation and S is càdlàg. Things become more delicate when we also consider jumps of φ: note that, for every stopping time τ the left and right limits φ_{τ_-} and φ_{τ_+} exist as φ is of bounded variation. The three values $\varphi_{\tau_-}, \varphi_\tau$ and φ_{τ_+} may very well be different. As in [2] we denote the increments by

$$\Delta\varphi_\tau = \varphi_\tau - \varphi_{\tau_-}, \qquad \Delta_+\varphi_\tau = \varphi_{\tau_+} - \varphi_\tau. \tag{8}$$

For totally inaccessible stopping times τ, the predictability of φ implies that $\Delta\varphi_\tau = 0$ almost surely, while for accessible stopping times τ it may happen that $\Delta\varphi_\tau \neq 0$ as well as $\Delta_+\varphi_\tau \neq 0$.

Equality (5) then has to be interpreted as

$$\Delta\varphi_\tau^{0,\uparrow} \leq (1-\lambda)S_{\tau_-}\Delta\varphi_\tau^{1,\downarrow}, \qquad \Delta\varphi_\tau^{0,\downarrow} \geq S_{\tau_-}\Delta\varphi_\tau^{1,\uparrow} \tag{9}$$

and in the case of right jumps

$$\Delta_+\varphi_\tau^{0,\uparrow} \leq (1-\lambda)S_\tau\Delta_+\varphi_\tau^{1,\downarrow}, \qquad \Delta_+\varphi_\tau^{0,\downarrow} \geq S_\tau\Delta_+\varphi_\tau^{1,\uparrow}, \tag{10}$$

holding true a.s. for all $[0,T]$-valued stopping times τ. Let us give an economic interpretation of the significance of (9) and (10). Think of a predictable event at time τ, say a speech of the chairman of the Fed, which does not come as a surprise but was announced some time before. It is to be expected that this speech will have a sudden effect on the price of a stock S, say a possible jump would be from $S_{\tau_-}(\omega) = 100$ to $S_\tau(\omega) = 110$ (recall that S is assumed to be càdlàg). A trader may want to follow the following strategy: she holds a position of $\varphi^1_{\tau_-}(\omega)$ stocks until "immediately before the speech". Then, one second before the speech starts, she changes the position from $\varphi^1_{\tau_-}(\omega)$ to $\varphi^1_{\tau}(\omega)$ causing an increment of $\Delta\varphi^1_{\tau}(\omega)$. Of course, the price $S_{\tau_-}(\omega)$ still applies, corresponding to (9). Subsequently, the speech starts and the jump $\Delta S_\tau(\omega) = S_\tau(\omega) - S_{\tau_-}(\omega)$ is revealed. The agent may now decide "immediately after learning the size of $\Delta S_\tau(\omega)$" to change her position from $\varphi^1_\tau(\omega)$ to $\varphi^1_{\tau_+}(\omega)$ on the base of the price $S_\tau(\omega)$ which corresponds to (10).

We have chosen to define the trading strategy φ by explicitly specifying both accounts, the holdings in stock φ^0 as well as the holdings in bond φ^1. It would be sufficient to only specify φ^1 similarly as in the frictionless theory where we usually only specify the process H in (1) which corresponds to φ^1 in the present notation. Given a predictable finite variation process $\varphi^1 = (\varphi^1_t)_{0\le t\le T}$ starting at $\varphi^1_0 = 0$, which we canonically decompose into the difference $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$, we may define the process φ^0 by

$$d\varphi^0_t = (1-\lambda)S_t d\varphi^{1,\downarrow}_t - S_t d\varphi^{1,\uparrow}_t.$$

The resulting pair (φ^0, φ^1) obviously satisfies (5) with equality holding true rather than inequality. Not withstanding, it is convenient in (5) to consider trading strategies (φ^0, φ^1) which allow for an inequality, i.e. for "throwing away money". But it is clear from the preceding argument that we may always pass to a dominating pair (φ^0, φ^1) where equality holds true in (5).

In the theory under transaction costs the super-martingale property of the value process is formulated in Proposition 1.6 below. First we have to recall a definition from [7] which extends the notion of a super-martingale beyond the framework of càdlàg processes.

Definition 1.5. *An optional process $X = (X_t)_{0\le t\le T}$ is called an* optional strong super-martingale *if, for all stopping times $0 \le \sigma \le \tau \le T$ we have*

$$\mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \le X_\sigma, \tag{11}$$

where we also impose that X_τ is integrable.

An optional strong super-martingale can be decomposed in the style of Doob-Meyer which is known under the name of Mertens decomposition (see [7]). X is an optional strong super-martingale if and only if it can be decomposed into

$$X = M - A, \tag{12}$$

where M is a local martingale (and therefore càdlàg) as well as a super-martingale, and A an increasing predictable process (which is làdlàg but has no reason to be càdlàg). This decomposition then is unique.

Proposition 1.6. *Fix S, transaction costs $1 > \lambda > 0$, and an admissible self-financing trading strategy $\varphi = (\varphi^0, \varphi^1)$ as above. Suppose that $(\widetilde{S}, Q)$ is a consistent price system under transaction costs λ. Then the process*

$$\widetilde{V}_t := \varphi_t^0 + \varphi_t^1 \widetilde{S}_t, \qquad\qquad 0 \le t \le T,$$

satisfies $\widetilde{V} \ge V^{liq}$ and is an optional strong super-martingale under Q.

Proof. The assertion $\widetilde{V} \ge V^{\text{liq}}$ is an obvious consequence of $\widetilde{S} \in [(1-\lambda)S, S]$.

We have to show that $\widetilde{V}$ decomposes as in (12). Arguing formally, we may apply the product rule to obtain

$$d\widetilde{V}_t = (d\varphi_t^0 + \widetilde{S}_t d\varphi_t^1) + \varphi_t^1 d\widetilde{S}_t \tag{13}$$

so that

$$\widetilde{V}_t = \int_0^t (d\varphi_u^0 + \widetilde{S}_u d\varphi_u^1) + \int_0^t \varphi_u^1 d\widetilde{S}_u. \tag{14}$$

The first term in (14) is decreasing by (5) and the fact that $\widetilde{S} \in [(1-\lambda)S, S]$. The second term defines, at least formally speaking, a local Q-martingale so that the sum of the two integrals is an (optional strong) super-martingale.

The justification of the above formal reasoning deserves some care (compare the proof of Lemma 8, in [2]). Suppose first that φ is continuous. In this case φ is a semi-martingale so that we are allowed to apply Itô calculus to $\widetilde{V}$. Formula (14) therefore makes perfect sense as an Itô integral, bearing in mind that φ has finite variation. The first integral in (14) now is a well-defined decreasing predictable process. As regards the second integral, note that by the admissibility of φ it is uniformly bounded from below. Hence by a result of Ansel-Stricker ([1], see also [15]) it is a local Q-martingale as well as a super-martingale. Hence $\widetilde{V}$ is indeed a super-martingale under Q (in the classical càdlàg sense).

Passing to the case when φ is allowed to have jumps, the process $\widetilde{V}$ need not be càdlàg anymore. It still is an optional process and we have to verify that it decomposes as in (12). Assume first that φ is of the form

$$\varphi_t = (f^0, f^1) \mathbb{1}_{]\!]\tau, T]\!]}(t), \tag{15}$$

where $(f^0, f^1) = \Delta_+(\varphi_\tau^0, \varphi_\tau^1)$ are $\mathcal{F}_\tau$-measurable bounded random variables verifying (10) and τ is a $[0, T]$-stopping time. We obtain

$$\begin{aligned} \widetilde{V}_t &= [\Delta_+\varphi_\tau^0 + (\Delta_+\varphi_\tau^1)\widetilde{S}_t] \mathbb{1}_{]\!]\tau, T]\!]}(t) \\ &= [\Delta_+\varphi_\tau^0 + (\Delta_+\varphi_\tau^1)\widetilde{S}_\tau] \mathbb{1}_{]\!]\tau, T]\!]}(t) + (\Delta_+\varphi_\tau^1)(\widetilde{S}_t - \widetilde{S}_\tau) \mathbb{1}_{]\!]\tau, T]\!]}(t) \end{aligned} \tag{16}$$

Again, the first term is a decreasing predictable process and the second term is a local martingale under Q.

Next assume that φ is of the form

$$\varphi_t = (f^0, f^1)\mathbb{1}_{[\![\tau,T]\!]}(t), \tag{17}$$

where τ is a predictable stopping time, and $(f^0, f^1) = \Delta(\varphi^0_\tau, \varphi^1_\tau)$ are bounded $\mathcal{F}_{\tau_-}$-measurable random variables verifying (9). Similarly as in (16) we obtain

$$\begin{aligned}\widetilde{V}_t &= [\Delta\varphi^0_\tau + (\Delta\varphi^1_\tau)\widetilde{S}_t]\mathbb{1}_{[\![\tau,T]\!]}(t) \\ &= [\Delta\varphi^0_\tau + (\Delta\varphi^1_\tau)\widetilde{S}_{\tau_-}]\mathbb{1}_{[\![\tau,T]\!]}(t) + (\Delta\varphi^1_\tau)(\widetilde{S}_t - \widetilde{S}_{\tau_-})\mathbb{1}_{[\![\tau,T]\!]}(t) \end{aligned} \tag{18}$$

Once more, the first term is a decreasing predictable process (this time it is even càdlàg) and the second term is a local martingale under Q.

Finally we have to deal with a general admissible self-financing trading strategy φ. To show that $\widetilde{V}$ is of the form (12) we first assume that the total variation of φ is uniformly bounded. We decompose φ into its continuous and purely discontinuous part $\varphi = \varphi^c + \varphi^{pd}$. We also may find a sequence $(\tau_n)_{n=1}^\infty$ of $[0,T] \cup \{\infty\}$-valued stopping times such that the supports $([\![\tau_n]\!])_{n=1}^\infty$ are mutually disjoint and $\bigcup\limits_{n=1}^\infty [\![\tau_n]\!]$ exhausts the right jumps of φ. Similarly, we may find a sequence $(\tau_n^p)_{n=1}^\infty$ of predictable stopping times such that their supports $([\![\tau_n^p]\!])_{n=1}^\infty$ are mutually disjoint and $\bigcup\limits_{n=1}^\infty [\![\tau_n^p]\!]$ exhausts the left jumps of φ. We apply the above argument to φ^c and to each $\tau_n, \Delta_+\varphi_{\tau_n}$ and $\tau_n^p, \Delta\varphi_{\tau_n^p}$, and sum up the corresponding terms in (14), (16) and (18). This sum converges to $\widetilde{V} = M - A$, where M is a local Q-martingale and A an increasing process, as we have assumed that the total variation of φ is bounded (compare [11] and the proof of Lemma 8 in [2]). By the boundedness from below we conclude that M is also a super-martingale.

Passing to the case where φ has only finite instead of uniformly bounded variation, we use the predictability of φ to find a localizing sequence $(\sigma_k)_{k=1}^\infty$ such that each stopped process φ^{σ_k} has uniformly bounded variation. Apply the above argument to each φ^{σ_k} to obtain the same conclusion for φ.

Summing up, we have shown that $\widetilde{V}$ admits a Mertens decomposition (12) and therefore is an optional strong super-martingale. $\square$

We can now state the analogous result to Proposition 1.2 in the presence of transaction costs.

Theorem 1.7. *Fix S and $1 > \lambda > 0$ as above, and suppose that S satisfies $(CPS^{\lambda'})$, for* each $1 > \lambda' > 0$.

Let $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ be an admissible, self-financing trading strategy under transaction costs λ, starting with zero endowment, and suppose that there is $x > 0$ s.t. for the terminal liquidation value V_T^{liq} we have a.s.

$$V_T^{liq}(\varphi^0, \varphi^1) = \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \ge -x. \tag{19}$$

We then also have that

$$V_\tau^{liq}(\varphi^0, \varphi^1) = \varphi_\tau^0 + (\varphi_\tau^1)^+(1-\lambda)S_\tau - (\varphi_\tau^1)^- S_\tau \ge -x, \tag{20}$$

a.s., for every stopping time $0 \le \tau \le T$.

Proof. Supposing that (20) fails, we may find $\frac{\lambda}{2} > \alpha > 0$, and a stopping time $0 \le \tau \le T$, such that either $A = A_+$ or $A = A_-$ satisfies $\mathbb{P}[A] > 0$, where

$$A_+ = \{\varphi_\tau^1 \ge 0,\ \varphi_\tau^0 + \varphi_\tau^1 \tfrac{1-\lambda}{1-\alpha} S_\tau < -x\}, \tag{21}$$
$$A_- = \{\varphi_\tau^1 \le 0,\ \varphi_\tau^0 + \varphi_\tau^1 (1-\alpha)^2 S_\tau < -x\}. \tag{22}$$

Choose $0 < \lambda' < \alpha$ and a λ'-consistent price system $(\widetilde{S}, Q)$. As $\widetilde{S}$ takes values in $[(1-\lambda')S, S]$, we have that $(1-\alpha)\widetilde{S}$ as well as $\frac{1-\lambda}{1-\alpha}\widetilde{S}$ take values in $[(1-\lambda)S, S]$ so that $((1-\alpha)\widetilde{S}, Q)$ as well as $(\frac{1-\lambda}{1-\alpha}\widetilde{S}, Q)$ are consistent price systems under transaction costs λ. By Proposition 1.6 we obtain that

$$\left(\varphi_t^0 + \varphi_t^1 (1-\alpha)\widetilde{S}_t\right)_{0 \le t \le T} \text{ and } \left(\varphi_t^0 + \varphi_t^1 \tfrac{1-\lambda}{1-\alpha}\widetilde{S}_t\right)_{0 \le t \le T}$$

are optional strong Q-super-martingales. Arguing with the second process using $\widetilde{S} \le S$, we obtain from (21) the inequality

$$\begin{aligned}
\mathbb{E}_Q[V_T^{liq} \mid A_+] &\le \mathbb{E}_Q\left[\varphi_T^0 + \varphi_T^1 \frac{1-\lambda}{1-\alpha}\widetilde{S}_T \Big| A_+\right] \\
&\le \mathbb{E}_Q\left[\varphi_\tau^0 + \varphi_\tau^1 \frac{1-\lambda}{1-\alpha}\widetilde{S}_\tau \Big| A_+\right] \\
&\le \mathbb{E}_Q\left[\varphi_\tau^0 + \varphi_\tau^1 \frac{1-\lambda}{1-\alpha}S_\tau \Big| A_+\right] < -x.
\end{aligned}$$

Arguing with the first process and using that $\widetilde{S} \ge (1-\lambda')S \ge (1-\alpha)S$ (which implies that $\varphi_\tau^1(1-\alpha)\widetilde{S}_\tau \le \varphi_\tau^1(1-\alpha)^2 S_\tau$ on A_-) we obtain from (22) the inequality

$$\begin{aligned}\mathbb{E}_Q[V_T^{liq} \mid A_-] &\le \mathbb{E}_Q\left[\varphi_T^0 + \varphi_T^1(1-\alpha)\widetilde{S}_T | A_-\right] \\ &\le \mathbb{E}_Q\left[\varphi_\tau^0 + \varphi_\tau^1(1-\alpha)\widetilde{S}_\tau | A_-\right] \\ &\le \mathbb{E}_Q\left[\varphi_\tau^0 + \varphi_\tau^1(1-\alpha)S_\tau^2 | A_-\right] < -x.\end{aligned}$$

Either A_+ or A_- has strictly positive probability; hence we arrive at a contradiction to $V_T^{liq} \ge -x$ almost surely. □

2 A Counter-Example

The assumption $(CPS^{\lambda'})$, for *each* $\lambda' > 0$, cannot be dropped in Proposition 1.7 as shown by the example presented in the next lemma.

Lemma 2.1. *Fix* $1 > \lambda \ge \lambda' > 0$ *and* $C > 1$. *There is a continuous process* $S = (S_t)_{0 \le t \le 1}$ *satisfying* $(CPS^{\lambda'})$, *and a* λ*-self-financing, admissible trading strategy* $(\varphi^0, \varphi^1) = (\varphi_t^0, \varphi_t^1)_{0 \le t \le 1}$ *such that*

$$V_1^{liq}(\varphi^0, \varphi^1) \ge -1, \qquad a.s. \tag{23}$$

while

$$\mathbb{P}\left[V_{\frac{1}{2}}^{liq}(\varphi^0, \varphi^1) \le -C\right] > 0. \tag{24}$$

Proof. In order to focus on the central (and easy) idea of the construction we first show the assertion for the constant $C = 2 - \lambda$ and under the assumption $\lambda = \lambda'$. In this case we can give a deterministic example, i.e. S, φ^0 and φ^1 will not depend on the random element $\omega \in \Omega$.

Define $S_0 = S_1 = 1$, and $S_{\frac{1}{2}} = 1 - \lambda$ where we fix $T = 1$.

To make $S = (S_t)_{0 \le t \le T}$ continuous, we interpolate linearly, i.e.

$$S_t = 1 - 2t\lambda, \qquad 0 \le t \le \tfrac{1}{2}, \tag{25}$$

$$S_t = 1 - 2(1-t)\lambda, \qquad \tfrac{1}{2} \le t \le 1. \tag{26}$$

Note that condition (CPS^{λ}) is satisfied, as the constant process $\widetilde{S}_t \equiv (1-\lambda)$ defines a λ-consistent price system: it trivially is a martingale (under any probability measure) and takes values in $[(1-\lambda)S, S]$.

Starting from the initial endowment $(\varphi_0^0, \varphi_0^1) = (0, 0)$, we might invest, at time $t = 0$, the maximal amount into the stock so that at time $t = 1$

condition (23) holds true. In other words, we let $\varphi^1_{0_+} = -\varphi^0_{0_+}$ be the biggest number such that

$$(1-\lambda)\varphi^1_{0_+} + \varphi^0_{0_+} \geq -1,$$

which clearly gives $\varphi^1_{0_+} = \frac{1}{\lambda}$. Hence $(\varphi^0_t, \varphi^1_t) = (-\frac{1}{\lambda}, \frac{1}{\lambda})$, for all $0 < t \leq T$, is a self-financing strategy, starting at $(\varphi^0_0, \varphi^1_0) = (0,0)$ for which (23) is satisfied. Looking at (24) we calculate

$$V_{\frac{1}{2}}(\varphi^0, \varphi^1) = (1-\lambda)\cdot(1-\lambda)\cdot\tfrac{1}{\lambda} - \tfrac{1}{\lambda} = -2+\lambda.$$

In order to replace $\lambda' = \lambda$ by an arbitrarily small constant $\lambda' > 0$, and $C = 2 - \lambda$ by an arbitrarily large constant $C > 1$, we make the following observation: if the initial endowment $(\varphi^0_0, \varphi^1_0) = (0,0)$ were replaced by $(\varphi^0_0, \varphi^1_0) = (M, 0)$, for some large M, the agent could play the above game on a larger scale: she could choose $(\varphi^0_t, \varphi^1_t) = (M - \frac{M+1}{\lambda}, \frac{M+1}{\lambda})$, for $0 < t \leq 1$, to still satisfy (23):

$$V_1(\varphi^0, \varphi^1) = M - \tfrac{M+1}{\lambda} + (1-\lambda)\tfrac{M+1}{\lambda} = -1.$$

As regards the liquidation value $V^{liq}_{\frac{1}{2}}$, we now assume $S_{\frac{1}{2}} = 1 - \lambda'$ (instead of $S_{\frac{1}{2}} = 1 - \lambda$ in (25) and (26)) to make sure that $(CPS^{\lambda'})$ holds true. The liquidation value at time $t = \frac{1}{2}$ then becomes

$$\begin{aligned} V^{liq}_{\frac{1}{2}}(\varphi^0, \varphi^1) &= M - \tfrac{M+1}{\lambda} + (1-\lambda)(1-\lambda')\tfrac{M+1}{\lambda} \\ &= M - (M+1)[1 + \lambda'(\frac{1}{\lambda} - 1)] \end{aligned}$$

which tends to $-\infty$, as $M \to \infty$ in view of $0 < \lambda' \leq \lambda < 1$.

Turning back to the original endowment $(\varphi^0_0, \varphi^1_0) = (0,0)$, the idea is that, during the time interval $[0, \frac{1}{4}]$, the price process S provides the agent with the opportunity to become rich with positive probability, i.e. $\mathbb{P}[(\varphi^0_{\frac{1}{4}}, \varphi^1_{\frac{1}{4}}) = (M, 0)] > 0$. We then play the above game, conditionally on the event $\{(\varphi^0_{\frac{1}{4}}, \varphi^1_{\frac{1}{4}}) = (M, 0)\}$ and with $[0,1]$ replaced by $[\frac{1}{4}, 1]$.

The subsequent construction makes this idea concrete. Let $(\mathcal{F}_t)_{0\leq t\leq 1}$ be generated by a Brownian motion $(W_t)_{0\leq t\leq 1}$. Fix disjoint sets A_+ and A_- in $\mathcal{F}_{\frac{1}{8}}$ such that $\mathbb{P}[A_+] = \frac{1}{2\widetilde{M}-1}$ and $\mathbb{P}[A_-] = 1 - \mathbb{P}[A_+]$, where $\widetilde{M} > 1$ is defined by $M = -1 + \widetilde{M}(1-\lambda')$. The set A_+ is split into two sets A_{++} and A_{+-} such that A_{++} and A_{+-} are in $\mathcal{F}_{\frac{1}{4}}$ and

$$\mathbb{P}\left[A_{++}\middle|\mathcal{F}_{\frac{1}{8}}\right] = \mathbb{P}\left[A_{+-}\middle|\mathcal{F}_{\frac{1}{8}}\right] = \frac{1}{2}\mathbb{1}_{A_+}.$$

We define $S_{\frac{1}{4}}$ by

$$S_{\frac{1}{4}} = \begin{cases} 2\widetilde{M} - 1 & \text{on } A_{++} \\ 1 & \text{on } A_{+-} \\ \frac{1}{2} & \text{on } A_- \end{cases}$$

and

$$S_t = \mathbb{E}\left[S_{\frac{1}{4}}|\mathcal{F}_t\right], \qquad 0 \le t \le \frac{1}{4}, \tag{27}$$

so that $(S_t)_{0\le t\le \frac{1}{4}}$ is a continuous $\mathbb{P}$-martingale. The numbers above were designed in such a way that

$$S_0 = 1,$$

and

$$S_{\frac{1}{8}} = \begin{cases} \widetilde{M} & \text{on } A_+ \\ \frac{1}{2} & \text{on } A_- \end{cases}$$

To define S_t also for $\frac{1}{4} < t \le 1$ we simply let $S_t = S_{\frac{1}{4}}$ on $A_{++} \cup A_-$ while, conditionally on A_{+-}, we repeat the above deterministic construction on $[\frac{1}{4}, 1]$:

$$\begin{aligned} S_t &= 1 - 4(t - \tfrac{1}{4})\lambda', & \tfrac{1}{4} \le t \le \tfrac{1}{2}, \\ S_t &= 1 - 2(1-t)\lambda', & \tfrac{1}{2} \le t \le 1. \end{aligned}$$

This defines the process S. Condition $(CPS^{\lambda'})$ is satisfied as $(\widetilde{S}_t)_{0\le t\le 1} := ((1-\lambda')S_{t\wedge\frac{1}{4}})_{0\le t\le 1}$ is a $\mathbb{P}$-martingale taking values in the bid-ask spread $[(1-\lambda')S_t, S_t]_{0\le t\le 1}$.

Let us now define the strategy (φ^0, φ^1) : starting with $(\varphi^0_0, \varphi^1_0) = (0,0)$ we define $(\varphi^0_t, \varphi^1_t) = (-1, 1)$, for $0 < t \le \frac{1}{8}$. In prose: the agent buys one stock at time $t = 0$ and holds it until time $t = \frac{1}{8}$. At time $t = \frac{1}{8}$ she sells the stock again, so that $(\varphi^0_{\frac{1}{8}}, \varphi^1_{\frac{1}{8}}) = (-1 + \frac{(1-\lambda)}{2}, 0)$ on A_-, while $(\varphi^0_{\frac{1}{8}+}, \varphi^1_{\frac{1}{8}+}) = (-1 + \widetilde{M}(1-\lambda'), 0) = (M, 0)$ on A_+.

On A_- we simply define $(\varphi^0_t, \varphi^1_t) = (-1 + \frac{1-\lambda}{2}, 0)$, for all $\frac{1}{8} < t \leq 1$ and note that (23) is satisfied on A_-.

On A_+ we define $(\varphi^0_t, \varphi^1_t) = (M, 0)$, for $\frac{1}{8} < t \leq \frac{1}{4}$. In prose: during $]\frac{1}{8}, \frac{1}{4}]$ the agent does not invest into the stock and is happy about the M bonds in her portfolio. At time $t = \frac{1}{4}$ we distinguish two cases: on A_{++} we continue to define $(\varphi^0_t, \varphi^1_t) = (M, 0)$, also for $\frac{1}{4} < t \leq 1$. On A_{+-} we let $(\varphi^0_t, \varphi^1_t) = (M - \frac{M+1}{\lambda}, \frac{M+1}{\lambda})$, for $\frac{1}{4} < t \leq 1$. As discussed above, inequality (23) then holds true almost surely, while $V_{\frac{1}{2}}(\varphi^0, \varphi^1)$ attains the value $M-(M+1)[1+\lambda'(\frac{1}{\lambda}-1)]$ which tends to $-\infty$ as M tends to ∞. This happens with positive probability $\mathbb{P}[A_{+-}] > 0$.

The construction of the example now is complete. $\square$

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