# A REGULARIZED KELLERER THEOREM IN ARBITRARY DIMENSION 

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#### Abstract

We present a multidimensional extension of Kellerer's theorem on the existence of mimicking Markov martingales for peacocks, a term derived from the French for stochastic processes increasing in convex order. For a continuous-time peacock in arbitrary dimension, after Gaussian regularization, we show that there exists a strongly Markovian mimicking martingale Itô diffusion. A novel compactness result for martingale diffusions is a key tool in our proof. Moreover, we provide counterexamples to show, in dimension $d \geq 2$, that uniqueness may not hold, and that some regularization is necessary to guarantee existence of a mimicking Markov martingale.


## 1. Introduction

Given a finite set of probability measures on $\mathbb{R}^{d}$ that are increasing in convex order, Strassen [31] showed in 1965 that there exists a Markov martingale whose marginals coincide with the given probability measures. We call this latter property mimicking. For a family of measures indexed by continuous time that are increasing in convex order, also called a peacock (Processus Croissant pour l'Ordre Convex), Kellerer [20] proved in 1972 that there exists a mimicking Markov martingale in dimension one. The questions of continuity, strong Markovianity, and uniqueness for Kellerer's mimicking martingale remained open until the work of Lowther [23, $24,26,27,25]$ completely clarified the situation. Lowther showed that, in dimension one, there exists a unique continuous strong Markov mimicking martingale when the peacock is weakly continuous and the marginals have convex support. It is also known that the strong Markov property is required to obtain uniqueness; Beiglböck et al. [4] construct a one-dimensional continuous Markov martingale whose marginals coincide with those of Brownian motion but which does not have the strong Markov property.

While the problem of finding mimicking Markov martingales is thus very well understood for one-dimensional peacocks, the higher dimensional case has remained wide open, although

[^0]50 years have passed since the publication of Kellerer's result. In this paper, to the best of our knowledge, we provide the first known multidimensional extension of Kellerer's theorem. Given a peacock in $\mathbb{R}^{d}$, we show that, after some Gaussian regularization, there exists a strongly Markovian martingale Itô diffusion that mimics the regularized peacock.

To prove our result, we construct a martingale Itô diffusion that mimics the regularized peacock on the dyadics, and then pass to a limit in finite dimensional distributions. In order to take such a limit, we prove a compactness result for martingale Itô diffusions.

Additionally, we show that uniqueness does not necessarily hold in higher dimensions. We consider an example of a martingale Itô diffusion studied by Robinson [29] and Cox-Robinson [8], and we show that this martingale mimics the marginals of a two-dimensional Brownian motion, while itself not being a Brownian motion. We also show, by means of counterexamples, that the Gaussian regularization is necessary to guarantee the existence of a mimicking Markov martingale.
Theorem 1.1 (existence of mimicking martingales). Let $\left(\mu_{t}\right)_{t \in[0,1]}$ be a weakly continuous peacock in $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Fix $\delta, \varepsilon>0$ and, for each $t \geq 0$, define the regularized measure $\mu_{t}^{\mathrm{r}}:=\mu_{t} * \gamma^{\varepsilon(t+\delta)}$. Then there exists a strongly Markovian martingale Itô diffusion $\left(M_{t}\right)_{t \in[0,1]}$ mimicking the regularized peacock $\left(\mu_{t}^{\mathrm{r}}\right)_{t \in[0,1]}$.

More precisely, there exists a measurable function $(t, x) \mapsto \sigma_{t}(x)$ on $[0,1] \times \mathbb{R}^{d}$, taking values in the set of positive definite matrices, and a standard $\mathbb{R}^{d}$-valued Brownian motion $\left(B_{t}\right)_{t \in[0,1]}$ such that the mimicking martingale $\left(M_{t}\right)_{t \in[0,1]}$ satisfies

$$
\mathrm{d} M_{t}=\sigma_{t}\left(M_{t}\right) \mathrm{d} B_{t}
$$

where $\sigma$ is locally Lipschitz continuous in the variable $x$, uniformly in $t \in[0,1]$ and, for every $x \in \mathbb{R}^{d}$, there exist constants $c_{x}, C_{x}>0$ such that, for $t \in[0,1]$, we have the bounds

$$
c_{x} \mathrm{id} \leq \sigma_{t}(x) \leq C_{x} \mathrm{id} .
$$

Moreover, the martingale $M$ is a strong Feller process.
Note that, in particular, the mimicking martingale that we construct in Theorem 1.1 is continuous and strongly Markovian. A key ingredient in the construction of this mimicking martingale is the following result which allows to pass to limits of martingale Itô diffusions, the details of which are presented in Section 5 .

Theorem 1.2 (compactness of martingale Itô diffusions). A set of martingale Itô diffusions satisfying Assumptions 5.1 (A1)-(A5) is precompact in the set of martingale Ito diffusions with respect to convergence in finite dimensional distributions.

Our next main result is that the mimicking martingale of Theorem 1.1 may not be unique.
Theorem 1.3 (non-uniqueness of mimicking martingales). Let $\left(B_{t}\right)_{t \in[0,1]}$ be a standard Brownian motion in $\mathbb{R}^{2}$ with initial law $\operatorname{Law}\left(B_{0}\right)=\eta$, where $\eta$ is rotationally invariant with finite second moment. Define a peacock $\mu$ by $\mu_{t}=\operatorname{Law}\left(B_{t}\right)$, for $t \in[0,1]$.

Then there exists a continuous strongly Markovian martingale diffusion $\left(M_{t}\right)_{t \in[0,1]}$, which is not a Brownian motion, such that $\operatorname{Law}\left(M_{t}\right)=\mu_{t}$, for all $t \in[0,1]$.

We further construct a series of counterexamples in dimension $d=4$ which show that, without regularization, Theorem 1.1 does not hold in full generality, even without imposing continuity of the mimicking martingale, let alone the Itô diffusion property.

Theorem 1.4 (necessity of regularization). There exists a weakly continuous square-integrable peacock $\left(\mu_{t}\right)_{t \geq 0}$ on $\mathbb{R}^{4}$ such that, for the peacock $\left(\mu_{t} * \gamma^{t}\right)_{t \geq 0}$, there exists no mimicking Markov martingale.

While previous authors have considered the problem of finding mimicking martingales in general dimensions, to the best of our knowledge the present work is the first to provide a
multidimensional extension of Kellerer's theorem. Prior to Kellerer's work, Doob [9] proved the existence of mimicking martingales taking values in an abstract compact space in continuous time, but notably did not consider the Markov property. More recently Hirsch and Roynette [16] proved existence for continuous-time peacocks on $\mathbb{R}^{d}, d \geq 1$, with right-continuous paths, again without the Markov property.

Juillet [18] considered generalizing Kellerer's theorem in two different directions, first showing that when a peacock on $\mathbb{R}$ is indexed by a two-parameter family with some partial order, mimicking martingales may not exist at all. Moreover, [18] provides an example of a peacock on $\mathbb{R}^{2}$ for which no Lipschitz-Markov mimicking martingale exists; the condition of LipschitzMarkovianity implies strong Markovianity and it is defined and used to prove continuity in the one-dimensional case in [3]. The Lipschitz-Markov property is also required in the proof of continuity in [27]. In light of the result of [18], the approaches of [3] and [27] do not lend themselves well to the higher-dimensional problem, and the strong Markovianity of mimicking martingales that follows from Theorem 1.1 cannot be improved in general.

We have seen that, in dimension one, uniqueness holds in the class of continuous strong Markov mimicking martingales when the marginals of the peacock have convex support. Theorem 1.3 shows that strong Markovianity is not sufficient to guarantee uniqueness in higher dimensions, by exhibiting a continuous two-dimensional strong Markov martingale with Brownian marginals that is not itself a Brownian motion. The question of the existence of martingales distinct from Brownian motion that have Brownian marginals goes back to Hamza and Klebaner [14], who showed that such a fake Brownian motion with discontinuous paths exists in one dimension. As already mentioned, the culmination of this one-dimensional investigation was the construction [4] of a continuous Markovian fake Brownian motion. Of course Brownian motion is the unique continuous strong Markov martingale with Brownian marginals in one dimension. In two dimensions however, we show in Theorem 1.3 that there exists a fake Brownian motion that is continuous and strongly Markovian.

We remark that the mimicking martingale of Theorem 1.1 is an Itô diffusion process with Markovian diffusion coefficient. Finding mimicking martingales of this form has also received extensive interest since the work of Krylov [21] and Gyöngy [13]. In fact we twice apply a more recent result of Brunick and Shreve [6] on mimicking Markovian diffusions in our construction in Section 2.

For a more detailed review of the existing literature, we refer the reader to the surveys of Hirsch and Roynette [17] and Beiglböck et al. [5], and the references therein.

The structure of the present article is as follows. In Section 2, we construct a strongly Markovian mimicking martingale Itô diffusion, thus proving Theorem 1.1. We then prove Theorem 1.3 in Section 3, by providing a counterexample to uniqueness of mimicking martingales. We present further examples in Section 4, which show that existence may fail without regularization, thus proving Theorem 1.4. Finally, in Section 5, we prove the compactness result Theorem 1.2 for martingale Itô diffusions, which is key to the proof of Theorem 1.1 in Section 2.

We introduce some notation and terminology that will be used throughout the paper. We denote by $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ the set of probability measures on $\mathbb{R}^{d}$ with finite second moment. We denote by $\mathcal{F}^{X}$ the natural filtration of a stochastic process $X$, enlarged as necessary to satisfy the usual conditions. For $\eta>0$, we write $\gamma^{\eta}$ for the centered Gaussian law on $\mathbb{R}^{d}$ with covariance matrix $\eta$ id. For measures $\mu, \nu$, we write $\mu \preceq \nu$ to denote that $\mu$ is dominated by $\nu$ in convex order; i.e. for any convex function $f, \int f \mathrm{~d} \mu \leq \int f \mathrm{~d} \nu$. A family of measures $\left(\mu_{t}\right)_{t \in I}$ is called a peacock if it is increasing in the convex order. ${ }^{1}$ We say that a process $\left(X_{t}\right)_{t \in I}$ mimics $\left(\mu_{t}\right)_{t \in I}$ if $\operatorname{Law}\left(X_{t}\right)=\mu_{t}$ for all $t \in I$.

For matrices $A, B \in \mathbb{R}^{d \times d}$ the notation $A \leq B$ denotes that the matrix $B-A$ is positive semidefinite. When working with matrices, we will always use the Hilbert-Schmidt norm (also

[^1]known as the Frobenius norm): for $A \in \mathbb{R}^{d \times d}$, we write $\|A\|=\operatorname{tr}\left(A A^{\top}\right)=\sum_{i, j=1}^{d} A_{i j}^{2}=\sum_{i=1}^{d} \lambda_{i}^{2}$, where $\lambda_{i}$ are the eigenvalues of $A$.

## 2. Construction of a mimicking martingale

Let $d \geq 2$ and let $\left(\mu_{t}\right)_{t \in[0,1]}$ be a weakly continuous peacock in $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$; i.e. $\mu_{t_{0}} \preceq \mu_{t_{1}}$ for all $t_{0} \leq t_{1}$, $\sup _{t \in[0,1]} \int x \mu_{t}(\mathrm{~d} x)<\infty$, and $t \mapsto \int f \mathrm{~d} \mu_{t}$ is continuous for any bounded continuous function $f$. Fix $\delta, \varepsilon>0$ and define the regularized peacock $\mu^{\mathrm{r}}$ by

$$
\begin{equation*}
\mu_{t}^{\mathrm{r}}=\mu_{t} * \gamma^{\varepsilon(t+\delta)}, \quad t \in[0,1] . \tag{2.1}
\end{equation*}
$$

Note that the process $\left(\mu_{t}^{\mathrm{r}}\right)_{t \in[0,1]}$ is a peacock satisfying $\mu_{t} \preceq \mu_{t}^{\mathrm{r}}$, for all $t \in[0,1]$.
Remark 2.1. The relevant feature of the function

$$
\begin{equation*}
\varphi(t)=\varepsilon(t+\delta), \quad t \in[0,1] \tag{2.2}
\end{equation*}
$$

is that $\phi(0)>0$ and $t \mapsto \phi(t)$ is strictly increasing. It will become clear from the construction below that we can replace (2.2) with any such function.

In this context, we also normalize the peacock $\left(\mu_{t}\right)_{t \in[0,1]}$ by making a deterministic time change so that $\int x^{2} \mu_{t+h}(\mathrm{~d} x)-\int x^{2} \mu_{t}(\mathrm{~d} x)=h$, for $t \in[0,1), h>0$. For convenience, we still take $\varphi$ as in (2.2) after the time change.

Fix $n \in \mathbb{N}$ and consider the dyadics $S^{n}:=\left\{2^{-n}, 2 \cdot 2^{-n}, \ldots, 2^{n} \cdot 2^{-n}\right\} \subseteq[0,1]$. For $k \in$ $\left\{1, \ldots, 2^{n}-1\right\}$, denote $t_{k}^{n}:=k 2^{-n}$. We will construct a martingale Itô diffusion that mimics $\mu^{\mathrm{r}}$ on the dyadics $S^{n}$. Theorem 5.2 will allow us to pass to a limit. We first construct martingale Itô diffusions on each interval $\left[t_{k}^{n}, t_{k+1}^{n}\right]$, before concatenating these intervals. This step is rather standard (cf. [15, 17]). For our purposes it is convenient to use the concept of stretched Brownian motion introduced in [1].

Lemma 2.2. Let $\left(\mu_{t}\right)_{t \in[0,1]}$ be a weakly continuous square-integrable peacock. Then there exists a martingale diffusion $\left(\bar{M}_{t}^{n, k}\right)_{t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]}$, with the representation

$$
\begin{equation*}
\mathrm{d} \bar{M}_{t}^{n, k}=\bar{\sigma}_{t}^{n, k}\left(\bar{M}_{t}^{n, k}\right) \mathrm{d} B_{t}, \quad \text { on } \quad\left[t_{k}^{n}, t_{k+1}^{n}\right), \tag{2.3}
\end{equation*}
$$

for some measurable function $(t, x) \mapsto \bar{\sigma}_{t}^{n, k}(x)$, taking values in the set of positive semidefinite matrices, such that $\operatorname{Law}\left(\bar{M}_{t_{k}^{n}}^{n, k}\right)=\mu_{t_{k}^{n}}$ and $\operatorname{Law}\left(\bar{M}_{t_{k+1}}^{n, k}\right)=\mu_{t_{k+1}^{n}}$.

Proof. Let $\left(\tilde{M}^{n, k}\right)_{t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]}$ be the stretched Brownian motion with $\operatorname{Law}\left(\tilde{M}_{t_{k}^{n}}^{n, k}\right)=\mu_{t_{k}^{n}}$ and $\operatorname{Law}\left(\tilde{M}_{t_{k+1}}^{n, k}\right)=\mu_{t_{k+1}^{n}}$, as defined in [1]. The represenation (2.3) follows from [1, Proposition 2.5 and Lemma 3.12].

We do not yet have a control on the matrix norm of $\left(\bar{\sigma}_{t}^{n, k}\left(\bar{M}^{n, R}\right)\right)_{t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]}$. In order to achieve an upper bound on the diffusion matrix, we make a first convolution with a Gaussian. This will have an averaging effect and allow us to control the diffusion from above almost surely. Namely, we take a centered Gaussian random variable $\Gamma^{n, k}$ with covariance matrix $\left(\varepsilon\left[t_{k}^{n}+\delta\right]\right) \mathrm{id}$, independent of $\mathcal{F}^{B}$ and $\mathcal{F}^{\bar{M}^{n, k}}$, and define

$$
\begin{equation*}
\hat{M}_{t}^{n, k}:=\bar{M}_{t}^{n, k}+\Gamma^{n, k}, \quad t \in\left[t_{k}^{n}, t_{k+1}^{n}\right] . \tag{2.4}
\end{equation*}
$$

Then, for the initial law in this interval, we have $\operatorname{Law}\left(\hat{M}_{t_{k}^{n}}^{n, k}\right)=\mu_{t_{k}^{n}} * \gamma^{\varepsilon\left(t_{k}^{n}+\delta\right)}=\mu_{t_{k}^{n}}^{\mathrm{r}}$, and for the terminal law, we have the ordering $\operatorname{Law}\left(\hat{M}_{t_{k+1}^{n}}^{n, k}\right)=\mu_{t_{k+1}^{n}}^{n} * \gamma^{\varepsilon\left(t_{k}^{n}+\delta\right)} \preceq \mu_{t_{k+1}^{n}} * \gamma^{\varepsilon\left(t_{k+1}^{n}+\delta\right)}=$ $\mu_{t_{k+1}^{r}}^{\mathrm{r}}$, where we recall the definition of $\mu^{\mathrm{r}}$ from (2.1). Later we will make a second Gaussian convolution, which will allow us to also bound the diffusion matrix $\sigma^{n, k}$ from below. We now prove that we have an upper bound.

Lemma 2.3. For $n, k \in \mathbb{N}$ and $\bar{\sigma}^{n, k}$ as in (2.3), define the positive semidefinite matrix-valued function $(t, x) \mapsto \tilde{\sigma}_{t}^{n, k}(x)$ by

$$
\begin{equation*}
\tilde{\sigma}_{t}^{n, k}(x):=\frac{\int \bar{\sigma}_{t}^{n, k}(y) g^{n, k}(x-y) \bar{m}_{t}^{n, k}(\mathrm{~d} y)}{\int g^{n, k}(x-y) \bar{m}_{t}^{n, k}(\mathrm{~d} y)}, \quad t \in\left[t_{k}^{n}, t_{k+1}^{n}\right], \tag{2.5}
\end{equation*}
$$

where $g^{n, k}$ is the density of a Gaussian with mean zero and covariance matrix $\left(\varepsilon\left[t_{k}^{n}+\delta\right]\right) \mathrm{id}$, and $\bar{m}_{t}^{n, k}=\operatorname{Law}\left(\bar{M}_{t}^{n, k}\right)$. Let $\left(\tilde{M}_{t}^{n, k}\right)_{t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]}$ be the martingale defined by choosing a random variable $\tilde{M}_{t_{k}^{n}}^{n, k}$ with $\operatorname{Law}\left(\tilde{M}_{t_{k}^{n}}^{n, k}\right)=\mu_{t_{k}^{r}}^{\mathrm{r}}$, and solving

$$
\mathrm{d} \tilde{M}_{t}^{n, k}=\tilde{\sigma}^{n, k}\left(\tilde{M}_{t}^{n, k}\right) \mathrm{d} B_{t} \quad \text { on } \quad\left[t_{k}^{n}, t_{k+1}^{n}\right],
$$

so that for each $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$, $\operatorname{Law}\left(\tilde{M}_{t}^{n, k}\right)=\operatorname{Law}\left(\hat{M}_{t}^{n, k}\right)=\bar{m}_{t}^{n, k} * \gamma^{\varepsilon\left(t_{k}^{n}+\delta\right)}$. In particular, $\operatorname{Law}\left(\tilde{M}_{t_{k+1}^{n}}^{n, k}\right)=\mu_{t_{k+1}^{n}} * \gamma^{\varepsilon\left(t_{k}^{n}+\delta\right)}$.

For every compact set $K \subseteq \mathbb{R}^{d}$, there exists a constant $C_{K}$, independent of $k$ and $n$, such that

$$
\left\|\tilde{\sigma}_{t}^{n, k}(x)\right\| \leq C_{K}, \quad(t, x) \in\left[t_{k}^{n}, t_{k+1}^{n}\right] \times K
$$

Moreover, there exists a Lipschitz constant $L_{K}$, independent of $t, n$ and $k$, such that the functions $x \mapsto \tilde{\sigma}_{t}^{n, k}(x)$ are Lipschitz with constant $L_{K}$ for $x \in K, t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$.

Proof. From (2.4), it is clear that

$$
\mathrm{d} \hat{M}_{t}^{n, k}=\bar{\sigma}_{t}^{n, k}\left(\hat{M}_{t}^{n, k}-\Gamma^{n, k}\right) \mathrm{d} B_{t} .
$$

We now make the following Markovian projection. Defining $\tilde{M}^{n, k}$ by

$$
\mathrm{d} \tilde{M}_{t}^{n, k}=\mathbb{E}\left[\bar{\sigma}_{t}^{n, k}\left(\hat{M}_{t}^{n, k}-\Gamma^{n, k}\right) \mid \tilde{M}_{t}^{n, k}\right] \mathrm{d} B_{t}
$$

we find that $\hat{M}^{n, k}$ and $\tilde{M}^{n, k}$ have the same marginals, by Corollary 3.7 of [6] on mimicking diffusions.

We can check that $\mathbb{E}\left[\bar{\sigma}_{t}^{n, k}\left(\hat{M}_{t}^{n, k}-\Gamma^{n, k}\right) \mid \tilde{M}_{t}^{n, k}\right]=\tilde{\sigma}_{t}^{n, k}\left(\tilde{M}_{t}^{n, k}\right)$ by the following elementary observations. For any $x \in \mathbb{R}^{d}$, we consider the density of the conditional law

$$
\operatorname{Law}\left(\hat{M}_{t}^{n, k}-\Gamma^{n, k} \mid \tilde{M}_{t}^{n, k}=x\right)=\operatorname{Law}\left(\tilde{M}_{t}^{n, k}-\Gamma^{n, k} \mid \tilde{M}_{t}^{n, k}=x\right),
$$

which must be proportional to $g^{n, k}(x-y) \bar{m}_{t}^{n, k}(\mathrm{~d} y)$. Computing the normalising constant leads us to the equality $\mathbb{E}\left[\bar{\sigma}_{t}^{n, k}\left(\hat{M}_{t}^{n, k}-\Gamma^{n, k}\right) \mid \tilde{M}_{t}^{n, k}=x\right]=\tilde{\sigma}_{t}^{n, k}(x)$.

Next we bound the quantity $\tilde{\sigma}_{t}^{n, k}(x)$ in the Hilbert-Schmidt norm . Note that the law $\left(\bar{m}_{t}^{n, k}\right)_{t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]}$ of the martingale $\bar{M}^{n, k}$ is tight. In particular, there exists a compact set $K_{0} \subseteq \mathbb{R}^{d}$ such that $\bar{m}_{t}^{n, k}\left(K_{0}\right) \geq \frac{1}{2}$, for all $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$ and all $k \in\left\{1, \ldots, 2^{n}-1\right\}$. Indeed, choosing $K_{0}$ such that $\left|K_{0}\right|^{-1} \int x \mathrm{~d} \mu_{1}(x) \leq \frac{1}{2}$, and applying Doob's inequality and the convex ordering of the marginals gives the desired bound. Then, for an arbitrary compact set $K \subseteq \mathbb{R}^{d}$, not necessarily coinciding with $K_{0}$, we can bound the normalising constant in the denominator of (2.5) by

$$
\begin{aligned}
\int_{\mathbb{R}^{d}} g^{n, k}(x-y) \bar{m}_{t}^{n, k}(\mathrm{~d} y) & \geq \int_{K_{0}} g^{n, k}(x-y) \bar{m}_{t}^{n, k}(\mathrm{~d} y) \\
& \geq \frac{\bar{C}_{K}}{2}\left(\varepsilon \pi\left[t_{k}^{n}+\delta\right]\right)^{-\frac{d}{2}}, \quad \text { for } \quad x \in K
\end{aligned}
$$

where $\bar{C}_{K}:=\inf \left\{\exp \left\{-\varepsilon^{-1} \delta^{-1}|x-y|^{2}\right\}: x \in K, y \in K_{0}\right\}>0$, independent of $t$ and $k$. We bound the numerator of (2.5) in the Hilbert-Schmidt norm by

$$
\left\|\int \bar{\sigma}_{t}^{n, k}(y) g^{n, k}(x-y) \bar{m}_{t}^{n, k}(\mathrm{~d} y)\right\|^{2} \leq\left(\varepsilon \pi\left[t_{k}^{n}+\delta\right]\right)^{d} \mathbb{E}\left[\left\|\bar{\sigma}_{t}^{n, k}\left(\bar{M}_{t}^{n, k}\right)\right\|^{2}\right] .
$$

Recall from Remark 2.1 that $\mathbb{E}\left[\left|\bar{M}_{t+h}^{n, k}\right|^{2}-\left|\bar{M}_{t}^{n, k}\right|^{2}\right]=h$, for all $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right)$ and $h$ sufficiently small. But, by the Itô isometry,

$$
\mathbb{E}\left[\left|\bar{M}_{t+h}^{n, k}\right|^{2}-\left|\bar{M}_{t}^{n, k}\right|^{2}\right]=\int_{t}^{t+h} \mathbb{E}\left[\left\|\bar{\sigma}_{t}^{n, k}\left(\bar{M}_{t}^{n, k}\right)\right\|^{2} \mathrm{~d} t,\right.
$$

and so $\mathbb{E}\left[\left\|\bar{\sigma}_{t}^{n, k}\left(\bar{M}_{t}^{n, k}\right)\right\|^{2}=1\right.$, for all $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$. Altogether, we have the upper bound

$$
\left\|\tilde{\sigma}_{t}^{n, k}(x)\right\| \leq \frac{2}{\bar{C}_{K}}, \quad \text { for } \quad x \in K
$$

It remains to prove continuity. To save notation in the following calculation, we suppress the dependency on $n$ and $k$, setting $g:=g^{n, k}, \bar{\sigma}:=\bar{\sigma}^{n, k}, \bar{m}:=\bar{m}^{n, k}$. Fix $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$, and $x_{0}, x_{1} \in K$ for some compact set $K \subseteq \mathbb{R}^{d}$. Then, from the definition (2.5), we calculate

$$
\begin{aligned}
& \left\|\tilde{\sigma}_{t}^{n, k}\left(x_{1}\right)-\tilde{\sigma}_{t}^{n, k}\left(x_{0}\right)\right\| \\
& \leq\left(\varepsilon \pi\left[t_{k}^{n}+\delta\right]\right)^{d} \bar{C}_{K}^{-2} \| \int g\left(x_{0}-y\right) \bar{m}_{t}(\mathrm{~d} y) \int \bar{\sigma}_{t} g_{t}\left(x_{1}-y\right) \bar{m}_{t}(\mathrm{~d} y) \\
& \quad-\int g_{t}\left(x_{1}-y\right) \bar{m}_{t}(\mathrm{~d} y) \int \bar{\sigma}_{t}(y) g\left(x_{0}-y\right) \bar{m}_{t}(\mathrm{~d} y) \| \\
& \leq\left(\varepsilon \pi\left[t_{k}^{n}+\delta\right]\right)^{d} \bar{C}_{K}^{-2}\left|\int g\left(x_{0}-y\right) \bar{m}_{t}(\mathrm{~d} y)\right|\left\|\int\left[g\left(x_{1}-y\right)-g\left(x_{0}-y\right)\right] \bar{\sigma}_{t}(y) \bar{m}_{t}(\mathrm{~d} y)\right\| \\
& \quad+\left(\varepsilon \pi\left[t_{k}^{n}+\delta\right]\right)^{d} \bar{C}_{K}^{-2}\left|\int\left[g\left(x_{0}-y\right)-g\left(x_{1}-y\right)\right] \bar{m}_{t}(\mathrm{~d} y)\right|\left\|\int g\left(x_{0}-y\right) \bar{\sigma}_{t}(y) \bar{m}_{t}(\mathrm{~d} y)\right\| .
\end{aligned}
$$

As in the proof of the upper bound, note that $\left\|\int \bar{\sigma}_{t}(y) \bar{m}_{t}(\mathrm{~d} y)\right\| \leq 1$, by Remark 2.1. We also see that each constant $\left(\varepsilon \pi\left[t_{k}^{n}+\delta\right]\right)^{\frac{d}{2}}$ cancels with a normalising constant from $g=g^{n, k}$. The Gaussian density $g$ is Lipschitz with Lipschitz constant $L$ that can be taken independent of $t, n, k$. Together, we find that

$$
\left\|\tilde{\sigma}_{t}^{n, k}\left(x_{1}\right)-\tilde{\sigma}_{t}^{n, k}\left(x_{0}\right)\right\| \leq 2 \bar{C}_{K}^{-2} L\left|x_{1}-x_{0}\right|,
$$

as required.
We now have a martingale $\tilde{M}^{n, k}$ on $\left[t_{k}^{n}, t_{k+1}^{n}\right]$, whose diffusion matrix $(t, x) \mapsto \tilde{\sigma}_{t}^{n, k}(x)$ is bounded from above on compact sets in $[0,1] \times \mathbb{R}^{d}$. To achieve a lower bound, we divide the interval $\left[t_{k}^{n}, t_{k+1}^{n}\right]$ in half and time-change the martingale by a factor of two in the first half of the interval. In the second half of the interval, we shall simply add a Brownian motion with an appropriately scaled covariance. This gives us a second Gaussian convolution to arrive at the measure $\mu_{t_{k+1}^{n}}^{\mathrm{r}}=\mu_{t_{k+1}^{n}} * \gamma^{\varepsilon\left(t_{k+1}^{n}+\delta\right)}$, rather than $\mu_{t_{k+1}^{n}} * \gamma^{\varepsilon\left(t_{k}^{n}+\delta\right)}$, at the terminal time $t_{k+1}^{n}$.

Define the function $(t, x) \mapsto \sigma_{t}^{n, k}(x)$ by

$$
\sigma_{t}^{n, k}(x):= \begin{cases}\sqrt{2} \tilde{\sigma}_{t_{k}^{n}+2\left(t-t_{k}^{n}\right)}^{n, k}(x), & t \in\left[t_{k}^{n}, t_{k}^{n}+2^{-(n+1)}\right] \\ \sqrt{2 \varepsilon} \mathrm{id}, & t \in\left[t_{k}^{n}+2^{-(n+1)}, t_{k+1}^{n}\right]\end{cases}
$$

and define the martingale $\left(M_{t}^{n, k}\right)_{t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]}$ by starting with a random variable $M_{t_{k}^{n}}^{n, k}$ with $\operatorname{Law}\left(M_{t_{k}^{n}}^{n, k}\right)=$ $\mu_{t_{k}^{n}}^{\mathrm{r}}$ and then solving

$$
\mathrm{d} M_{t}^{n, k}=\sigma_{t}^{n, k}\left(M_{t}^{n, k}\right) \mathrm{d} B_{t}, t \in\left[t_{k}^{n}, t_{k+1}^{n}\right] .
$$

Then we see that

$$
\operatorname{Law}\left(M_{t_{k+1}^{n}}^{n, k}\right)=\left(\mu_{t_{k+1}^{n}} * \gamma^{\varepsilon\left(t_{k}^{n}+\delta\right)}\right) * \gamma^{\varepsilon 2^{-n}}=\mu_{t_{k+1}^{n}} * \gamma^{\varepsilon\left(t_{k+1}^{n}+\delta\right)}=\mu_{t_{k+1}^{n}}^{\mathrm{r}},
$$

and we have the lower bound

$$
\int_{t_{k}^{n}}^{t_{k+1}^{n}} \sigma_{t}^{n, k}(x) \mathrm{d} t \geq \sqrt{\varepsilon} 2^{-(n+1)} \mathrm{id}, \quad x \in \mathbb{R}^{d}
$$

We finally paste everything together and define $(t, x) \mapsto \sigma_{t}^{n}(x)$ by

$$
\sigma_{t}^{n}(x):=\sum_{k=1}^{2^{n}-1} \sigma_{t}^{n, k}(x) \mathbb{1}_{\left[t_{k}^{n}, t_{k+1}^{n}\right]}(t)
$$

and define a martingale $\left(M_{t}^{n}\right)_{t \in[0,1]}$ by first choosing a random variable $M_{0}^{n}$ with $\operatorname{Law}\left(M_{0}^{n}\right)=\mu_{0}^{\mathrm{r}}$ and then solving

$$
\mathrm{d} M_{t}^{n}=\sigma_{t}^{n}\left(M_{t}^{n}\right) \mathrm{d} B_{t}, \quad \text { on } \quad[0,1] .
$$

For each dyadic $t_{k}^{n} \in S^{n}$, we thus have $\operatorname{Law}\left(M_{t_{k}^{n}}^{n}\right)=\mu_{t_{k}^{n}}^{\mathrm{r}}$, by construction. Let us summarise what we have achieved so far in the following proposition.

Proposition 2.4. For each $n \in \mathbb{N}$, there exists a martingale diffusion $\left(M_{t}^{n}\right)_{t \in[0,1]}$ satisfying

$$
\mathrm{d} M_{t}^{n}=\sigma_{t}^{n}\left(M_{t}^{n}\right) \mathrm{d} B_{t}
$$

with $\operatorname{Law}\left(M_{r}^{n}\right)=\mu_{r}^{\mathrm{r}}$, for all $r \in S^{n}$. Moreover, $(t, x) \mapsto \sigma_{t}^{n}(x)$ is locally Lipschitz in $x$, uniformly in $t \in[0,1]$ and satisfies the following bounds. For any compact $K \subseteq \mathbb{R}^{d}$,

$$
\sigma_{t}^{n}(x) \leq C_{K} \mathrm{id}, \quad(t, x) \in[0,1] \times K
$$

and for any $s, t \in S^{n}$ with $t-s=2^{-m}$, for any $m \leq n$, we have the lower bound

$$
\int_{s}^{t} \sigma_{r}^{n}(x) \mathrm{d} r \geq \sqrt{\varepsilon} 2^{-(m+1)} \mathrm{id}, \quad x \in \mathbb{R}^{d}
$$

The next step will be to find a limiting martingale, which mimics the peacock $\mu^{\mathrm{r}}$ at every time $t \in[0,1]$. In Section 5 , we will prove a result on compactness of Itô diffusions with respect to convergence in finite dimensional distributions, which is tailor-made for the present application. We now use this result to allow us to pass to a limit and complete the proof of Theorem 1.1.

Proof of Theorem 1.1 (admitting Theorem 5.2). Take the sequence of functions $\sigma^{n}:[0,1] \times$ $\mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}, n \in \mathbb{N}$, to be as in Proposition 2.4. Then Assumptions 5.1 (A1)-(A5) are satisfied. By Theorem 5.2, there exists a function $(t, x) \mapsto \sigma_{t}(x)$, which is locally Lipschitz in $x$, uniformly in $t \in[0,1]$, such that $M^{n}$ converges in finite dimensional distributions to $M$, the unique strong solution of the $\operatorname{SDE~} \mathrm{d} M_{t}=\sigma_{t}\left(M_{t}\right) \mathrm{d} B_{t}$, with $\operatorname{Law}\left(M_{0}\right)=\mu_{0}^{\mathrm{r}}$. For each $n \in \mathbb{N}$, we have that $\operatorname{Law}\left(M_{t}^{n}\right)=\mu_{t}^{\mathrm{r}}$, for any dyadic $t \in S^{n}$. Therefore, taking the limit in finite dimensional distributions, we have $\operatorname{Law}\left(M_{t}\right)=\mu_{t}^{\mathrm{r}}$ for all $t \in[0,1]$. That is, $M$ is a mimicking martingale for the regularized peacock $\mu^{\mathrm{r}}$.

From the conclusion of Theorem 5.2, we also obtain the required bounds on $\sigma$. That is, for each $x \in \mathbb{R}^{d}$, there exist constants $c_{x}, C_{x}>0$ such that, uniformly in $t \in[0,1]$, we have

$$
\begin{equation*}
c_{x} \mathrm{id} \leq \sigma_{t}(x) \leq C_{x} \mathrm{id} \tag{2.6}
\end{equation*}
$$

It remains to verify the strong Feller property of $M$. The law of $\left(M_{t}\right)_{t \in[0,1]}$ is a solution of the associated martingale problem of Stroock and Varadhan [32]. Moreover, we proved above that the diffusion coefficient $\sigma$ satisfies the bounds (2.6), and that $\sigma$ is locally Lipschitz in $x$, uniformly in $t \in[0,1]$. Under these conditions, Theorem 10.1.3 of [32] implies that the martingale problem admits at most one solution. We now have that the martingale problem is well posed and, by applying Corollary 10.1.4 of [32], the unique solution has the strong Feller property. We conclude that the martingale $\left(M_{t}\right)_{t \in[0,1]}$ also has the strong Markov property.

## 3. Non-Uniqueness

We shall show that, even for the simple example of a two-dimensional Brownian motion, uniqueness does not hold in the class of continuous strong Markov mimicking martingales. In other words, the one-dimensional uniqueness result of Lowther [23] cannot be extended directly to higher dimensions.

Considering the problem of mimicking the marginals of a standard two-dimensional Brownian motion, the Brownian motion itself is of course a continuous strong Markov martingale with the required marginals. In order to disprove uniqueness, we seek another mimicking process with these properties. We will thus construct a two-dimensional continuous fake Brownian motion (see [14]) that is strongly Markovian.

Proposition 3.1. For every peacock $\mu=\left(\mu_{t}\right)_{t \in[0,1]}$ on $\mathbb{R}^{2}$ defined as in Theorem 1.3, there exist two distinct continuous strong Markov martingale diffusions mimicking $\mu$.

Proof. Let $B$ be a standard 2-dimensional Brownian motion started in some rotationally invariant law $\eta \in \mathcal{P}_{2}\left(\mathbb{R}^{2}\right) \backslash\{0\}$, and write $\left(\mu_{t}\right)_{t \in[0,1]}$ for its marginals. Then $B$ is a continuous strong Markov martingale mimicking $\left(\mu_{t}\right)_{t \in[0,1]}$. We now construct a continuous strong Markov martingale $M$ that mimics $\left(\mu_{t}\right)_{t \in[0,1]}$ and is not itself a Brownian motion.

For any $x \in \mathbb{R}^{2}$, let us denote $x^{\perp}:=\left(-x_{2}, x_{1}\right)^{\top}$, so that $x \cdot x^{\perp}=0$ and $|x|=\left|x^{\perp}\right|$. Let $W$ be a standard $\mathbb{R}$-valued Brownian motion and consider the SDE

$$
\begin{equation*}
\mathrm{d} M_{t}=\frac{1}{\left|M_{t}\right|}\left(M_{t}+M_{t}^{\perp}\right) \mathrm{d} W_{t} ; \quad \operatorname{Law}\left(M_{0}\right)=\eta . \tag{3.1}
\end{equation*}
$$

It is shown in [8, Proposition 3.2] that any solution of (3.2) almost surely does not hit the origin. Therefore, by applying standard arguments for SDEs with Lipschitz coefficients, one can show that there exists a unique strong solution $M$ of this SDE that is a continuous strong Markov martingale. A simulated trajectory of $M$ is shown in Figure 1b.

By [8, Proposition 3.2] again, the radius of $M$, denoted by $R_{t}=\left|M_{t}\right|$ for all $t \geq 0$, is a 2-dimensional Bessel process satisfying

$$
R_{t}=W_{t}+\frac{1}{2 R_{t}} \mathrm{~d} t, \quad t>0 ; \quad \operatorname{Law}\left(R_{0}\right)=\operatorname{Law}\left(\left|M_{0}\right|\right)
$$

Hence the radius of $M$ coincides with the radius of the 2 -dimensional Brownian motion $B$ in law (see e.g. [28, Chapter XI]). Moreover, the marginals of both the processes $M$ and $B$ have rotational symmetry. Hence we conclude that these marginals coincide. However, we can see that $M$ is not itself a 2-dimensional Brownian motion, since the components of $M$ in the two coordinate directions are not independent. We have thus shown that there exist at least two distinct continuous strong Markov martingales that mimic the marginals $\left(\mu_{t}\right)_{t \in[0,1]}$.
Remark 3.2. Note that $M$ solves the $\operatorname{SDE}$ (3.1) if and only if the time-changed process $\left(X_{t}^{\lambda}\right)_{t \in[0,1]}:=\left(M_{\lambda^{2} t}\right)_{t \in[0,1]}$ solves

$$
\begin{equation*}
\mathrm{d} X_{t}^{\lambda}=\frac{1}{\left|X_{t}^{\lambda}\right|}\left(\lambda X_{t}^{\lambda}+\sqrt{1-\lambda^{2}}\left(X_{t}^{\lambda}\right)^{\perp}\right) \mathrm{d} W_{t} ; \quad X_{0}=x_{0} \tag{3.2}
\end{equation*}
$$

with $\lambda=\frac{\sqrt{2}}{2}$. The $\operatorname{SDE}(3.2)$ with $\lambda \in[0,1]$ is as studied by Cox and Robinson in $[7,8]$ and in [29]. For $\lambda=1$, the martingale solving (3.2) acts as a one-dimensional Brownian motion on a fixed line through the origin - see Figure 1c. For $\lambda=0$, the martingale follows what is dubbed tangential motion in $[7,8]$. In this case, the process moves on a tangent to its current position, increasing the radius of the process deterministically - see Figure 1a. Such a martingale already appeared in [11] and [22] in the context of stochastic portfolio theory. In [8, Theorem 1.1] Cox and Robinson showed that there is no strong solution to (3.2) with $\lambda=0$ started from the origin, i.e. $\eta=\delta_{0}$, drawing parallels with Tsirelson's famous one-dimensional example [33] and the circular Brownian motion of Émery and Schachermayer [10]. In fact [8,

Theorem 1.2] also shows that there is no strong solution to the $\operatorname{SDE}$ (3.2) started from the origin for any $\lambda \in[0,1)$.

We will repeatedly refer to the $\operatorname{SDE}$ (3.2) with $\lambda=0$ in the examples of Section 4 below.


Figure 1. Simulations of the solution $X^{\lambda}$ of (3.2), up to the first exit of a ball, for different values of $\lambda$. Figure 1a and Figure 1c show the extreme behaviours within the class of martingales $\left\{X^{\lambda}: \lambda \in[0,1]\right\}$ (as already appeared in [7]). Figure 1b shows the midpoint between these cases, where we set $\lambda=\frac{\sqrt{2}}{2}$ then rescale time so that the martingale mimics the marginals of a Brownian motion.

## 4. Necessity of regularization

In this section, we will construct a series of counterexamples, showing that a mimicking Markov martingale may not exist without the regularization of Theorem 1.1. We present the examples in increasing order of complexity, first showing that there may not exist a continuous mimicking Markov martingale. We then remove the continuity assumption, and finally add some (partial) regularization, in both cases showing that mimicking Markov martingales may not exist.

The following examples build on the $\operatorname{SDE}(3.2)$, started from the origin, with $\lambda=0$; i.e.

$$
\begin{equation*}
\mathrm{d} X_{t}=\frac{1}{\left|X_{t}\right|} X_{t}^{\perp} \mathrm{d} W_{t}, \quad X_{0}=0 \tag{4.1}
\end{equation*}
$$

We recall some important properties of (4.1).
Remark 4.1. There exists a weak solution of (4.1) by [22, Theorem 4.3]. Moreover, [8, Theorem 1.1] shows that, at any time $t \in(0,1]$, the law of a weak solution $X$ is a uniform measure on the circle of radius $\sqrt{t}$, and so uniqueness in law holds for (4.1). In particular, a weak solution $X$ has deterministically increasing radius

$$
\left|X_{t}\right|=\sqrt{t}, \quad t \in[0,1] .
$$

We construct each of the below examples ${ }^{2}$ on $\mathbb{R}^{4} \equiv \mathcal{X}^{1} \times \mathcal{X}^{2}$, where $\mathcal{X}^{1}, \mathcal{X}^{2}$ are copies of $\mathbb{R}^{2}$. We also denote $S_{t}^{1}:=\left\{\left(x_{1}, x_{2}\right) \in \mathcal{X}^{1}: x_{1}^{2}+x_{2}^{2}=t\right\} \times\{(0,0)\}, S_{t}^{2}:=\{(0,0)\} \times\left\{\left(x_{3}, x_{4}\right) \in \mathcal{X}^{2}:\right.$ $\left.x_{1}^{2}+x_{2}^{2}=t\right\}$, and $S^{i}:=\cup_{t \geq 0} S_{t}^{i}$ for $i=1,2$. Note that $S^{1}$ and $S^{2}$ only intersect at the origin.

We emphasise that, throughout the following sections, the usual conditions of right-continuity and completeness are in force for all filtrations that we consider, and $\sigma(X)$ denotes the completion of the sigma-algebra generated by a given random variable $X$.

[^2]4.1. The continuous case. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we can define two independent copies $M^{1}, M^{2}$ of the weak solution of (4.1), as well as an independent Bernoulli $\left(\frac{1}{2}\right)$ random variable $\xi$. Now define a process $X$ taking values in $\mathbb{R}^{4}$ by
\[

X_{t}=\left\{$$
\begin{array}{ll}
\left(M_{t}^{1}, 0\right), & \xi=0,  \tag{4.2}\\
\left(0, M_{t}^{2}\right), & \xi=1,
\end{array}
$$ \quad t \in[0,1]\right.
\]

and write $\mu_{t}=\operatorname{Law}\left(X_{t}\right)$. Thus, following Remark 4.1, the measure $\mu_{t}$ is a uniform measure on $S_{t}^{1} \cup S_{t}^{2} \subset \mathbb{R}^{4}$. Note that the process $X$ is a martingale, and so $\mu$ is a peacock.

Proposition 4.2. There exists a peacock $\mu$ on $\mathbb{R}^{4}$ such that there does not exist any continuous Markov process mimicking $\mu$.

Proof. Let $\left(\mu_{t}\right)_{t \in[0,1]}=\left(\operatorname{Law}\left(X_{t}\right)\right)_{t \in[0,1]}$, where $X$ is defined by (4.2). Suppose that there exists a continuous process $Y$ that mimics $\mu$. We will show that $Y$ is not Markovian at time 0.

By definition of the peacock $\mu$ and continuity of the paths, we have

$$
\mathbb{P}\left[\left\{Y_{t} \in S_{t}^{1}, \forall t \in[0,1]\right\} \cup\left\{Y_{t} \in S_{t}^{2}, \forall t \in[0,1]\right\}\right]=1
$$

In particular, for $t_{0}>0$ the events $A^{1}:=\left\{Y_{t} \in S_{t}^{1}, \forall t \in[0,1]\right\}$ and $\left\{Y_{t_{0}} \in S_{t_{0}}^{1}\right\} \in \mathcal{F}_{t_{0}}$ differ just by a null set, hence $\mathbb{P}\left[A^{1}\right]=\frac{1}{2}$. On the one hand, we have by completeness and right-continuity of the filtration that $A^{1} \in \mathcal{F}_{0}$. On the other hand, $A^{1}$ can not be in $\sigma\left(Y_{0}\right)$ since the former has probability $\frac{1}{2}$ whereas the latter is the completion of the trivial $\sigma$-algebra. Now define a function $f: \mathbb{R}^{4} \rightarrow \mathbb{R}$ by $f(x)=\sqrt{x_{3}^{2}+x_{4}^{2}}$. Then, for any $t \in(0,1)$ we find

$$
\frac{\sqrt{t}}{2}=\mathbb{E}\left[f\left(Y_{t}\right) \mid \sigma\left(Y_{0}\right)\right] \neq \mathbb{E}\left[f\left(X_{t}\right) \mid \mathcal{F}_{0}\right]= \begin{cases}0 & \text { on } A^{1} \\ \sqrt{t} & \text { on } \Omega \backslash A^{1}\end{cases}
$$

We conclude that $Y$ is not Markovian at time 0 .
Remark 4.3. For the peacock $\mu$ defined via (4.2), the continuity assumption in Proposition 4.2 is required in order to show non-existence of mimicking Markov processes. In the following we construct a càdlàg strong Markov martingale $X$ that mimics the peacock $\mu$. The process behaves similarly to a compensated Poisson process: at time $t$ a particle $X_{t}$ starting either in $x=\left(x_{1}, x_{2}, 0,0\right) \in \mathbb{R}^{4}$ or $x=\left(0,0, x_{3}, x_{4}\right) \in \mathbb{R}^{4}$ drifts in the direction $x$ with speed $|x|^{2}$. This drift is compensated with jumps of rate $\frac{1}{2|x|^{2}}$ to a uniform distribution on $S_{|x|^{2}}^{2}\left(\right.$ resp. $S_{\left.|x|\right|^{2}}^{1}$ ). For $t \in \mathbb{R}_{+}$define the rate function $\lambda$ and its anti-derivative $\Lambda$ by

$$
\lambda_{t}:=\frac{1}{2 t} \quad \text { and } \quad \Lambda_{t}:=\frac{1}{2} \log (t) .
$$

To construct the process $X$, we first consider the peacock $\left(\mu_{t}\right)_{t \in\left[t_{0}, 1\right]}$ where $t_{0} \in(0,1)$ and define a mimicking process $X^{t_{0}}$. To this end, let $\left(\xi_{n}\right)_{n \in \mathbb{N}},\left(U_{n}\right)_{n \in \mathbb{N}}$, and $\left(V_{n}\right)_{n \in \mathbb{N}}$ be families of independent random variables such that $\xi_{n} \sim \exp (1), U_{n} \sim \operatorname{Unif}\left(S_{1}^{1}\right)$, and $V_{n} \sim \operatorname{Unif}\left(S_{1}^{2}\right)$. Given that $X_{t_{0}}^{t_{0}}=x \in S_{|x|^{2}}^{1}, U_{0}:=x /|x|$, and $t \in\left(t_{0}, 1\right]$, we set

$$
X_{t}^{t_{0}}=\sqrt{t} \sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{1}_{\left\{\sum_{k=1}^{n} \xi_{k} \leq \Lambda_{t}-\Lambda_{t_{0}}<\sum_{k=1}^{n+1} \xi_{k}\right\}}\left(\mathbb{1}_{2 \mathbb{Z}}(n) V_{n}+\mathbb{1}_{2 \mathbb{Z}+1}(n) U_{n}\right),
$$

where we use the convention that the sum over an empty index set is $-\infty$. Similarly, when starting in $S_{|x|^{2}}^{2}$, we define $X_{t}^{t_{0}}$ analogously to the display above but with the roles of odd and even integers reversed. It is straightforward to show that this process is a Feller process and thus has the strong Markov property.

By Lemma B.1, which we postpone to the Appendix, $X^{t_{0}}$ mimics $\left(\mu_{t}\right)_{t \in\left[t_{0}, 1\right]}$ and there exists a process $X$ with the property that, for any $t_{1} \in(0,1]$,

$$
\begin{equation*}
\left(X_{t}\right)_{t \in\left[t_{1}, 1\right]} \sim\left(X_{t}^{t_{1}}\right)_{t \in\left[t_{1}, 1\right]} . \tag{4.3}
\end{equation*}
$$

Hence $X$ mimics $\mu$. We deduce from (4.3) and the strong Markov property of $X^{t_{1}}$ that $X$ has the strong Markov property for stopping times $\tau$ with $\tau \geq t_{1}$ and $t_{1}>0$. By Lemma B.2, we also have that $X$ is Markovian at time 0 . Hence, by Lemma A. 2 below, $X$ is a strong Markov process.
4.2. The general case. We now generalize the example given in Proposition 4.2 to find a peacock for which there is no mimicking Markov martingale, even if we allow for jumps. We construct such a peacock by modifying the previous example in the following way. Let us partition the time interval $[0,1]$ into the intervals $I_{1}:=\bigcup_{\substack{n \in \mathbb{N} \\ n}}\left[2^{-(n+1)}, 2^{-n}\right], I_{2}:=\bigcup_{\substack{n \in \mathbb{N} \\ n \text { odd }}}\left[2^{-(n+1)}, 2^{-n}\right]$. Introduce the notation $n_{t}:=\min \left\{n \in \mathbb{N}: t \geq 2^{-n}\right\}$ and define the functions

$$
a_{1}(t)=\int_{0}^{t} \mathbb{1}_{I_{1}}(s) \mathrm{d} s, \quad a_{2}(t)=\int_{0}^{t} \mathbb{1}_{I_{2}}(s) \mathrm{d} s
$$

Now time-change the processes $M^{1}, M^{2}$ from (4.2) to define a process $X$ by

$$
X_{t}=\left\{\begin{array}{ll}
\left(M_{a_{1}(t)}^{1}, 0\right), & \xi=0,  \tag{4.4}\\
\left(0, M_{a_{2}(t)}^{2}\right), & \xi=1,
\end{array} \quad t \in[0,1],\right.
$$

and write $\mu_{t}=\operatorname{Law}\left(X_{t}\right)$. Then, at time $t \in[0,1], \mu_{t}$ is the uniform measure on $S_{a_{1}(t)}^{1} \cup S_{a_{2}(t)}^{2} \subset$ $\mathbb{R}^{4}$. Note that, for $t \in I_{1}$, the radius of $S_{a_{1}(t)}^{1}$ is increasing deterministically at rate $\sqrt{t}$, while the radius of $S_{a_{2}(t)}^{2}$ remains constant, with the roles reversed on the set of times $I_{2}$.

We will show that there is no Markov martingale mimicking $\mu$.
Proposition 4.4. There exists a peacock $\mu$ on $\mathbb{R}^{4}$ such that there does not exist any Markov martingale mimicking $\mu$.

Proof. Let $\mu_{t}=\operatorname{Law}\left(X_{t}\right), t \in[0,1]$, where $X$ is defined by (4.4). Suppose that there exists a Markov martingale $Y$ mimicking $\mu$. As in Proposition 4.2, we will show that $Y$ is not Markovian at time 0 .

Fix $n \in \mathbb{N}$ even and $t_{0} \in\left[2^{-(n+1)}, 2^{-n}\right) \subset I_{1}$. . Then, for all $t \in\left[2^{-(n+1)}, 2^{-n}\right], \mu_{t}$ is supported on $S_{a_{1}(t)}^{1} \cup S_{a_{2}\left(t_{0}\right)}^{2}$. Define a function $f: \mathbb{R}^{4} \rightarrow \mathbb{R}$ by $f(x)=\sqrt{x_{3}^{2}+x_{4}^{2}}$, and note that $f$ is convex. Also note that, for any $t \in[0,1], f(x)=0$ for $x \in S_{a_{1}(t)}^{1}$, and $f(x)=\sqrt{a_{2}(t)}$ for $x \in S_{a_{2}(t)}^{2}$. Let $t \in\left(t_{0}, 2^{-n}\right]$. Then, by convex ordering,

$$
\begin{align*}
& \mathbb{E}\left[f\left(Y_{t}\right) \mid Y_{t_{0}} \in S_{a_{2}\left(t_{0}\right)}^{2}\right] \geq \mathbb{E}\left[f\left(Y_{t_{0}}\right) \mid Y_{t_{0}} \in S_{a_{2}\left(t_{0}\right)}^{2}\right] \geq \sqrt{a_{2}\left(t_{0}\right)}, \\
\text { and } & \mathbb{E}\left[f\left(Y_{t}\right) \mid Y_{t_{0}} \in S_{a_{1}\left(t_{0}\right)}^{1}\right] \geq \mathbb{E}\left[f\left(Y_{t_{0}}\right) \mid Y_{t_{0}} \in S_{a_{1}\left(t_{0}\right)}^{1}\right] \geq 0 . \tag{4.5}
\end{align*}
$$

This implies the lower bound $\mathbb{E}\left[f\left(Y_{t}\right)\right] \geq \sqrt{a_{2}\left(t_{0}\right)}$. We also have

$$
\begin{aligned}
\mathbb{E}\left[f\left(Y_{t}\right)\right] & =\sqrt{a_{2}(t)} \mathbb{P}\left[Y_{t} \in S_{a_{2}(t)}^{2}\right] \\
& =\sqrt{a_{2}\left(t_{0}\right)} \mathbb{P}\left[Y_{t_{0}} \in S_{a_{2}\left(t_{0}\right)}^{2}\right]=\mathbb{E}\left[f\left(Y_{t_{0}}\right)\right] \leq \sqrt{a_{2}\left(t_{0}\right)},
\end{aligned}
$$

where the second equality holds since both $a_{2}(t)$ and $\left.\mu_{t}\right|_{\mathcal{X}^{2}}$ are constant for $t \in\left[2^{-(n+1)}, 2^{-n}\right]$. Hence equality holds in each inequality in (4.5) and, in particular,

$$
\mathbb{P}\left[Y_{t} \in S_{a_{1}\left(t_{0}\right)}^{1} \mid Y_{t_{0}} \in S_{a_{1}\left(t_{0}\right)}^{1}\right]=1
$$

This holds for any subinterval of $I_{1}$, and a symmetric argument applies to $I_{2}$. Since the peacock $\mu$ is weakly continuous, we can suppose that $Y$ has càdlàg paths. Therefore

$$
\mathbb{P}\left[Y_{t} \in S_{a_{1}(t)}^{1}, \forall t \geq t_{0} \mid Y_{t_{0}} \in S_{a_{1}\left(t_{0}\right)}^{1}\right]=1 .
$$

For $t \in(0,1]$, define $A_{t}^{1}:=\left\{Y_{s} \in S_{a_{1}(s)}^{1}, \forall s \geq t\right\}$, and let $A^{1}:=\bigcap_{t>0} A_{t}^{1}$. Then, for each $t \in(0,1], \mathbb{P}\left(A_{t}^{1}\right)=\frac{1}{2}$, and so $\mathbb{P}\left(A^{1}\right)=\frac{1}{2}$. Since $\sigma\left(Y_{0}\right)$ is trivial, $A^{1} \notin \sigma\left(Y_{0}\right)$. On the other hand,
we have $A_{t}^{1} \in \mathcal{F}_{t}$, where $\mathcal{F}$ is the usual augmentation of the filtration generated by $Y$. Hence $A^{1} \in \mathcal{F}_{0}:=\bigcap_{s>0} \mathcal{F}_{s}$.

We conclude as in Proposition 4.2. Observe that, with $f$ defined as above, for any $t \in(0,1)$, we get

$$
\frac{\sqrt{a_{2}(t)}}{2}=\mathbb{E}\left[f\left(Y_{t}\right) \mid \sigma\left(Y_{0}\right)\right] \neq \mathbb{E}\left[f\left(Y_{t}\right) \mid \mathcal{F}_{0}\right]= \begin{cases}0 & \text { on } A^{1} \\ \sqrt{a_{2}(t)} & \text { on } \Omega \backslash A^{1}\end{cases}
$$

Hence $Y$ is not Markovian at time 0 .
4.3. A partially regularized case. In this section, we present a final example, in which we regularize a peacock defined similarly as in Proposition 4.4 by convolving the peacock at time $t \in[0,1]$ with a centered Gaussian with covariance $t \mathrm{id}$. We will show that, even after such regularization, there exists no Markov martingale mimicking this peacock. Therefore, in order to guarantee existence of a mimicking Markov martingale, some further regularization is required, as in Theorem 1.1.

Throughout this section, we use the notation $\gamma^{\sigma}$ for the 4 -dimensional centered Gaussian measure with covariance $\sigma$ id.

Proposition 4.5. There exists a peacock $\mu$ on $\mathbb{R}^{4}$ such that there is no Markov martingale mimicking $\mu^{\mathrm{r}}$, which is defined by $\mu_{t}^{\mathrm{r}}:=\mu_{t} * \gamma^{t}$, for $t \in[0,1]$.

Proof. Let $\nu_{t}=\operatorname{Law}\left(X_{t}\right)$, where $X_{t}$ is defined by (4.4), and define a regularized peacock $\nu^{\mathrm{r}}$ by $\nu_{t}^{\mathrm{r}}:=\nu_{t} * \gamma^{t^{14}}$, for $t \in[0,1]$. Suppose that there exists a martingale $Y$ mimicking $\nu^{\mathrm{r}}$. In the following, we will show that $Y$ can not be Markovian at 0 . Finally, we will time-change $\nu$ in order to find a peacock $\mu$ such that any martingale mimicking $\mu^{\mathrm{r}}:=\left(\mu_{t} * \gamma^{t}\right)_{t \in[0,1]}$ is not Markovian.

Due to the convolution with a Gaussian $\nu_{t}^{\mathrm{r}}$ is no longer concentrated on $S_{a_{1}(t)}^{1} \cup S_{a_{2}(t)}^{2}$, it rather has full support on $\mathbb{R}^{4}$, for all $t \in[0,1]$. Since $Y$ mimics $\nu^{\mathrm{r}}$ we have, for each $t \in[0,1]$, that $\operatorname{Law}\left(Y_{t}\right)=\operatorname{Law}\left(X_{t}+N_{t^{14}}\right)$ where $X_{t} \sim \operatorname{Unif}\left(S_{a_{1}(t)}^{1} \cup S_{a_{2}(t)}^{2}\right)$ and $N_{t^{14}} \sim \mathcal{N}\left(0, t^{14} \mathrm{id}\right)$ are independent. Note that, from the definitions of $a_{1}$ and $a_{2}$, we can find constants $c, C>0$ such that $c t \leq a_{i}(t) \leq C t$, for $t \in[0,1], i=1,2$. We also have the estimate

$$
\mathbb{P}\left[\left|N_{t^{14}}\right| \geq a\right] \leq \frac{t^{14}}{a^{2}}, \quad a>0
$$

Define the events

$$
\overline{\mathcal{S}}_{t}^{i}:=\left\{\exists x \in S_{a_{i}(t)}^{i},\left|x-Y_{t}\right|<t\right\}, \quad \hat{\mathcal{S}}_{t}^{i}:=\left\{\exists x \in S_{a_{i}(t)}^{i},\left|x-Y_{t}\right|<t^{2}\right\},
$$

and write $t_{k}:=2^{-k}$ for $k \in \mathbb{N}$. By Lemma B. 3 we have that for $t \in[0,1], i=1,2$,

$$
\mathbb{P}\left[\overline{\mathcal{S}}_{t_{k}}^{i} \triangle \overline{\mathcal{S}}_{t_{k+1}}^{i}\right] \leq C t_{k},
$$

where $A \triangle B:=(A \backslash B) \cup(B \backslash A)$ denotes the symmetric difference between events $A$ and $B$. Therefore, for $m, n \in \mathbb{N}, m \geq n$,

$$
\begin{aligned}
\mathbb{P}\left[\bigcap_{k=n}^{m} \overline{\mathcal{S}}_{t_{k}}^{1}\right] & =\mathbb{P}\left[\overline{\mathcal{S}}_{t_{m}}^{1} \backslash \bigcup_{k=n}^{m-1} \overline{\mathcal{S}}_{t_{k}}^{1} \triangle \overline{\mathcal{S}}_{t_{k+1}}^{1}\right] \geq \mathbb{P}\left[\overline{\mathcal{S}}_{t_{m}}^{1}\right]-\sum_{k=n}^{m-1} \mathbb{P}\left[\overline{\mathcal{S}}_{t_{k}}^{1} \triangle \overline{\mathcal{S}}_{t_{k+1}}^{1}\right] \\
& \geq \frac{1}{2}-\mathbb{P}\left[\left|N_{t^{14}}\right| \geq t_{m}\right]-C \sum_{k=n}^{m-1} t_{k} \geq \frac{1}{2}-C \sum_{k=n}^{m} t_{k} \xrightarrow{m, n \rightarrow \infty} \frac{1}{2}
\end{aligned}
$$

since the sequence $\left(t_{k}\right)_{k \in \mathbb{N}}$ is summable. On the other hand,

$$
\mathbb{P}\left[\bigcap_{k=n}^{m} \overline{\mathcal{S}}_{t_{k}}^{1}\right] \leq \inf _{n \leq k \leq m} \mathbb{P}\left[\overline{\mathcal{S}}_{t_{k}}^{1}\right] \leq \frac{1}{2}+t_{m} \xrightarrow{m \rightarrow \infty} \frac{1}{2} .
$$

Defining an increasing sequence of events $\mathcal{S}_{n}, n \in \mathbb{N}$, and its limit $\mathcal{S}$ by

$$
\mathcal{S}_{n}:=\bigcap_{k=n}^{\infty} \overline{\mathcal{S}}_{t_{k}}^{1}, \quad \text { and } \quad \mathcal{S}:=\bigcup_{n=1}^{\infty} \mathcal{S}_{n},
$$

we conclude that

$$
\mathbb{P}[\mathcal{S}]=\lim _{n \rightarrow \infty} \mathbb{P}\left[\mathcal{S}_{n}\right]=\frac{1}{2}
$$

Hence $\mathcal{S} \notin \sigma\left(Y_{0}\right)$. However, we have that, for each $k \in \mathbb{N}, \overline{\mathcal{S}}_{t_{k}}^{1} \in \mathcal{F}_{t_{k}}$, and so for $n \in \mathbb{N}, \mathcal{S}_{n} \in \mathcal{F}_{t_{k}}$ for all $k \geq n$. Hence $\mathcal{S} \in \bigcap_{k \geq n} \mathcal{F}_{t_{k}}=\mathcal{F}_{0}$.

Now choose $k \in \mathbb{N}$ sufficiently large that $\mathbb{P}\left[\mathcal{S} \backslash \mathcal{S}_{k}\right] \leq \frac{1}{8}$ and $t_{k} \leq \frac{1}{8}$. Then we have

$$
\mathbb{P}\left[\overline{\mathcal{S}}_{t_{k}}^{1} \mid \mathcal{S}\right]=\mathbb{P}\left[\overline{\mathcal{S}}_{t_{k}}^{1} \cap \mathcal{S}\right] / \mathbb{P}[\mathcal{S}] \geq 2 \mathbb{P}\left[\mathcal{S}_{k} \cap \mathcal{S}\right]=2\left(\mathbb{P}[\mathcal{S}]-\mathbb{P}\left[\mathcal{S} \backslash \mathcal{S}_{k}\right]\right) \geq 2\left(\frac{1}{2}-\frac{1}{8}\right)=\frac{3}{4}
$$

while on the other hand, $\mathbb{P}\left[\overline{\mathcal{S}}_{t_{k}}^{1} \mid \sigma\left(Y_{0}\right)\right] \leq \frac{1}{2}+t_{k} \leq \frac{5}{8}<\frac{3}{4}$. Therefore $Y$ cannot be Markovian at time 0 .

Now define a peacock $\mu$ by a time-change of $\nu$ such that $\mu_{t}:=\nu_{t^{14-1}}$, and define a regularized peacock $\mu^{\mathrm{r}}$ by $\mu_{t}^{\mathrm{r}}:=\mu_{t} * \gamma^{t}, t \in[0,1]$. Then, rescaling time by $t \mapsto t^{14^{-1}}$ in all of the above arguments, we obtain the result that any martingale mimicking $\mu^{\mathrm{r}}$ cannot be Markovian at time 0 .

## 5. Compactness of martingale Itô diffusions

In this section we prove a compactness result for martingale diffusions with respect to convergence in finite dimensional distributions (Theorem 5.2). We applied this result in the proof of Theorem 1.1 in order to pass to a limit when constructing a mimicking martingale diffusion. This parallels the approach of Lowther [27] to the one-dimensional case.

From here on, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a standard $\mathbb{R}^{d}$-valued Brownian motion $B$ is defined. We consider a sequence $\left(\sigma^{k}\right)_{k \in \mathbb{N}}$ of positive semidefinite matrix-valued measurable functions $\sigma^{k}:[0,1] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$. We use the notation $\Sigma^{k}:[0,1] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ to denote the integral

$$
\Sigma_{t}^{k}(x):=\int_{0}^{t}\left(\sigma_{s}^{k}(x)\right)^{2} d s
$$

where $\left(\sigma_{s}^{k}(x)\right)^{2}=\sigma_{s}^{k}(x) \sigma_{s}^{k}(x)=\sigma_{s}^{k}(x) \sigma_{s}^{k}(x)^{\top}$ is defined by matrix multiplication. Moreover, we fix a sequence of initial distributions $\left(\mu_{0}^{k}\right)_{k \in \mathbb{N}}$ and consider, for $t \in[0,1]$, strong solutions of the SDE

$$
d X_{t}^{k}=\sigma_{t}^{k}\left(X_{t}^{k}\right) d B_{t}, \quad \text { where } X_{0}^{k} \sim \mu_{0}^{k} .
$$

Existence and uniqueness of the solution will be guaranteed by Assumptions 5.1 (A1) and (A2) below (see, e.g., Theorem V. 12.1 of [30]). We write $\left(\mu_{t}^{k}\right)_{t \in[0,1]}$ for the marginal distributions of $X^{k}$. In the following we will use combinations of the following assumptions, which were typically satisfied in the setting of the previous sections.

## Assumption 5.1.

(A1) The map $x \mapsto \sigma_{t}^{k}(x)$ is locally Lipschitz continuous, uniformly in $k \in \mathbb{N}$ and $t \in[0,1]$.
(A2) For every $x \in \mathbb{R}^{d}$ the value of $\left\|\sigma_{t}^{k}(x)\right\|$ is bounded, uniformly in $k \in \mathbb{N}$ and $t \in[0,1]$.
(A3) The family of random variables $\left\{\left|X_{1}^{k}\right|^{2}: k \in \mathbb{N}\right\}$ is uniformly integrable.
(A4) The matrix $\left(\sigma_{t}^{k}(x)\right)^{2}$ is positive definite with eigenvalues bounded away from zero, locally in $x \in \mathbb{R}^{d}$, uniformly in $t \in \bigcup_{j=0}^{2^{k}-1}\left[j 2^{-k}, j 2^{-k}+2^{-k-1}\right]$, and $k \in \mathbb{N}$.
(A5) The set of initial distributions $\left\{\mu_{0}^{k}: k \in \mathbb{N}\right\}$ converges to $\mu \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$.
We start by noting that, due to Assumptions 5.1 (A1) and (A2) and the Arzelà-Ascoli theorem, we can assume without loss of generality, by passing to subsequences, that $\left(\sum_{t}^{k}(x)\right)_{k \in \mathbb{N}}$ converges for every $(t, x) \in[0,1] \times \mathbb{R}^{d}$.

Theorem 5.2. Under Assumptions 5.1 (A1)-(A5), suppose that $\left(\Sigma_{t}^{k}(x)\right)_{k \in \mathbb{N}}$ converges pointwise for $(t, x) \in[0,1] \times \mathbb{R}^{d}$ to $\Sigma:[0,1] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$.

Then there exists a function $(t, x) \mapsto \sigma_{t}(x)$ taking values in the set of positive definite $d \times d$ matrices that is locally Lipschitz continuous in $x \in \mathbb{R}^{d}$, uniformly in $t \in[0,1]$, with $\Sigma_{t}(x)=$ $\int_{0}^{t} \sigma_{s}(x)^{2} d s$, such that $\left(X^{k}\right)_{k \in \mathbb{N}}$ converges in finite dimensional distributions to $X$, the unique strong solution of the $S D E d X_{t}=\sigma_{t}\left(X_{t}\right) d B_{t}$.

Moreover, for each $x \in \mathbb{R}^{d}$, there exist constants $c, C>0$ such that $c \mathrm{id} \leq \sigma_{t}(x) \leq C \mathrm{id}$, for $t \in[0,1]$.

As a simple corollary we have the following compactness result.
Corollary 5.3. Under Assumptions 5.1 (A1)-(A5), the set of martingale Itô diffusions $\left\{X^{k}: k \in\right.$ $\mathbb{N}\}$ is precompact w.r.t. convergence in finite dimensional distributions in the set of martingale Itô diffusions.

We start with two auxiliary lemmas.
Lemma 5.4. Under Assumptions 5.1 (A1)-(A3), the sequence of curves $t \mapsto \mu_{t}^{k}, t \in[0,1]$, of marginal distributions of $\left(X^{k}\right)_{k \in \mathbb{N}}$ is equicontinuous in $C\left([0,1], \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)\right)$ if we equip $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ with the $\mathcal{W}_{2}$-metric.

Proof. Since by Assumption 5.1 (A3) the set of terminal distributions is $\mathcal{W}_{2}$-precompact, the set $\left\{\eta \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right): \exists k \in \mathbb{N}\right.$ with $\left.\eta \leq_{c x} \mu_{1}^{k}\right\}$ is also $\mathcal{W}_{2}$-precompact. Applying Doob's maximal $\mathcal{L}^{2}$-inequality, for each $\varepsilon>0$, we can find a ball $B_{R} \subseteq \mathbb{R}^{d}$ of radius $R>0$ such that for all $t \in[0,1]$ and $k \in \mathbb{N}, \mathcal{W}_{2}\left(\mu_{t}^{k}, \mu_{t}^{k, R}\right)<\varepsilon$ where

$$
X_{t}^{k, R}:=X_{\tau^{k} \wedge t}^{k}, \quad \mu_{t}^{k, R}:=\operatorname{Law}\left(X_{t}^{k, R}\right), \quad \tau^{k}:=\inf \left\{s>0: X_{s}^{k} \notin B_{R}\right\} .
$$

Next, we show that the curves $\left(\mu_{t \in[0,1]}^{k, R}\right)_{k \in \mathbb{N}}$ are $\Lambda_{\varepsilon}$-Lipschitz continuous, for some $\Lambda_{\varepsilon}>0$. Indeed, by the Itô isometry, Assumptions 5.1 (A1) and (A2), we get, for $0 \leq t_{0} \leq t_{1} \leq 1, k \in \mathbb{N}$,

$$
\begin{aligned}
\mathcal{W}_{2}^{2}\left(\mu_{t_{0}}^{k, R}, \mu_{t_{1}}^{k, R}\right) & \leq \mathbb{E}\left|X_{\tau^{k} \wedge t_{1}}^{k}-X_{\tau^{k} \wedge t_{0}}^{k}\right|^{2}=\mathbb{E}\left[\int_{t_{0} \wedge \tau^{k}}^{t_{1} \wedge \tau^{k}}\left\|\sigma_{s}^{n}\left(X_{s}^{k}\right)\right\|_{\mathrm{HS}}^{2} d s\right] \\
& \leq 3 \mathbb{E}\left[\int_{t_{0} \wedge \tau^{k}}^{t_{1} \wedge \tau^{k}}\left\|\sigma_{s}^{k}(0)\right\|_{\mathrm{HS}}^{2}+\left\|\sigma_{s}^{k}(0)-\sigma_{s}^{k}\left(X_{t}^{k}\right)\right\|_{\mathrm{HS}}^{2}+\left\|\sigma_{s}^{k}\left(X_{t}^{k}\right)-\sigma_{s}^{k}\left(X_{s}^{k}\right)\right\|_{\mathrm{HS}}^{2}\right] \\
& \leq 3\left(t_{1}-t_{0}\right)\left(C+L_{R}^{2} \mathbb{E}\left|X_{t}^{k}\right|^{2}+L_{R}^{2} \mathbb{E}\left|X_{t_{0}}^{k}-X_{t_{1}}^{k}\right|^{2}\right)
\end{aligned}
$$

where $L_{R}$ denotes the Lipschitz constant on $B_{R}$ provided by Assumption 5.1 (A1). Setting $\Lambda_{\varepsilon}:=3\left(C+2 L_{R}^{2} \sup _{k \in \mathbb{N}} \mathbb{E}\left|X_{1}^{k}\right|^{2}\right)$ we obtain a uniform Lipschitz bound.

Now we can define a modulus of continuity $\omega: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$via

$$
\omega(\delta):=\inf _{\varepsilon>0}\left\{2 \varepsilon+\sqrt{\Lambda_{\varepsilon} \delta}\right\},
$$

which is, as an infimum over concave functions, a concave function that vanishes at 0 . We find by the triangle inequality that, for $0 \leq t_{0} \leq t_{1} \leq 1$,

$$
\mathcal{W}_{2}\left(\mu_{t_{0}}^{k}, \mu_{t_{1}}^{k}\right) \leq \inf _{R>0}\left\{\mathcal{W}_{2}\left(\mu_{t_{0}}^{k}, \mu_{t_{0}}^{k, R}\right)+\mathcal{W}_{2}\left(\mu_{t_{0}}^{k, R}, \mu_{t_{1}}^{k, R}\right)+\mathcal{W}_{2}\left(\mu_{t_{1}}^{k, R}, \mu_{t_{1}}^{k}\right)\right\} \leq \omega\left(t_{1}-t_{0}\right),
$$

whence $\omega$ is in fact a modulus of continuity for the sequence $\left(\mu_{t \in[0,1]}^{k}\right)_{k \in \mathbb{N}}$. Hence the sequence is equicontinuous.

For the following lemma, compare to Beiglböck, Huesmann, Stebegg [3, Theorem 1].
Lemma 5.5. Let $\Lambda$ be a $\mathcal{W}_{2}$-compact subset of $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ and denote by $\mathcal{M}(\Lambda)$ the set of probability measures $\pi$ on the Skorokhod space $\mathcal{D}\left(\mathbb{R}^{d}\right)$ consisting of

$$
\mathcal{M}(\Lambda):=\left\{\pi=\operatorname{Law}(M):\left(M_{t}\right)_{t \in[0,1]} \text { is a càdlàg martingale with } \operatorname{Law}\left(M_{1}\right) \in \Lambda\right\} .
$$

Then, for any sequence $\left(\pi^{k}\right)_{k \in \mathbb{N}}$ in $\mathcal{M}(\Lambda)$ there exists $\pi \in \mathcal{M}(\Lambda)$ and a subsequence $\left(\pi^{k_{j}}\right)_{j \in \mathbb{N}}$ that converges to $\pi=\operatorname{Law}(M)$ in finite dimensional distributions on the set of continuity points w.r.t. the weak topology of the function $t \mapsto \operatorname{Law}\left(M_{t}\right)$.

Proof. Let $\left(\pi^{k}\right)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{M}(\Lambda)$ and write $M^{k}$ for a martingale with law $\pi^{k}$. Since $\Lambda$ is compact, it follows easily that $\left\{\mu \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right): \exists \nu \in \Lambda\right.$ with $\left.\mu \leq_{c} \nu\right\}$ is also compact. Therefore we find a subsequence $\left(\pi^{k_{j}}\right)_{j \in \mathbb{N}}$ such that, for any finite subset $S \subseteq[0,1] \cap \mathbb{Q}$,

$$
\operatorname{Law}\left(M_{t}^{k_{j}}\right)_{t \in S} \rightarrow \tilde{\pi}^{S} \text { weakly for } j \rightarrow \infty
$$

where $\tilde{\pi}^{S}$ is the law of a discrete-time martingale in $|S|$ time steps with values in $\mathbb{R}^{d}$. The family $\left(\tilde{\pi}^{S}\right)_{S \subset[0,1] \cap \mathbb{Q},|S|<\infty}$ is a consistent family and we can apply Kolmogorov's extension theorem to obtain a probability $\tilde{\pi}$ on $\prod_{t \in[0,1] \cap \mathbb{Q}} \mathbb{R}^{d}$. Note that, for any $S \subseteq[0,1] \cap \mathbb{Q}$, the projection of $\tilde{\pi}$ onto the $S$-coordinates coincides with $\tilde{\pi}^{S}$. Hence $\tilde{\pi}$ is the law of a martingale $\tilde{M}=\left(\tilde{M}_{t}\right)_{t \in[0,1] \cap \mathbb{Q}}$ with terminal distribution $\operatorname{Law}\left(\tilde{M}_{1}\right) \in \Lambda$. By standard arguments, there exists $M$ where $M_{t}:=$ $\lim _{q \backslash t, q \in \mathbb{Q} \cap[0,1]} \tilde{M}_{q}$ for $t \in[0,1]$ which is a càdlàg martingale (in the right-continuous version of the filtration). We claim that $\pi:=\operatorname{Law}(M)$ has the desired properties.

As $t \mapsto \operatorname{Var}\left(M_{t}\right)$ is non-decreasing there are at most countably many points of discontinuity. Let $S$ be a finite subset of the continuity points of $t \mapsto \operatorname{Var}\left(M_{t}\right)$, which coincide with the continuity points of $t \mapsto \operatorname{Law}\left(M_{t}\right)$. Fix $N \in \mathbb{N}$ and note that, as all involved processes are martingales, $\left(M^{k_{j}}\right)_{t \in \tilde{S}}$ converges for $j \rightarrow \infty$ in $\mathcal{W}_{2}$ to $\left(\tilde{M}_{t}\right)_{t \in \tilde{S}}$ uniformly for all $\tilde{S} \subseteq[0,1] \cap \mathbb{Q}$ with $|\tilde{S}| \leq N$. Moreover, by Doob's martingale convergence theorem, we have, for any $t \in S$, that $\lim _{q \backslash t, \in[0,1] \cap \mathbb{Q}} \tilde{M}_{q}$ almost surely. We conclude that $\left(M_{t}^{k_{j}}\right)_{t \in S}$ converges in $\mathcal{W}_{2}$ to $\left(M_{t}\right)_{t \in S}$.
Proposition 5.6. In addition to Assumptions 5.1 (A1)-(A5), suppose that there exists $R>0$ such that $\left(\sigma^{k}\right)_{k \in \mathbb{N}}$ satisfies $\sigma^{k}(x)=\sigma^{k}\left(\frac{R x}{|x| \vee R}\right)$ for $x \in \mathbb{R}^{d}, k \in \mathbb{N}$, and that $\left(\Sigma_{t}^{k}(x)\right)_{k \in \mathbb{N}}$ converges pointwise for $(t, x) \in[0,1] \times \mathbb{R}^{d}$ to $\Sigma:[0,1] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$.

Then the conclusion of Theorem 5.2 holds.
We break the proof of Proposition 5.6 into the following lemmas.
Lemma 5.7. In the setting of Proposition 5.6, there exists $\sigma:[0,1] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$, with $\Sigma_{t}(x)=$ $\int_{0}^{t} \sigma_{s}(x)^{2} \mathrm{~d} s$, such that $x \mapsto \sigma_{t}(x)$ is locally Lipschitz continuous, uniformly in $t \in[0,1]$, and for fixed $x \in \mathbb{R}^{d}$, there exist constants $c_{x}, C_{x}>0$ such that $c_{x} \mathrm{id} \leq \sigma_{t}(x) \leq C_{x} \mathrm{id}$, for $t \in[0,1]$.
Proof. As the limit of Lipschitz functions, $t \mapsto \Sigma_{t}(x)$ is Lipschitz continuous on the ball $B_{R}:=$ $\left\{x \in \mathbb{R}^{d}:|x|<R\right\}$ of radius $R>0$, and so are the entries $\left(\Sigma^{i, j}\right)_{i, j=1}^{d}$ of $\Sigma$. Therefore there exist densities

$$
\rho_{t}(x):=\left(\rho_{t}^{i, j}(x)\right)_{i, j=1}^{d} \in \mathbb{R}^{d \times d}, \text { where } \int_{t_{0}}^{t_{1}} \rho_{t}(x) d t=\Sigma_{t_{1}}(x)-\Sigma_{t_{0}}(x), \quad x \in \mathbb{R}^{d}
$$

We define $\sigma$ as the matrix square root of $\rho$, which is possible as $\rho$ is a.s. positive semidefinite.
Next, we define, Lipschtitz norm of a function $g:[0,1] \times B_{R} \rightarrow \mathbb{R}^{d \times d}$ as

$$
F(g):=\operatorname{esssup}_{x, y \in B_{R}, t \in[0,1]} \frac{\left\|g_{t}(x)-g_{t}(y)\right\|_{\mathrm{HS}}}{|x-y|}
$$

where the essential supremum is taken w.r.t. $d t \otimes d x$. Since $F: L^{2}\left([0,1] \times B_{R} ; \mathbb{R}^{d \times d}\right) \rightarrow \mathbb{R}_{+} \cup\{\infty\}$ is lower semicontinuous and convex, we have by [2, Theorem 9.1] that $F$ is weakly lower semicontinuous, and in particular

$$
\liminf _{j \rightarrow \infty} F\left(\left(\sigma^{k_{j}}\right)^{2}\right) \geq F(\rho)=: L
$$

which implies that $x \mapsto \rho_{t}(x)$ is $d t \otimes d x$-almost everywhere $L$-Lipschitz continuous. Thus, by choosing a suitable $\mathcal{L}^{2}$-representative of $\rho$ we can assume w.l.o.g. that $x \mapsto \rho_{t}(x)$ is, for every $t \in$
$[0,1], L$-Lipschitz continuous in $x$, and that $\sup _{t \in[0,1]}\left\|\rho_{t}(x)\right\|_{\text {HS }}<\infty$. By Lipschitz continuity of $t \mapsto \Sigma_{t}^{k}(x)$ and $x \mapsto \rho_{t}(x)$, we have that, for every $x \in B_{R}, \xi \in \mathbb{R}^{d}$ and $0 \leq t_{0} \leq t_{1} \leq 1$,

$$
\int_{t_{0}}^{t_{1}} \xi^{\mathrm{T}} \rho_{t}(x) \xi d t=\lim _{k \rightarrow \infty} \xi^{\mathrm{T}}\left(\Sigma_{t_{1}}^{k}(x)-\Sigma_{t_{0}}^{k}(x)\right) \xi
$$

Therefore, by Assumption 5.1 (A4), there exists a constant $c>0$ such that $\rho_{t}(x) \geq c$ id for every $x \in B_{R}$ and Lebesgue-almost every $t \in[0,1]$. As the matrix square root is Lipschitz continuous on the set of symmetric, positive matrices $M$ such that $M \geq c$ id, we deduce Lipschitz continuity of $x \mapsto \sigma_{t}(x):=\sqrt{\rho_{t}(x)}$ on $B_{R}$.

The final result that we will make use of in the proof of Proposition 5.6 is the Lipschitz continuity of the $\mathcal{W}_{2}$-distance between Gaussian laws with respect to covariance matrices.

Lemma 5.8. Let $C, \delta>0$. Then the map

$$
\left(\sigma, \sigma^{\prime}\right) \mapsto \mathcal{W}_{2}\left(\mathcal{N}(0, \sigma), \mathcal{N}\left(0, \sigma^{\prime}\right)\right)
$$

is Lipschitz continuous on the set $\left\{\sigma \in \mathbb{R}^{d \times d} \text { symmetric: } \delta \mathrm{id} \leq \sigma \leq C \mathrm{id}\right\}^{2}$ equipped with the product of the Hilbert-Schmidt norm.

Proof. As shown in [12, Proposition 7], the Wasserstein-2-distance between two centered Gaussians with covariance matrices $\sigma, \sigma^{\prime}$ is explicitly given by

$$
\operatorname{tr}\left(\sigma+\sigma^{\prime}-2\left(\sigma^{\frac{1}{2}} \sigma^{\prime} \sigma^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)
$$

from which the assertion follows.
Proof of Proposition 5.6. By Lemma 5.4 there exists a subsequence, still denoted by $\left(X^{k}\right)_{k \in \mathbb{N}}$, such that the curves $\left(\mu_{t}^{k}\right)_{t \in[0,1]}, k \in \mathbb{N}$ converge to a $\mathcal{W}_{2}$-continuous curve $\left(\mu_{t}\right)_{t \in[0,1]}$. After a deterministic time-change, if necessary, we can assume w.l.o.g. that $t \mapsto \int|x|^{2} \mu_{t}(d x)$ is 1-Lipschitz. By Lemma 5.7, we can find a diffusion coefficient $\sigma=\sigma_{t}(x)$, with the desired properties. Combining Lemma 5.7 with the particular form of the $\left(\sigma^{k}\right)_{k \in \mathbb{N}}$, we see that $x \mapsto$ $\sigma_{t}(x)$ is in fact Lipschitz continuous, uniformly in $t \in[0,1]$. We write $L$ for a suitable Lipschitz constant and $C$ for a uniform bound of the Hilbert-Schmidt norms of $\sigma_{t}^{k}, k \in \mathbb{N}$. For $m \in \mathbb{N}$ and $k=0, \ldots, 2^{m}$, we define $t_{k}^{m}:=k 2^{-m}$ and consider the kernels

$$
\begin{aligned}
& \pi_{m, k}^{n}(x):=\operatorname{Law}\left(X_{t_{k+1}^{m}}^{n} \mid X_{t_{k}^{m}}^{n}=x\right), \quad d X_{t}^{n}=\sigma_{t}^{n}\left(X_{t}^{n}\right) d B_{t}, \quad X_{t_{k}^{m}}^{n}=x, \\
& \bar{\pi}_{m, k}^{n}(x):=\operatorname{Law}\left(Y_{t_{k+1}^{m}}^{n} \mid Y_{t_{k}^{m}}^{n}=x\right), \quad d Y_{t}^{n}=\sigma_{t}^{n}(x) d B_{t}, \quad Y_{t_{k}^{m}}^{n}=x .
\end{aligned}
$$

Observe that

$$
\begin{equation*}
\bar{\pi}_{m, k}^{n}(x)=\mathcal{N}\left(x, \int_{t_{k}^{m}}^{t_{k+1}^{m}}\left(\sigma_{t}^{n}(x)\right)^{2} d t\right)=\mathcal{N}\left(x, \sum_{t_{k+1}^{m}}^{n}(x)-\sum_{t_{k}^{m}}^{n}(x)\right) . \tag{5.1}
\end{equation*}
$$

In a similar manner we define $\pi_{m, k}, X, \bar{\pi}_{m, k}$, and $Y$. Since $\left(\Sigma^{n}\right)_{n \in \mathbb{N}}$ converges uniformly on $B_{R}$ and therefore on $\mathbb{R}^{d}$ to $\Sigma$, we find by (5.1) and Lemma 5.8 that

$$
\lim _{n \rightarrow \infty} \bar{\pi}_{m, k}^{n}(x)=\bar{\pi}_{m, k}(x) \quad \text { uniformly in } x \in \mathbb{R}^{d} \text {, uniformly in } m \text { and } k .
$$

Combining this with the bound $\left\|\Sigma_{t}\right\|_{\text {HS }} \leq C$ gives

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sum_{k=0}^{2^{m}-1} \mathbb{E}\left[\mathcal{W}_{2}^{2}\left(\bar{\pi}_{m, k}^{n}\left(X_{t_{k}^{m}}^{n}\right), \bar{\pi}_{m, k}\left(X_{t_{k}^{m}}^{n}\right)\right)\right]=0 \tag{5.2}
\end{equation*}
$$

In the following we choose $n:=n(m) \in \mathbb{N}, n(m) \geq m$ sufficiently large such that for this particular $n$ the sum in (5.2) is smaller than $2^{-m}$. We estimate

$$
\begin{aligned}
\mathcal{W}_{2}^{2}\left(\pi_{m, k}^{n}(x), \bar{\pi}_{m, k}^{n}(x)\right) & \leq \mathbb{E}\left[\left|X_{t_{k+1}^{m}}^{n}-Y_{t_{k+1}^{m}}^{n}\right|^{2} \mid X_{t_{k}^{m}}^{n}=x=Y_{t_{k}^{m}}^{n}\right] \\
& =\mathbb{E}\left[\int_{t_{k}^{m}}^{t_{k+1}^{m}}\left|\sigma_{t}^{n}\left(X_{s}^{n}\right)-\sigma_{t}^{n}(x)\right|^{2} d s \mid X_{t_{k}^{m}}^{n}=x\right] \\
& \leq L^{2} \mathbb{E}\left[\int_{t_{k}^{m}}^{t_{k+1}^{m}}\left|X_{s}^{n}-X_{t_{k}^{m}}^{n}\right|^{2} d s \mid X_{t_{k}^{m}}^{n}=x\right]
\end{aligned}
$$

using the Itô isometry and the Lipschitz continuity of $\sigma^{n}$. Now, since $X^{n}$ is a square-integrable martingale, we have

$$
\begin{align*}
\mathcal{W}_{2}^{2}\left(\pi_{m, k}^{n}(x), \bar{\pi}_{m, k}^{n}(x)\right) & \leq L^{2} \mathbb{E}\left[\int_{t_{k}^{m}}^{t_{k+1}^{m}}\left|X_{t_{k+1}^{m}}^{n}-X_{t_{k}^{m}}^{n}\right|^{2} d s \mid X_{t_{k}^{m}}^{n}=x\right] \\
& =\frac{L^{2}}{2^{m}} \mathbb{E}\left[\left|X_{t_{k+1}^{m}}^{n}-X_{t_{k}^{m}}^{n}\right|^{2} \mid X_{t_{k}^{m}}^{n}=x\right] \\
& =\frac{L^{2}}{2^{m}} \mathbb{E}\left[\int_{t_{k}^{m}}^{t_{k+1}^{m}}\left\|\sigma_{s}^{n}\left(X_{s}^{n}\right)\right\|_{\text {HS }}^{2} d s \mid X_{t_{k}^{m}}^{n}=x\right] \\
& \leq \frac{L^{2} C^{2}}{2^{2 m}}, \tag{5.3}
\end{align*}
$$

where we use the Itô isometry again, as well as the bound on the norm of $\sigma^{n}$. By the same line of reasoning we find that $\mathcal{W}_{2}^{2}\left(\pi_{m, k}(x), \bar{\pi}_{m, k}(x)\right)$ admits the very same bound as in (5.3). Hence the triangle inequality together with the above estimates yields

$$
\begin{equation*}
\sum_{k=0}^{2^{m}-1} \mathbb{E}\left[\mathcal{W}_{2}^{2}\left(\pi_{m, k}^{n(m)}\left(X_{t_{k}^{m}}^{n(m)}\right), \pi_{m, k}\left(X_{t_{k}^{m}}^{n(m)}\right)\right)\right] \leq \frac{L^{2} C^{2}+1}{2^{m-2}} \tag{5.4}
\end{equation*}
$$

Next, we define for every $m \in \mathbb{N}$ an auxiliary process $S^{m}$ that has càdlàg paths and the same marginals as $X^{n(m)}$ at the $m$-dyadics, where $n(m)$ is fixed after (5.2). For $k=0, \ldots, 2^{m}-1$, the process is given by

$$
S_{0}^{m}=X_{0}^{n(m)}, \quad d S_{t}^{m}=\sigma_{t}\left(S_{t}^{m}\right) d B_{t}, \quad S_{t_{k+1}^{m}}^{m}=T_{k}^{m}\left(S_{t_{m, k+1-}}^{m}\right),
$$

where $T_{k}^{m}$ is the $\mathcal{W}_{2}$-optimal map between $\pi_{m, k}\left(S_{t_{k}^{m}}^{m}\right)$ and $\pi_{m, k}^{n}\left(S_{t_{k}^{m}}^{m}\right)$. The discrete-time jump process $Z_{t}^{m}:=\sum_{l=1}^{2^{m}} \mathbb{1}_{[0, t]}\left(t_{m, l}\right)\left(S_{t_{m, l}}^{m}-S_{t_{m, l-}}^{m}\right)$ is a martingale in the underlying filtration and $\tilde{S}^{m}:=S^{m}-Z^{m}$ is a continuous martingale. Indeed

$$
\begin{aligned}
\mathbb{E}\left[Z_{\hat{t}}^{m} \mid \mathcal{F}_{t}\right] & =Z_{t}^{m}+\sum_{l=1}^{2^{m}} \mathbb{1}_{(t, \hat{t}]}\left(t_{m, l}\right) \mathbb{E}\left[S_{t_{m, l}}^{m}-S_{t_{m, l-}}^{m} \mid \mathcal{F}_{t}\right] \\
& =Z_{t}^{m}+\sum_{l=1}^{2^{m}} \mathbb{1}_{(t, \hat{t}]}\left(t_{m, l}\right) \mathbb{E}\left[S_{t_{m, l-1}}^{m}-S_{t_{m, l-1}}^{m} \mid \mathcal{F}_{t}\right]=Z_{t}^{m} .
\end{aligned}
$$

Moreover, by (5.4), $Z^{m}$ admits the following estimate

$$
\mathbb{E}\left[\left|Z_{1}^{m}\right|^{2}\right]^{\frac{1}{2}}=\mathbb{E}\left[\sum_{k=0}^{2^{m}-1} \mathcal{W}_{2}^{2}\left(\pi_{m, k}\left(X_{t_{k}^{m}}^{n}\right), \pi_{m, k}^{n}\left(X_{t_{k}^{m}}^{n}\right)\right)\right]^{\frac{1}{2}} \leq \frac{\sqrt{L^{2} C^{2}+1}}{2^{m / 2}}
$$

whence, by Doob's maximal inequality, the term $\mathbb{E}\left[\sup _{t \in[0,1]}\left|Z_{t}^{m}\right|\right]$ also vanishes as $m \rightarrow \infty$. We claim that $\left(\tilde{S}^{m}\right)_{m \in \mathbb{N}}$ is a Cauchy sequence. Indeed, for $m, \hat{m} \in \mathbb{N}, \hat{m} \geq m$ we have

$$
\begin{aligned}
\mathbb{E}\left[\left|\tilde{S}_{t}^{m}-\tilde{S}_{t}^{\hat{m}}\right|^{2}\right] & \leq 2\left(\mathbb{E}\left[\left|X_{0}^{n(m)}-X_{0}^{n(\hat{m})}\right|^{2}\right]+\mathbb{E}\left[\int_{0}^{t}\left|\sigma_{t}\left(S_{\hat{t}}^{m}\right)-\sigma_{t}\left(S_{\hat{t}}^{\hat{m}}\right)\right|^{2} d \hat{t}\right]\right) \\
& \leq 2\left(\mathbb{E}\left[\left|X_{0}^{n(m)}-X_{0}^{n(\hat{m})}\right|^{2}\right]+L^{2} \mathbb{E}\left[\int_{0}^{t}\left|S_{\hat{t}}^{m}-S_{\hat{t}}^{\hat{m}}\right|^{2} d \hat{t}\right]\right) \\
& \leq 2\left(\mathbb{E}\left[\left|X_{0}^{n(m)}-X_{0}^{n(\hat{m})}\right|^{2}\right]+3 L^{2}\left(\mathbb{E}\left[\int_{0}^{t}\left|\tilde{S}_{\hat{t}}^{m}-\tilde{S}_{\hat{t}}^{\hat{m}}\right|^{2} d \hat{t}\right]+\frac{L^{2} C^{2}+1}{2^{m-3}}\right)\right)
\end{aligned}
$$

By Grönwall's lemma, we have that $\left(\tilde{S}_{1}^{m}\right)_{m \in \mathbb{N}}$ is an $\mathcal{L}^{2}$-Cauchy sequence. Thus, there exists a continuous $\mathcal{L}^{2}$-martingale $S$ such that $\left(\tilde{S}_{1}^{m}\right)_{m \in \mathbb{N}}$ converges in $\mathcal{L}^{2}$ to $S_{1}$. As $Z^{m}$ vanishes uniformly, we get

$$
\lim _{m \rightarrow \infty} \mathbb{E}\left[\sup _{t \in[0,1]}\left|S_{t}^{m}-S_{t}\right|^{2}\right]=0
$$

Since $S_{t}^{m} \sim \mu_{t}$ for $t \in\left\{0,2^{-m}, \ldots, 1\right\}$, we have by continuity of $t \mapsto \mu_{t}$ that $S_{t} \sim \mu_{t}$ for every $t \in[0,1]$. By $L$-Lipschitz continuity of $x \mapsto \sigma_{t}(x)$ we find that

$$
\lim _{m \rightarrow \infty} \mathbb{E}\left[\int_{0}^{1}\left|\sigma_{t}\left(S_{t}^{m}\right)-\sigma_{t}\left(S_{t}\right)\right|^{2} d t\right]=0
$$

This means that $\left(\sigma \circ S_{t}^{m}\right)_{m \in \mathbb{N}}$ converges to $\sigma \circ S_{t}$ in $L^{2}\left(d t \otimes \mathbb{P} ; \mathbb{R}^{d \times d}\right)$, which yields by the Itô isometry that

$$
\int \sigma_{t}\left(S_{t}\right) d B_{t}=\lim _{m \rightarrow \infty} \int \sigma_{t}\left(S_{t}^{m}\right) d B_{t}=\lim _{m \rightarrow \infty} \tilde{S}^{m}=S
$$

whence $d S_{t}=\sigma_{t}\left(S_{t}\right) d B_{t}$.
In order to prove convergence of $\left(X^{n}\right)_{n \in \mathbb{N}}$ in finite dimensional distributions, it suffices to show for every $\hat{m} \in \mathbb{N}$ that $\left(X_{0}^{n}, X_{t_{\hat{m}, 1}}^{n}, \ldots, X_{1}^{n}\right)_{n \in \mathbb{N}}$ converges in distribution to $\left(S_{0}, S_{t_{\hat{m}, 1}}, \ldots, S_{1}\right)$. Fix $\varepsilon>0$ and $\hat{m} \in \mathbb{N}$. Then there exists $M \geq \hat{m}$ such that for all $m \geq M$

$$
\mathbb{E}\left[\sup _{t \in[0,1]}\left|S_{t}^{m}-S_{t}\right|\right]<\varepsilon .
$$

Recall that by construction we have $\left(S_{0}^{m}, S_{t_{\hat{m}, 1}}^{m}, \ldots, S_{1}^{m}\right)=\left(X_{0}^{n(m)}, X_{t_{\hat{m}, 1}}^{n(m)}, \ldots, X_{1}^{n(m)}\right)$ in law, from which we deduce that $\left(X^{n(m)}\right)_{m \in \mathbb{N}}$ converges in finite dimensional distributions to $S$.

Finally, to see uniqueness of the limit, note that by Lemma 5.5 the sequence $\left(\operatorname{Law}\left(X^{n}\right)\right)_{n \in \mathbb{N}}$ is precompact w.r.t. f.d.d. convergence. By the first part of the proof any subsequence of $\left(X^{n}\right)_{n \in \mathbb{N}}$ admits an f.d.d.-convergent subsequence with limit $S$. Hence, we conclude that Law $(S)$ is the unique limit by recalling that the finite dimensional distributions separate points on $\mathcal{P}\left(\mathcal{D}\left([0,1] ; \mathbb{R}^{d}\right)\right)$.

Having established Proposition 5.6, we now extend the result from compact subsets to the whole of $\mathbb{R}^{d}$ in order to complete the proof of Theorem 5.2. For this step, we require the following lemma on domination of stochastic integrals in convex order by Brownian motion $B$.
Lemma 5.9. Let $\sigma$ be a process adapted to $\mathcal{F}^{B}$, and let $\left(X_{t}\right)_{t \in[0,1]}$ be defined by the stochastic integral $X_{t}=\int_{0}^{t} \sigma_{s} d B_{s}, t \in[0,1]$. Let $C>0$ and suppose that $\sigma_{t} \leq C \mathrm{id}$, for all $t \in[0,1]$. Then $\operatorname{Law}\left(X_{1}\right) \preceq \operatorname{Law}\left(B_{C}\right)=\mathcal{N}(0, C \mathrm{id})$.
Proof of Lemma 5.9. The statement is obvious for simple $\sigma$. Noting that convex order can be checked against convex functions with linear growth, the assertion follows for general $\sigma$ by a limiting argument.

Proof of Theorem 5.2. In order to apply Proposition 5.6, we will define for any radius $R>0$ the diffusion coefficients $\left(\sigma^{n, R}\right)_{n \in \mathbb{N}}$ by

$$
\sigma_{t}^{n, R}(x):= \begin{cases}\sigma_{t}^{n}(x) & |x| \leq R, \\ \sigma_{t}^{n}\left(R \frac{x}{|x|}\right) & \text { else } .\end{cases}
$$

Since in Hilbert spaces projections onto convex sets are contractions, we find that $x \mapsto \sigma_{t}^{n, R}(x)$ is both Lipschitz continuous and bounded, uniformly in $(t, n) \in[0,1] \times \mathbb{N}$. Denote by $X^{n, R}$ the unique strong solution of the SDE $d X_{t}^{n, R}=\sigma_{t}^{n, R}\left(X_{t}^{n, R}\right) d B_{t}$ with initial condition $X^{n, R} \sim \mu_{0}^{n}$. By Lemma 5.9, $\left\{\mu_{1}^{n, R}: n \in \mathbb{N}\right\}$ is a precompact subset of $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. We can apply Proposition 5.6, which yields a diffusion coefficient $\sigma^{R}=\sigma_{t}^{R}(x)$ such that $\left(X^{n, R}\right)_{n \in \mathbb{N}}$ converges in finite dimensional distributions to $X^{R}$, the unique strong solution of the $\operatorname{SDE} d X^{R}=\sigma_{t}^{R}\left(X_{t}^{R}\right) d B_{t}$. Observe that, for $R^{\prime} \geq R$, the corresponding diffusion coefficients are compatible, in the sense that

$$
\sigma_{t}^{R}(x)=\sigma_{t}^{R^{\prime}}(x) \quad \text { for every } x \in B_{R} \text { and almost every } t \in[0,1] .
$$

Defining $\sigma_{t}(x):=\sum_{R=0}^{\infty} \mathbb{1}_{[R, R+1)}(|x|) \sigma_{t}^{R}(x)$, the properties of $\sigma^{R}$ given by Proposition 5.6 imply that $\sigma$ is locally Lipschitz continuous, uniformly in $t \in[0,1]$, and that, for each $x \in \mathbb{R}^{d}$, there exist constants $c, C>0$ such that $c \mathrm{id} \leq \sigma_{t}(x) \leq C$ id, for $t \in[0,1]$.

As a consequence of uniqueness of the solutions of the $\operatorname{SDE} d X^{R}=\sigma_{t}^{R}\left(X_{t}^{R}\right) d B_{t}$ and $d X_{t}=$ $\sigma_{t}\left(X_{t}\right) d B_{t}$, we have that

$$
\begin{equation*}
X_{t \wedge \tau^{R}}^{R}=X_{t \wedge \tau^{R}} \quad \forall t \in[0,1], \tag{5.5}
\end{equation*}
$$

where $\tau^{R}:=\inf \left\{t>0:\left|X_{t}^{R}\right| \geq R\right\}$. An analogous relation holds between $X^{n, R}$ and $X^{n}$, where $\tau^{n, R}:=\inf \left\{t>0:\left|X_{t}^{n, R}\right| \geq R\right\}$. By the optional stopping theorem we therefore get $\mathbb{E}\left[\left|X_{\tau^{n, R}}^{n, R}\right|^{2}\right] \leq \mathbb{E}\left[\left|X_{1}^{n}\right|^{2}\right]$ and, by Assumption 5.1 (A3), $C:=\sup _{n \in \mathbb{N}} \mathbb{E}\left[\left|X_{1}^{n}\right|^{2}\right]<\infty$.

Using (5.5) and applying Doob's martingale inequality yields

$$
\begin{equation*}
\mathbb{E}\left[\sup _{t \in[0,1]}\left|X_{t}^{n, R}-X_{t}^{n}\right| \wedge 1\right] \leq \mathbb{P}\left(\sup _{t \in[0,1]}\left|X_{t}^{n}\right| \geq R\right) \leq \frac{C}{R^{2}} \tag{5.6}
\end{equation*}
$$

Moreover, we have

$$
\begin{aligned}
\mathbb{P}\left(\sup _{t \in[0,1]}\left|X_{t}\right|>R\right) & =\mathbb{P}\left(\sup _{t \in[0,1]}\left|X_{t}^{R}\right|>R\right)=\sup _{I \subseteq \mathbb{Q} \cap[0,1]} \mathbb{P}\left(\sup _{t \in I}\left|X_{t}^{R}\right|>R\right) \\
& \leq \sup _{I \subseteq \mathbb{Q} n[0,1]} \sup _{n \in \mathbb{N}} \mathbb{P}\left(\sup _{t \in I}\left|X_{t}^{n, R}\right|>R\right) \leq \frac{C}{R^{2}},
\end{aligned}
$$

where we used (5.5) for the first equality, continuity of the paths of $X^{R}$ for the second equality and f.d.d. convergence of $\left(X^{n, R}\right)_{n \in \mathbb{N}}$ to $X^{R}$ in the first inequality. Therefore, similarly to (5.6), we get that

$$
\begin{equation*}
\mathbb{E}\left[\sup _{t \in[0,1]}\left|X_{t}^{R}-X_{t}\right| \wedge 1\right] \leq \mathbb{P}\left(\sup _{t \in[0,1]}\left|X_{t}\right| \geq R\right) \leq \frac{C}{(R-1)^{2}} \tag{5.7}
\end{equation*}
$$

By combining (5.6) and (5.7) we find, for any finite subset $I \subseteq \mathbb{Q} \cap[0,1]$, that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbb{E}\left[\sup _{t \in I}\left|X_{t}^{n}-X_{t}\right| \wedge 1\right]  \tag{5.8}\\
& \leq \liminf _{n \rightarrow \infty} \mathbb{E}\left[\sup _{t \in[0,1]}\left|X_{t}^{n}-X_{t}^{n, R}\right| \wedge 1+\sup _{t \in I}\left|X_{t}^{n, R}-X_{t}^{R}\right|+\sup _{t \in[0,1]}\left|X_{t}^{R}-X_{t}\right| \wedge 1\right] \\
& \leq \frac{2 C}{(R-1)^{2}}+\liminf _{n \rightarrow \infty} \mathbb{E}\left[\sup _{t \in I}\left|X_{t}^{n, R}-X_{t}^{R}\right| \wedge 1\right]=\frac{2 C}{(R-1)^{2}},
\end{align*}
$$

where we used f.d.d. convergence distributions of $\left(X^{n, R}\right)_{n \in \mathbb{N}}$ to $X^{R}$ in the final line. Hence, $\left(X^{n}\right)_{n \in \mathbb{N}}$ converges in finite dimensional distributions to $X$.

## Appendix A. The strong Markov property

We take the following definition from Karatzas and Shreve [19, Chapter 2, Definition 6.2].
Definition A.1. An $\mathbb{R}^{d}$-valued stochastic process $X$ with right-continuous filtration $\mathcal{F}$ is strong Markov if, for every bounded measurable function $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ and every $t \in \mathbb{R}_{+}$, there exists a measurable function $f: \mathbb{R}_{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that for every finite stopping time $\tau$

$$
\begin{equation*}
f\left(\tau, X_{\tau}\right)=\mathbb{E}\left[g\left(X_{\tau+t}\right) \mid \mathcal{F}_{\tau}\right] . \tag{A.1}
\end{equation*}
$$

Lemma A.2. Let $X$ be a càdlàg process on a filtered probability space with the usual conditions. Suppose that $X$ satisfies the Markov property for all times $t \in[0,1]$ and satisfies the strong Markov property (A.1) for all $\varepsilon>0$ and all finite stopping times $\tau \geq \varepsilon>0$. Then $X$ is strong Markov.

Proof. Let $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be measurable and bounded, $t \in \mathbb{R}_{+}$, and fix $\varepsilon>0$. For a given finite stopping time $\tau$, we consider the event $A_{\varepsilon}:=\{\tau \leq \varepsilon\} \in \mathcal{F}_{\varepsilon}$ and define the finite stopping time

$$
\tau_{\varepsilon}:=\varepsilon \mathbb{1}_{A_{\varepsilon}}+\tau \mathbb{1}_{A_{\varepsilon}^{c}} .
$$

On the one hand, by assumption, there exists a measurable map $f: \mathbb{R}_{+} \times R^{d} \rightarrow \mathbb{R}$ such that, for all $\varepsilon>0$ and finite stopping times $\tau$, we have on $A_{\varepsilon}^{\mathrm{c}}$ that

$$
f\left(\tau, X_{\tau}\right)=f\left(\tau_{\varepsilon}, X_{\tau_{\varepsilon}}\right)=\mathbb{E}\left[g\left(X_{\tau_{\varepsilon}+t}\right) \mid \mathcal{F}_{\tau_{\varepsilon}}\right] .
$$

In particular, since $A_{\varepsilon}^{c} \nearrow\{\tau>0\}$ as $\varepsilon \searrow 0$ we find on $\{\tau>0\}$ that

$$
f\left(\tau, X_{\tau}\right)=\mathbb{E}\left[g\left(X_{\tau+t}\right) \mid \mathcal{F}_{\tau}\right] .
$$

On the other hand, we have by the Markov property of $X$ that there is a measurable function $f_{0}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ with $f_{0}\left(X_{0}\right)=\mathbb{E}\left[g\left(X_{t}\right) \mid \mathcal{F}_{0}\right]$. Combining these two observation yields that for all finite stopping times $\tau$

$$
\begin{aligned}
\mathbb{1}_{\{\tau=0\}} f_{0}\left(X_{0}\right)+\mathbb{1}_{\{\tau>0\}} f\left(\tau, X_{\tau}\right) & =\mathbb{1}_{\{\tau=0\}} \mathbb{E}\left[g\left(X_{t}\right) \mid \mathcal{F}_{0}\right]+\mathbb{1}_{\{\tau>0\}} \mathbb{E}\left[g\left(X_{\tau+t}\right) \mid \mathcal{F}_{\tau}\right] \\
& =\mathbb{E}\left[g\left(X_{\tau+t}\right) \mid \mathcal{F}_{\tau}\right],
\end{aligned}
$$

hence $X$ has the strong Markov property.

## Appendix B. Auxiliary results for the examples of Section 4

Lemma B.1. In the setting of Remark 4.3, for each $t_{0} \in(0,1]$, the process $X^{t_{0}}$ mimics $\mu$ on $\left[t_{0}, 1\right]$. Moreover, as $t_{0} \rightarrow 0$, there exists a limit $X$ of $X^{t_{0}}$ in distribution such that $\left(X_{t}\right)_{t \in\left[t_{1}, 1\right]} \sim$ $\left(X_{t}^{t_{1}}\right)_{t \in[t, 1]}$, for any $t_{1} \in(0,1]$.

Proof. For $t_{0} \in(0,1)$ and $n \in \mathbb{N}$, denote $A_{n, t_{0}}:=\left\{\sum_{k=1}^{n} \xi_{k}<\Lambda_{t}-\Lambda_{t_{0}} \leq \sum_{k=1}^{n+1} \xi_{k}\right\}$. Note that the sum of $n$ independent exponential distributions with rate parameter 1 is distributed according to a Gamma distribution with shape paramter $n$ and rate parameter 1. Therefore we have

$$
\begin{equation*}
\mathbb{P}\left[A_{n, t_{0}}\right]=\int_{0}^{\Lambda_{t}-\Lambda_{t_{0}}} \frac{x^{n-1}}{(n-1)!} e^{-x+x+\Lambda_{t}-\Lambda_{t_{0}}} d x=\frac{\left(\Lambda_{t}-\Lambda_{t_{0}}\right)^{n}}{n!} e^{-\left(\Lambda_{t}-\Lambda_{t_{0}}\right)} \tag{B.1}
\end{equation*}
$$

For $i=1,2$, we compute

$$
\begin{aligned}
\mathbb{P}\left[X_{t}^{t_{0}} \in S_{t}^{i} \mid X_{t_{0}}^{t_{0}} \in S_{t_{0}}^{2}\right] & =\sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{1}_{2 \mathbb{Z}+i}(n) \mathbb{P}\left[A_{n, t_{0}}\right] \\
& =\sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{1}_{2 \mathbb{Z}+i}(n) \frac{\left(\Lambda_{t}-\Lambda_{t_{0}}\right)^{n}}{n!} e^{-\left(\Lambda_{t}-\Lambda_{t_{0}}\right)} \\
& = \begin{cases}\frac{\sinh \left(\Lambda_{t}-\Lambda_{t_{0}}\right)}{\left.e^{\left(\Lambda_{t}-\Lambda_{t_{0}}\right.}\right)} & i=1, \\
\frac{\cosh \left(\Lambda_{t}-\Lambda_{t_{0}}\right)}{e^{\left(\Lambda_{t}-\Lambda_{t_{0}}\right)}} & i=2 .\end{cases}
\end{aligned}
$$

Therefore, by rotational symmetry, we have that $X_{t_{0}}^{t_{0}} \sim \mu_{t_{0}}$ implies that $X_{t}^{t_{0}} \sim \mu_{t}$ for all $t \in\left[t_{0}, 1\right]$. Now note that by the memoryless property of the exponential distribution we have, for $0<t_{0}<t_{1} \leq 1$,

$$
\left(X_{t}^{t_{0}}\right)_{t \in\left[t_{1}, 1\right]} \sim\left(X_{t}^{t_{1}}\right)_{t \in\left[t_{1}, 1\right]} .
$$

Hence, for $t_{0} \searrow 0$ we have that the law of $\left(X_{t}^{t_{0}}\right)_{t \in[0,1]}$ converges to some law $\eta \in \mathcal{P}\left(\mathcal{D}\left(\mathbb{R}^{4}\right)\right)$ with the property that $\left(X_{t}\right)_{t \in\left[t_{1}, 1\right]} \sim\left(X_{t}^{t_{1}}\right)_{t \in\left[t_{1}, 1\right]}$, for any $t_{1} \in(0,1]$, where $X=\left(X_{t}\right)_{t \in[0,1]}$ denotes the canonical process on the probability space $\mathcal{D}\left(\mathbb{R}^{4}\right)$ equipped with its Borel $\sigma$-algebra, the probability measure $\eta$, and the right-continuous $\eta$-complete filtration $\left(\mathcal{F}_{t}\right)_{t \in[0,1]}$ generated by $X$.

Lemma B.2. Let $X, \mathcal{F}$ be as in Remark 4.3 and let $f: \mathbb{R}^{d} \rightarrow[0,1]$ be measurable. Then we claim that, for $t \in[0,1]$,

$$
\begin{equation*}
\mathbb{E}\left[f\left(X_{t}\right) \mid \mathcal{F}_{0}\right]=\lim _{t_{0} \searrow 0} \mathbb{E}\left[f\left(X_{t}\right) \mid \mathcal{F}_{t_{0}}\right]=\mathbb{E}\left[f\left(X_{t}\right)\right] \quad \text { a.s. } \tag{B.2}
\end{equation*}
$$

Proof. Note that the first equality in (B.2) is due the martingale convergence theorem. By the Markov property at $t_{0}$, we have that, for $0<t_{0}<t$,

$$
\mathbb{E}\left[f\left(X_{t}\right) \mid \mathcal{F}_{t_{0}}\right]=\mathbb{E}\left[f\left(X_{t}\right) \mid X_{t_{0}}\right] .
$$

By construction, given the starting point $X_{t_{0}}=x \in S_{t_{0}}^{2}$, we have $X_{t} \sim X_{t}^{t_{0}}$. Now use the independence of $\left(U_{n}\right)_{n \in \mathbb{N}}$ and $\left(\xi_{k}\right)_{k \in \mathbb{N}}$ to compute that

$$
\begin{aligned}
\mathbb{E}\left[f\left(X_{t}\right) \mid X_{t_{0}}=x\right] & =\sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{E}\left[\mathbb{1}_{A_{n, t_{0}}}\left(\mathbb{1}_{2 \mathbb{Z}}(n) f\left(V_{n}\right)+\mathbb{1}_{2 \mathbb{Z}+1}(n) f\left(U_{n}\right)\right]\right. \\
& =\sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{P}\left[A_{n, t_{0}}\right]\left(\mathbb{1}_{2 \mathbb{Z}}(n) \mathbb{E}[f(V)]+\mathbb{1}_{2 \mathbb{Z}+1}(n) \mathbb{E}[f(U)]\right),
\end{aligned}
$$

where $U \sim \operatorname{Unif}\left(S_{1}^{1}\right)$ and $V \sim \operatorname{Unif}\left(S_{1}^{2}\right)$. Since the law of $X_{t}$ is given explicitly by $\mu_{t}=$ $\frac{1}{2}(\mathcal{L}((0,0, \sqrt{t} U))+\mathcal{L}(\sqrt{t} U, 0,0))$, it remains to prove that

$$
\sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{P}\left[A_{2 n, t_{0}}\right] \rightarrow \frac{1}{2} \quad \text { as } t_{0} \searrow 0 .
$$

Using (B.1) and noting that $\Lambda_{t}-\Lambda_{t_{0}}$ diverges to $+\infty$ for $t_{0} \searrow 0$, we find that

$$
\sum_{n \in \mathbb{N} \cup\{0\}} \mathbb{P}\left[A_{2 n, t_{0}}\right]=\frac{\cosh \left(\Lambda_{t}-\Lambda_{t_{0}}\right)}{e^{\Lambda_{t}-\Lambda_{t_{0}}}} \rightarrow \frac{1}{2},
$$

and conclude that (B.2) holds.
Lemma B.3. Let $n \in \mathbb{N}$ and set $t_{0}=2^{-(n+1)}, t_{1}=2^{-n}$. Then, in the setting of Proposition 4.5, there exists a constant $C>0$ such that

$$
\mathbb{P}\left[\overline{\mathcal{S}}_{t_{0}}^{i} \triangle \overline{\mathcal{S}}_{t_{1}}^{i}\right] \leq C t_{0} .
$$

Proof. Fix $t_{0}=2^{-(n+1)}, t_{1}=2^{-n}$ for some even integer $n \in \mathbb{N} \cup\{0\}$. Then we have that $\left[2^{-(n+1)}, 2^{-n}\right] \in I_{1}$, and so $a_{2}(t)$ and $\mu_{t} \mid \mathcal{X}_{2}$ are constant on this interval. Subsequently, we will repeatedly employ the estimates

$$
\begin{align*}
& \mathbb{E}\left[\left|X_{t}\right|^{4}+\left|N_{t^{14}}\right|^{4}\right] \leq \frac{a_{1}(t)^{2}+a_{2}(t)^{2}}{2}+3 t^{28} \leq C t^{2},  \tag{B.3}\\
& \mathbb{P}\left[\left|N_{t^{14}}\right| \geq a\right] \leq \frac{t^{14}}{a^{2}}, \quad a>0
\end{align*}
$$

Similarly as in Proposition 4.4, we split the vector $Y$ into two parts where $Y_{t}^{1}$ denotes the first two coordinates and $Y_{t}^{2}$ the last two components of $Y_{t}$, for $t \in[0,1]$. We aim to bound the probability of $Y^{i}$ leaving a small ball around $Y_{t_{0}}^{i}$ in the time interval $\left[2^{-(n+1)}, 2^{-n}\right]$.

First we compute the upper bound

$$
\begin{equation*}
\mathbb{E}\left[\left|Y_{t}^{i}\right|^{2}\right] \leq \mathbb{E}\left[\left|X_{t}^{i}\right|^{2}+\left|N_{t^{14}}\right|^{2}\right] \leq \frac{1}{2} a_{i}(t)+t^{14} \tag{B.4}
\end{equation*}
$$

In the case that $i=2$ we get the lower bound

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_{t_{0}}^{2}\right|^{2} \mathbb{1}_{\hat{\mathcal{S}}_{t_{0}}^{2}}\right] & \geq \mathbb{E}\left[\left|X_{t_{0}}+N_{t_{0}^{14}}\right|^{2} \mathbb{1}_{\hat{\mathcal{S}}_{t_{0}}^{2}} ;\left|N_{t_{0}^{14}}\right|<t_{0}^{2}\right] \\
& \geq \mathbb{E}\left[\left|X_{t_{0}}+N_{t_{0}^{14}}\right|^{2} \mathbb{1}_{\mathcal{S}_{t_{0}}^{2}}^{2}\right]-\mathbb{E}\left[\left|X_{t_{0}}+N_{t_{0}^{14}}\right|^{2} ;\left|N_{t_{0}^{14}}\right| \geq t_{0}^{2}\right] \\
& \geq \mathbb{E}\left[\left|X_{t_{0}}\right|^{2} \mathbb{1}_{\hat{\mathcal{S}}_{t_{0}}^{2}}\right]-\mathbb{E}\left[\left|X_{t_{0}}\right|^{2}+\left|N_{t_{0}^{14}}\right|^{2} ;\left|N_{t_{0}^{14}}\right| \geq t_{0}^{2}\right],
\end{aligned}
$$

using the independence of $X_{t_{0}}$ and $N_{t_{0}^{14}}$. We use this independence again, together with the Cauchy-Schwarz inequality and the estimates (B.3), to show that

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_{t_{0}}^{2}\right|^{2} \mathbb{1}_{\hat{\mathcal{S}}_{t_{0}}^{2}}\right. & \geq \frac{1}{2} a_{2}\left(t_{0}\right)-\mathbb{E}\left[\left|X_{t_{0}}\right|^{2}+\left|N_{t_{0}^{14}}\right|^{2} ;\left|N_{t_{0}^{14}}\right| \geq t_{0}^{2}\right] \\
& \geq \frac{1}{2} a_{2}\left(t_{0}\right)-\left(\mathbb{P}\left[\left|N_{t_{0}^{14}}\right| \geq t_{0}^{2}\right] \mathbb{E}\left[\left|X_{t_{0}}\right|^{4}+\left|N_{t_{0}^{14}}\right|^{4}\right]\right)^{\frac{1}{2}} \\
& \geq \frac{1}{2} a_{2}\left(t_{0}\right)-C t_{0}^{6} .
\end{aligned}
$$

Recalling that $a_{2}$ is constant on $\left[t_{0}, t_{1}\right]$, (B.4) implies that $\mathbb{E}\left|Y_{t_{1}}\right|^{2} \leq \frac{1}{2} a_{2}\left(t_{0}\right)+t_{1}^{14}$. Since $Y$ is a martingale we have

$$
\mathbb{E}\left[\left|Y_{t_{1}}^{2}-Y_{t_{0}}^{2}\right|^{2}\right]=\mathbb{E}\left[\left|Y_{t_{1}}^{2}\right|^{2}-\left|Y_{t_{0}}^{2}\right|^{2}\right] \leq \mathbb{E}\left|Y_{t_{1}}^{2}\right|^{2}-\mathbb{E}\left[\left|Y_{t_{0}}^{2}\right|^{2} \mathbb{1}_{\hat{\mathcal{S}}_{0}}^{2}\right] .
$$

Thus combining the upper and lower bounds above gives us

$$
\begin{equation*}
\mathbb{E}\left[\left|Y_{t_{1}}^{2}-Y_{t_{0}}^{2}\right|^{2}\right] \leq \frac{1}{2} a_{2}\left(t_{0}\right)+t_{1}^{14}-\left(\frac{1}{2} a_{2}\left(t_{0}\right)-C t_{0}^{6}\right) \leq C_{0} t_{0}^{6} \tag{B.5}
\end{equation*}
$$

for some $C_{0}>0$. By Doob's maximal inequality we obtain

$$
\begin{equation*}
\mathbb{P}\left[\sup _{t \in\left[t_{0}, t_{1}\right]}\left|Y_{t}^{2}-Y_{t_{0}}^{2}\right| \geq t_{0} / 2\right] \leq 4 C_{0} t_{0}^{4} . \tag{B.6}
\end{equation*}
$$

Next, we show an analogous result to (B.5) when $i=1$. We claim that, for some constant $C_{1}>0$,

$$
\begin{equation*}
\mathbb{E}\left[\left|Y_{t_{1}}^{1}-Y_{t_{0}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c}}\right] \leq C_{1} t_{0}^{3} . \tag{B.7}
\end{equation*}
$$

Since $Y$ is a martingale, we have

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_{t_{1}}^{1}-Y_{t_{0}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c}}\right] & =\mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2}-\left|Y_{t_{0}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c}}\right] \leq \mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c}}\right] \\
& \leq \mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cup \hat{\mathcal{S}}_{t_{0}}^{2}\right)^{\mathrm{c}}\right]+\mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ; \hat{\mathcal{S}}_{t_{0}}^{2}\right]
\end{aligned}
$$

We estimate each term separately, using the Cauchy-Schwarz inequality and the estimates (B.3) in each case. We first bound

$$
\mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cup \hat{\mathcal{S}}_{t_{0}}^{2}\right)^{\mathrm{c}}\right] \leq \mathbb{E}\left[\left|Y_{t_{1}}\right|^{4}\right]^{\frac{1}{2}} \mathbb{P}\left[\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cup \hat{\mathcal{S}}_{t_{0}}^{2}\right)^{\mathrm{c}}\right]^{\frac{1}{2}} \leq C t_{1} \mathbb{P}\left[\left|N_{t_{0}^{14}}\right| \geq t_{0}^{2}\right]^{\frac{1}{2}} \leq C_{2} t_{0}^{6} .
$$

In bounding the second term, we additionally apply (B.6) to get

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ; \hat{\mathcal{S}}_{t_{0}}^{2}\right] & =\mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ; \hat{\mathcal{S}}_{t_{0}}^{2},\left|Y_{t_{1}}^{2}-Y_{t_{0}}^{2}\right| \leq t_{0} / 2\right]+\mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ;\left|Y_{t_{1}}^{2}-Y_{t_{0}}^{2}\right| \geq t_{0} / 2\right] \\
& \leq \mathbb{E}\left[\left|Y_{t_{1}}^{1}\right|^{2} ;\left(\hat{\mathcal{S}}_{t_{1}}^{1}\right)^{c}\right]+\left(\mathbb{E}\left[\left|Y_{t_{1}}\right|^{4}\right] \mathbb{P}\left[\left|Y_{t_{1}}^{2}-Y_{t_{0}}^{2}\right| \geq t_{0} / 2\right]\right)^{\frac{1}{2}} \\
& \leq C_{2} t_{0}^{6}+\left(C t_{0}^{2} \cdot 4 C_{0} t_{0}^{4}\right)^{\frac{1}{2}} \leq C_{3} t_{0}^{3} .
\end{aligned}
$$

Combining the two preceding inequalities yields (B.7). As before, Doob's maximal inequality implies

$$
\begin{equation*}
\mathbb{P}\left[\sup _{t \in\left[t_{0}, t_{1}\right]}\left|Y_{t}^{1}-Y_{t_{0}}^{1}\right| \geq t_{0} / 4 ;\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c}}\right] \leq 16 C_{1} t_{0} . \tag{B.8}
\end{equation*}
$$

Note that, for $i=1,2$,

$$
\mathbb{P}\left[\overline{\mathcal{S}}_{t_{0}}^{i} \triangle \overline{\mathcal{S}}_{t_{0}}^{i}\right] \leq \mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{i} \triangle \overline{\mathcal{S}}_{t_{1}}^{i}\right]+\mathbb{P}\left[\overline{\mathcal{S}}_{t_{0}}^{i} \backslash\left(\hat{\mathcal{S}}_{t_{0}}^{i} \cup \overline{\mathcal{S}}_{t_{1}}^{i}\right)\right],
$$

and

$$
\mathbb{P}\left[\overline{\mathcal{S}}_{t_{0}}^{i} \backslash\left(\hat{\mathcal{S}}_{t_{0}}^{i} \cup \overline{\mathcal{S}}_{t_{1}}^{i}\right)\right]=\mathbb{P}\left[t_{0}^{2} \leq\left|N_{t_{0}^{14}}\right|<t_{0},\left|N_{t_{1}^{14}}\right| \geq t_{1}\right] \leq t_{0}^{10} .
$$

Hence, to prove the conclusion of the lemma, we only require bounds on $\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{i} \triangle \overline{\mathcal{S}}_{t_{1}}^{i}\right]$, for $i=1,2$.
Now consider the events

$$
A_{t_{0}}:=\left\{\left|Y_{t_{1}}^{2}-Y_{t_{0}}^{2}\right| \leq t_{0} / 4\right\} \quad \text { and } \quad B_{t_{0}}:=\hat{\mathcal{S}}_{t_{0}}^{2} \cap\left\{\left|Y_{t_{1}}^{1}-Y_{t_{0}}^{1}\right| \leq t_{0} / 4\right\},
$$

and observe that (B.6) and (B.8) imply that

$$
\mathbb{P}\left[A_{t_{0}}^{\mathrm{c}}\right] \leq 16 C_{0} t_{0}^{4} \quad \text { and } \quad \mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{2} \backslash B_{t_{0}}\right] \leq 16 C_{1} t_{0} .
$$

We calculate that

$$
\begin{aligned}
\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1} \triangle \overline{\mathcal{S}}_{t_{1}}^{1}\right] & =\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1} \backslash\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right)\right]+\mathbb{P}\left[\overline{\mathcal{S}}_{t_{1}}^{1} \backslash\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right)\right] \\
& =\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1}\right]+\mathbb{P}\left[\overline{\mathcal{S}}_{t_{1}}^{1}\right]-2 \mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right] \\
& \leq 2\left(\mathbb{P}\left[\overline{\mathcal{S}}_{t_{1}}^{1}\right]-\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right]\right)+t_{0}^{10},
\end{aligned}
$$

since $\mathbb{P}\left[\overline{\mathcal{S}}_{t_{1}}^{1}\right] \geq \mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1},\left|N_{t_{0}^{14}}\right|<t_{0}^{2}\right] \geq \mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1}\right]-t_{0}^{10}$. Note that we have the inclusions

$$
\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c} \cap \overline{\mathcal{S}}_{t_{1}}^{1} \subseteq\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cup \hat{\mathcal{S}}_{t_{0}}^{2}\right)^{\mathrm{c}} \cup\left(\hat{\mathcal{S}}_{t_{0}}^{2} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right), \quad \hat{\mathcal{S}}_{t_{0}}^{2} \cap \overline{\mathcal{S}}_{t_{1}}^{1} \subseteq A_{t_{0}}^{\mathrm{c}} . . . . . . . . . . . . ~}
$$

Hence

$$
\begin{aligned}
\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{1} \triangle \overline{\mathcal{S}}_{t_{1}}^{1}\right] \leq 2 \mathbb{P}\left[\left(\hat{\mathcal{S}}_{t_{0}}^{1}\right)^{\mathrm{c}} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right]+t_{0}^{10} & \leq 2 \mathbb{P}\left[\left(\hat{\mathcal{S}}_{t_{0}}^{1} \cup \hat{\mathcal{S}}_{t_{0}}^{2}\right)^{\mathrm{c}} \cup\left(\hat{\mathcal{S}}_{t_{0}}^{2} \cap \overline{\mathcal{S}}_{t_{1}}^{1}\right)\right]+t_{0}^{10} \\
& \leq 2 \mathbb{P}\left[\left|N_{t^{14}}\right| \geq t_{0}^{2}\right]+2 \mathbb{P}\left[A_{t_{0}}^{\mathrm{c}}\right]+t_{0} \\
& \leq 3 t_{0}^{10}+32 C_{0} t_{0}^{4} .
\end{aligned}
$$

On the other hand, we have

$$
\mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{2} \triangle \overline{\mathcal{S}}_{t_{1}}^{2}\right] \leq \mathbb{P}\left[\hat{\mathcal{S}}_{t_{0}}^{2} \backslash B_{t_{0}}\right]+\mathbb{P}\left[\left|N_{t_{0}^{14}}\right| \geq t_{0}\right] \leq C t_{0}
$$

Thus we have shown the claim in the case that $n$ is even. Due to symmetry of the arguments we also obtain the desired result when $n$ is odd.

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