BOUNDDED OPERATORS ON $L^p$ SPACES

PART II

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ABSTRACT:

As in the first part of the paper, we deal with some problems posed in [1]. In the present paper we give solutions to the problems 3.12, 8.2, 17.6 and 8.4.

1. INTRODUCTION:

For definitions and notations we refer to the first part of the paper [3]. But we shall not use the concept of a Halmos-function in this second part.

2. SOLUTION TO PROBLEM [1], 3.12:

The solution is a relatively straightforward application of the Banach-Steinhaus theorem and the closed graph theorem. We first need, however, a definition and a preliminary result.

2.1. Definition: Given a Banach space $(G, \| \cdot \|)$ we shall call a Banach space $(E, \| \cdot \|)$ together with a continuous injection $j : E \to G$, a Banach subspace of $G$. Similarly, given an $F$-space $G$ (i.e. a completely metrisable topological vector space) we shall call an $F$-space $E$ together with a continuous injection $j : E \to G$ an $F$-subspace of $G$.

2.2. Proposition: Let $E$ be a Banach subspace of $L^1(X)$ and let $k(x,y)$ be a measurable function on $X \times Y$ such that
(1) \( \forall g \in E \quad k(x, \cdot), g(\cdot) \in L^1(\nu) \quad \text{for } \mu\text{-a.e. } x \in X. \)

Then the operator

\[
\text{Int}(k) : E \longrightarrow L^\infty(\mu),
\]

\[
g \longmapsto f(x) = \int k(x, y)g(y)d\nu(y)
\]

is well defined and continuous.

**Proof:** Let \( k_n \) be the truncation of \( k \) at \( n \), i.e.

\[
k_n(x, y) = k(x, y) \cdot \chi_{\{|k(x, y)| \leq n\}}(x, y).
\]

The operator

\[
\text{Int}(k_n) : L^1(\nu) \longrightarrow L^\infty(\mu)
\]

\[
g \longmapsto f(x) = \int k_n(x, y)g(y)d\nu(y)
\]

is continuous from \( L^1(\nu) \) to \( L^\infty(\mu) \) (its norm is at most \( n \)).

In particular \( \text{Int}(k_n) \) restricts to a continuous operator from \( E \) to \( L^\infty(\mu) \), as the injection from \( E \) to \( L^1(\nu) \) as well as the injection from \( L^\infty(\mu) \) to \( L^\infty(\mu) \) are continuous.

Given \( g \in E \), note that \( \text{Int}(k_n)(g) \) converges \( \mu \)-almost everywhere to \( \text{Int}(k)(g) \) because of the integrability condition (1). Hence \( \text{Int}(k_n)(g) \) converges to \( \text{Int}(k)(g) \) in measure, i.e. with respect to the topology of \( L^\infty(\mu) \). Therefore the map \( \text{Int}(k) \) is the pointwise limit of the sequence \( \text{Int}(k_n) \) of continuous operators from \( E \) to \( L^\infty(\mu) \). We may apply the Banach-Steinhaus theorem in its form for F-spaces ([4], th. III, 4.6) to infer that \( \text{Int}(k) \) is continuous.

2.3. **Remark:** The idea of cutting \( k \) down to \( k_n \) and to apply the Banach-Steinhaus theorem in the above proof is due to J.B. Cooper, who thus replaced a cumbersome gliding-hump-argument, that I had applied previously.
2.4. Corollary: Let $E$ be a Banach subspace of $L^1(\mathcal{V})$, $F$ an $F$-subspace of $L^0(\mathcal{U})$ and $k(x,y)$ a measurable function such that

1. for $g \in E$ \( k(x,.)g(.) \in L^1(\mathcal{V}) \) for $\mu$-a.e. $x \in X$ and

2. for $g \in E$ \( f(x) = \int k(x,y)g(y)du(y) \in F. \)

Then $\text{Int}(k)$ induces a continuous operator from $E$ to $F$.

2.5. Remark: The corollary applies in particular to the case, where $E$ is a closed subspace of $L^2(\mathcal{V})$ and $F = L^2(\mathcal{U})$, thus answering problem 3.12 of [1] in the positive.

Proof of 2.4: By 2.2 and condition (1) the graph of $\text{Int}(k)$ is a closed subspace of $E \times L^0(\mathcal{U})$. By condition (2) the graph of $\text{Int}(k)$ is contained in $E \times F$ and, as $F$ injects continuously into $L^0(\mathcal{U})$, it is closed in $E \times F$.

The closed graph theorem ([4], th. III, 2.3) implies that $\text{Int}(k)$ is a continuous operator from $E$ to $F$. \(\square\)

3. SOLUTION TO PROBLEM [1], 8.2:

We have already indicated in [3] one possible way to recapture the kernel $k$ from the values of the operator $\text{Int}(k)$ "effectively".

We now present a different "effective procedure", which uses only the (scalar-valued) Radon-Nikodym theorem. Of course, by the vagueness of the term "effective" it will depend on the taste of the reader if he accepts the following construction as a satisfactory answer to problem 8.2.

Let $k$ be a kernel inducing an operator $\text{Int}(k)$ from $L^2(\mathcal{V})$ to $L^2(\mathcal{U})$ and
suppose for the moment \( k \in L^1(\mu \times \nu) \). Given measurable sets \( A \subseteq X, B \subseteq Y \),
\[
\iint k(x,y) d\nu(y) d\mu(x) = (\text{Int}(k)\chi_B,\chi_A)
\]
The right hand side depends only on the values of \( \text{Int}(k) \). If we denote
by \( \lambda \) the measure \( k(x,y)(\mu \times \nu) \) on \( X \times Y \), the above expression equals
\( \lambda(A \times B) \). By the integrability of \( k \) the measure \( \lambda \) is finite and absolutely
continuous with respect to \( \mu \times \nu \).

The above formula gives the values of \( \lambda \) on the rectangles, the usual
Caratheodory procedure extends \( \lambda \) to the product \( \sigma \)-algebra \( \mathcal{X} \times \mathcal{Y} \) and
the Radon-Nikodym theorem gives
\[
k(x,y) = \frac{d\lambda}{d(\mu \times \nu)}.
\]

Unfortunately a kernel \( k \) may induce a continuous (even compact) operator
from \( L^2(\nu) \) to \( L^2(\mu) \), without \( k \) being integrable (although the measure
spaces \( (X,\mu) \) and \( (Y,\nu) \) are assumed to be finite).

In this case we may not apply brutally the above construction. But,
given a kernel \( k \), observe that for each \( \varepsilon > 0 \) there is \( X_\varepsilon \subseteq X, \mu(X \setminus X_\varepsilon) < \varepsilon \)
such that \( k \) restricted to \( X_\varepsilon \times Y \) is integrable. Indeed the function
\[
x \mapsto \|k(x,.)\|_{L^1(\nu)}
\]
is \( \mu \)-measurable and \( \mu \)-almost everywhere finite,
hence we only have to take \( X_\varepsilon = \{x : \|k(x,.)\|_{L^1(\nu)} \leq M\} \) for \( M \) large
enough. Having this in mind we may present our construction.

3.1. \textbf{Proposition}: Let \( T : L^2(\nu) \to L^2(\mu) \) be an operator of the form
\[
T = \text{Int}(k).
\]
Then there is an "effective procedure" to recapture \( k \)
from the values of \( T \).

\textbf{Proof}: For measurable sets \( A \subseteq X, B \subseteq Y \) define
\[
\lambda(A \times B) = (\text{Int}(k)\chi_B,\chi_A).
\]
Clearly \( \lambda \) may be extended to a finitely additive set function on the
algebra generated by the rectangles. Let
\[
\|\lambda\|_{A \times B} = \sup \left\{ \sum_{i=1}^n \lambda(A_i \times B_i), A_i \times B_i \text{ disjoint subrectangles of } A \times B \right\}.
\]
It is easily seen, that
\[ |\lambda|(A \times B) = \int \int |k(x, y)| \, dv(y) \, du(x). \]

This expression may be equal $+\infty$. But we know from the discussion preceding the proposition that for $\varepsilon > 0$ there is $X_\varepsilon$ with $\mu(X \setminus X_\varepsilon) < \varepsilon$ and $|\lambda|(X_\varepsilon \times Y) < \infty$. Hence we may extend the restriction of $\lambda$ to $X_\varepsilon \times Y$ to the product $\sigma$-algebra and by the Radon–Nikodym theorem we may find the values of $k$ on $X_\varepsilon \times Y$ (to be exact: almost everywhere on $X_\varepsilon \times Y$).

Finally, it is clear how to find $k$ on all of $X \times Y$. Let $\varepsilon = n^{-1}$ and find successively $k$ on $X_{n^{-1}} \times Y$.

\[ \square \]

4. SOLUTION to [1], 17.6:

The answer is no: There exists an integral operator $\text{Int}(k) : L^2(v) \rightarrow L^2(\mu)$, an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in $L^2(v)$ and a square summable sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive scalars such that $\sum_{n=1}^{\infty} |\alpha_n \cdot \text{Int}(k)(e_n)|$ is infinite on a set of positive measure.

Actually it is easy to provide such an example in view of the remark in [1], 17.6 that a positive solution to problem 17.6 would solve positively problem [1], 11.8. As we have seen in [3], the answer to [1], 11.8 is negative and a close look at the counterexamples [3], 6.6, 6.8 and 6.9 shows that they also provide counterexamples to [1], 17.6.

However we prefer not to repeat these examples but rather to give a very easy counterexample, taylormade for problem [1], 17.6.

Fix a sequence $\{\beta_n\}_{n=-\infty}^{\infty}$ of scalars, which is not square summable but such that $\sum_{n=-\infty}^{\infty} |\beta_n| < \infty$. Set $s(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{2\pi i nt}$, which converges in $L^1(T)$, where $T$ denotes the onedimensional torus equipped with
Lebesgue measure. For example \( \{ n^p \}_{n=\infty}^{\infty} \) for \( \frac{1}{2} < p < 0 \) is such a sequence (c.f. [2], ex. II, 1.3).

Denote by \( C \) the convolution operator on \( L^2(T) \) induced by \( c \) (c.f. [1], th. 12.2). Clearly \( C \) is an absolutely bounded integral operator and the kernel corresponding to \( C \) is given by:

\[
k(t,s) = c(t-s) = \sum_{n=\infty}^{\infty} \beta_n e^{2\pi i n(t-s)}.
\]

Note that the operator \( C \) maps \( e^{2\pi i n} \) to \( \beta_n e^{2\pi i n} \). Indeed, as \( e^{2\pi i n} \) is an element of \( L^1(T) \), it defines a continuous linear functional on \( L^1(T) \), hence the following equations hold true.

\[
\int_T k(t,s) e^{2\pi i n s} ds = \sum_{m=\infty}^{\infty} \int_T e^{2\pi i m s} \beta_m e^{2\pi i n(t-s)} ds.
\]

\[
= \sum_{m=\infty}^{\infty} \beta_m e^{2\pi i n} \int_T e^{2\pi i (n-m) s} ds.
\]

\[
= \beta_n e^{2\pi i n}.
\]

Find a square-summable sequence \( \{ \alpha_n \}_{n=\infty}^{\infty} \) such that \( \sum_{n=\infty}^{\infty} |\alpha_n \beta_n| = \infty \). Then

\[
\sum_{n=\infty}^{\infty} |\alpha_n C e^{2\pi i n}(t)| = \sum_{n=\infty}^{\infty} |\alpha_n \beta_n| = \infty
\]

for almost every \( t \in T \).

5. SOLUTION TO PROBLEM [1], B.4:

5.1. Proposition: There is an \( \mathbb{R}_+ \)-valued Lebesgue measurable function \( k(x,y) \) on \([0,1] \times [0,1]\) such that for every Lebesgue measurable \( \mathbb{R}_+ \)-valued function \( g \) on \([0,1]\), which is different from 0 on a set of positive measure,
\[ \int k(x,y)g(y)dy = \infty \]

for almost each \( x \in [0,1] \).

Whence, in the language of [1], all nontrivial subkernels of \( k \) have domain \( \{0\} \).

**Proof:** Let \( h \) be the function on \([0,1] \times [0,1], \)

\[
\begin{align*}
h(x,y) &= |x - y|^{-1} & \text{if } x \neq y \\
h(x,y) &= 0 & \text{if } x = y.
\end{align*}
\]

It is shown in [1], ex. 3.2, that for any positive, measurable function \( g \) on \([0,1], \) not vanishing almost everywhere, the set

\[ A_h = \{ x : \int g(y)h(x,y)dy = \infty \} \]

has strictly positive measure. Our task is to replace \( h \) by some \( k \) such that this set is always of measure 1.

Let \( \{ r_n \}_{n=1}^{\infty} \) be an enumeration of the rationals in \([0,1] \) and let \( h_n(x,y) \)

be the \( r_n \)-th translate of \( h \), i.e.

\[ h_n(x,y) = h(x - r_n, y), \]

where \( \pm \) denotes subtraction modulo 1. Let \( \{ p_n \}_{n=1}^{\infty} \) be a sequence of

strictly positive numbers, such that

\[ m_2\{ (x,y) : p_n h_n(x,y) \leq 2^{-n} \} \leq 2^{-n}, \]

where \( m_2 \) denotes Lebesgue measure on \([0,1] \times [0,1] \).

Define

\[ k(x,y) = \sum_{n=1}^{\infty} p_n h_n(x,y). \]

By the Borel-Cantelli lemma \( k \) is \( m_2 \)-almost everywhere finite.

By changing \( k \) on a set of measure zero, we may assume that \( k \) is everywhere \( \mathbb{R}_+ \)-valued.

Let \( g \) be a positive function on \( Y \), different from zero on a set of positive measure. The set
\[ A_n = \{ x : \int g(y)h(x,y)dy = \infty \} \]

is of strictly positive measure. As the set

\[ A_k = \{ x : \int g(y)k(x,y)dy = \infty \} \]

contains all rational translates of \( A_n \) (modulo 1), \( A_k \) has measure 1.

REFERENCES:


