THE CLASS OF BANACH SPACES, WHICH DO NOT HAVE $c_0$ AS A SPREADING MODEL, IS NOT $L^2$-HEREDITARY

by

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1. INTRODUCTION: In [4] the problem was raised whether the fact, that a Banach space $E$ does not have $c_0$ as a spreading model, implies that $L^2([0,1]; E)$ has the same property. It was conjectured that the answer is no, as the property of not having $c_0$ as a spreading model is somewhat dual to the Banach-Saks property (see [2]) and for this latter property J. Bourgain has constructed a counterexample ([3]).

The present author has constructed independently of J. Bourgain another space $E$ with the Banach-Saks property and $L^2(E)$ failing it ([6]) and it turns out that the dual $E'$ gives a counterexample to the problem raised in the title.

2. THE EXAMPLE: Let $\gamma = (n_1,n_2,\ldots,n_k)$ an increasing finite sequence of natural numbers. Write $n_i = 2^{u_i} + v_i$ where this expression is unique, if we require that $0 \leq v_i < 2^{u_i}$. As in [6] we associate to every $n_i$ the real number $t(n_i) = v_i/2^{u_i} \in [0,1[$ and call $\gamma$ admissible if

1. $k \leq n_1$
2. For every $0 \leq j < 2^{u_1}+1$ there is only one $i$ such that $t(n_i) \in [(j+1)/2^{u_1}+1, (j+1)/2^{u_1}+1[$.

For an admissible $\gamma = (n_1,\ldots,n_k)$ and $x \in \mathbb{R}(\mathbb{N})$, the space of finite sequences, we define

$$\|x\|_{\gamma} = \sum_{i=1}^{k} |x_{n_i}|.$$
For our purposes it will this time be convenient, not to use interpolation but to follow Baernstein’s original definition ([1]): For $x \in \mathbb{R}^{(\mathbb{N})}$ define

$$\|x\|_E = \sup \left\{ \left( \sum_{\ell=1}^{\infty} \|x\|_{\gamma_\ell}^2 \right)^{1/2} \right\}$$

where the sup is taken over all increasing sequences \( \{\gamma_\ell\}_{\ell=1}^\infty \) of admissible sets (i.e. the last member of \( \gamma_\ell \) is smaller than the first member of \( \gamma_{\ell+1} \)).

Let \((E, \| \cdot \|_E)\) be the completion of \( \mathbb{R}^{(\mathbb{N})} \) with respect to this norm. In an analogous way as in [6] one shows that E has the Banach-Saks property.

**Proposition 1:**

E’ does not have \( c_0 \) as a spreading model.

**Proof:** As E does not have \( \ell^1 \) as spreading model ([2]), no quotient of E' has \( c_0 \) as spreading model ([5]), hence in particular E' does not have \( c_0 \) as spreading model.

To show that \( L^2(E') \) does have \( c_0 \) as spreading model we need a trivial probabilistic lemma, whose proof is left to the reader.

**Lemma:** Let \( k \in \mathbb{N} \) and \( \varepsilon > 0 \); there is \( N(k, \varepsilon) \) such that for \( M > N(k, \varepsilon) \) and for independent random variables \( X_1, \ldots, X_k \) taking their values in \( \{1, \ldots, M\} \) in a uniformly distributed way, we have

$$P \left\{ \omega : \text{there is } 1 \leq i < j \leq k \text{ with } X_i(\omega) = X_j(\omega) \right\} < \varepsilon$$

**Proposition 2:** \( L^2(0,1)(E) \) has \( c_0 \) isometrically as spreading model.

**Proof:** Similarly as in [6] we let \( \{f_u\}_{u=1}^\infty \) be an independent sequence in \( L^2(E') \) such that \( f_u \) takes the value \( e_{2^u+v}^{u+v} \) with probability \( 1 - u \) (for \( v=0, \ldots, 2^u-1 \)). This time the \( e_{2^u+v} \) are unit-vectors in E'.
Clearly \( \| f u \|_{L^2(E')} = 1 \) and for every sequence 
\( u_1 < u_2 < \ldots < u_k \) and \( \epsilon_i = \pm 1 \)

\[
\limsup_{u \to \infty} \left\{ \left\| \sum_{i=1}^{k} \epsilon_i f_{u_i} \right\|_{L^2(E')} : u \leq u_1 < \ldots < u_k \right\} = 1
\]

Hence the following claim will prove the proposition.

CLAIM: For every \( k \in \mathbb{N} \)

\[
\limsup_{u \to \infty} \left\{ \left\| u_1 < \ldots < u_k \right\|_{L^2(E')} : \epsilon_1 = \pm 1 \right\} = 1
\]

To prove the claim fix \( k \) and \( \epsilon > 0 \) and let \( u \) be such that \( 2^u > \max(k, N(k, \epsilon)) \), where the \( N(k, \epsilon) \) is defined in the preceding lemma. Now fix \( u \leq u_1 < u_2 < \ldots < u_k \) and a sequence of signs \( \epsilon_1, \ldots, \epsilon_k \).

To apply the above lemma let \( X_1, \ldots, X_k \) be the random variables with values in \( \{1, \ldots, 2^u\} \) defined by

\[
X_i(\omega) = m \text{ if } f_{u_i}(\omega) = e_{2^u_i + v} \text{ and } t(2^u_i + v) = v/2^u_i \in [(m-1)/2^u_i, m/2^u_i] + 1
\]

It follows form the above lemma and the definition of admissible sets \( \gamma \) that there is a subset \( A \subseteq [0, 1] \) of measure greater than \( 1 - \epsilon \) such that for \( \omega \in A \) the set \( \gamma_\omega = \{n_1, \ldots, n_k\} \) corresponding to the indices of the unit vectors \( \{f_{u_1}, \ldots, f_{u_k}\} \) is admissible. Hence for \( \omega \in A \) we have

\[
\left. \sum_{i=1}^{k} \epsilon_i f_{u_i}(\omega) \right\|_{E'} = \sup \left\{ \left\langle \sum_{i=1}^{k} \epsilon_i f_{u_i}(\omega), x \right\rangle : \|x\|_{E} \leq 1 \right\}
\]

\[
\leq \sup \left\{ \left\langle \sum_{i=1}^{k} \epsilon_i f_{u_i}(\omega), x \right\rangle : \|x\|_{\gamma_\omega} \leq 1 \right\}
\]

\[
= 1.
\]
Integrating we obtain
\[
\sum_{i=1}^{k} \varepsilon_i \| \mathbf{f}_{u_i} \|_{L^2(E')}^2 \leq \int \sum_{A_i=1}^{k} \mathbf{f}_{u_i}(\omega) \|_{E}^2 \, d\omega + \int \left( \sum_{A_i=1}^{k} \mathbf{f}_{u_i} \|_{E}^2 \right)^2 \, d\omega \in (0,1) \setminus A \, \sum_{i=1}^{k} \mathbf{u}_i \|_{E}, \quad d\omega
\]
\leq 1 + k^2 \varepsilon.

This proves the claim and therefore proposition 2.

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**REFERENCES**


