Hilbert-Schmidt integral operators vs. systems of elliptic Calogero-Moser type

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1. Introduction

- Assume F is a bounded subset of $\mathbb{R}^N$ and $K(x,y)$ is a function that is continuous on $\overline{F} \times \overline{F}$ (with $\overline{F}$ the closure of F). Thus we have
$$\int_{F^2} |K(x,y)|^2 dxdy < \infty.$$
As a consequence the integral operator
$$(\mathcal{I}f)(x) := \int_F K(x,y)f(y)dy,$$
is a Hilbert-Schmidt operator on $\mathcal{H} := L^2(F, dx)$.
- Recall this entails that there exist two sets of pairwise orthogonal unit vectors $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$ such that
$$K(x,y) = \sum_{m=0}^{M} s_m f_m(x)\overline{g_m(y)}, \quad s_0 \geq s_1 \geq s_2 \geq \cdots > 0, \quad M \leq \infty,$$
with the so-called singular values s_m satisfying $\sum s_m^2 < \infty$.

- The self-adjoint operators $\mathcal{I}^*\mathcal{I}$ and $\mathcal{I}\mathcal{I}^*$ are then given by

$$\mathcal{I}^*\mathcal{I} = \sum_{m=0}^{M} s_m^2 g_m \otimes \overline{g_m}, \quad \mathcal{I}\mathcal{I}^* = \sum_{m=0}^{M} s_m^2 f_m \otimes \overline{f_m},$$

  so they are trace class and non-negative.
- Let us call an HS operator complete when it has trivial null space and dense range. Equivalently, the vectors $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$ are ONBs (orthonormal bases) for $\mathcal{H}$.
- It seems there is no useful general way to recognize a complete HS operator when you meet one.
- For large classes of special HS operators, however, completeness can be shown. The proofs only involve elementary Fourier analysis (S. R., 2013). These results apply in particular to the elliptic kernel functions of my survey talk, provided the parameters are suitably restricted.

- A long-standing goal is to reinterpret the $2N$ commuting A_{N-1} AΔOs $\mathcal{A}_{k,\delta}(x)$ from my survey talk as commuting self-adjoint operators on the Hilbert space

$$\mathcal{H}_A := L^2(F_A, dx),$$

$$F_A := \{-\pi/2r < x_N < \cdots < x_1 \leq \pi/2r\}.$$

  (They are at least formally self-adjoint on $\mathcal{H}_A$, by contrast to $A_{k,\delta}(x) = C(x)\mathcal{A}_{k,\delta}(x)C(x)^{-1}$.)
- Likewise, the 2 commuting BC_N AΔOs $\mathcal{A}_\delta(x)$ ought to be promoted to commuting self-adjoint operators on the Hilbert space

$$\mathcal{H}_B := L^2(F_B, dx),$$

$$F_B := \{0 < x_N < \cdots < x_1 \leq \pi/2r\}.$$

- To this end, we need 'only' show existence of an ONB of joint eigenfunctions with real eigenvalues.

- Under suitable restrictions on the parameters, the kernel functions from my survey talk give rise to complete HS integral operators $\mathcal{I}_\xi, \xi \in \mathbb{C}$, and $\mathcal{I}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, resp. Thus the operators

$$\mathcal{T}_\xi := \mathcal{I}_\xi \mathcal{I}_\xi^*, \quad \mathcal{T} := \mathcal{I}\mathcal{I}^*,$$

  are positive trace class operators.
- Crux: it can be expected that the $\mathcal{T}_\xi$- and $\mathcal{T}$-eigenvectors extend to meromorphic eigenfunctions of the AΔOs $\mathcal{A}_{k,\delta}$ and $\mathcal{A}_\delta$ with real eigenvalues.
- Reason: the AΔOs are formally self-adjoint and formally satisfy

$$[\mathcal{A}_{k,\delta}, \mathcal{T}_\xi] = 0, \quad [\mathcal{A}_\delta, \mathcal{T}] = 0,$$

  due to the kernel identities. Thus the eigenvector ONB of the trace class operators 'should' yield an ONB of joint eigenfunctions of the commuting AΔOs.

- This approach is easily understood and formally convincing, but a lot of analysis is needed to make it work. This involves in particular complex analysis to prove the meromorphy of the $\mathcal{T}$-eigenfunctions, and functional analysis to control dense domains for the AΔOs. (No general Hilbert space theory for AΔOs exists to date.)
- It can be expected that a similar approach applies to the nonrelativistic (PDO) case. A difficulty in this setting is that the eigenfunctions are (generically) not meromorphic. However, for the rank-one cases (i. e. Lamé and Heun), one can invoke Sturm-Liouville and Frobenius theory to push it through. This gives rise to a novel S_4 spectral invariance of Heun Hamiltonians (S. R., 2009). In this seminar we only supply some further information about the AΔO case.

2. The A_{N-1} case

- For the $N = 2$ (relativistic Lamé) case and special couplings, the 'expected' results for the AΔOs $\mathcal{A}_{\pm}$ were shown to hold true (without using kernel functions) some 15 years ago (S. R., 2003). Specifically, letting
$$b = (N_+ + 1)a_+ - N_- a_- \in (0, a_+ + a_-),$$
with
$$N_+, N_- \in \mathbb{N} := \{0, 1, 2, \ldots\}, \quad a_+/a_- \notin \mathbb{Q},$$
the Hilbert space $\mathcal{H} = L^2((-\pi/2r, \pi/2r], dx)$ has an ONB that consists of restrictions of meromorphic joint eigenfunctions with real eigenvalues to $(-\pi/2r, \pi/2r]$. A crucial ingredient in the proof is a coupled system of Bethe Ansatz equations.
- Specializing my recent results for the BC_1 case, this joint eigenfunction ONB admits an interpolation to any $b \in (0, a_+ + a_-)$. These BC_1 results hinge on suitable use of the BC_1 kernel function and are sketched below.

- For $N > 2$ there is work in progress; requiring once more $b \in (0, a_+ + a_-)$, there is circumstantial evidence for the conjecture that the joint eigenfunction ONB can be labelled by

$$n \in \mathbb{Z}^N_\geq \equiv \{n \in \mathbb{Z}^N \mid n_1 \geq \cdots \geq n_N\},$$

in such a way that when the minimum of the gaps $n_j - n_{j+1}$, $j = 1, \ldots, N-1$, tends to ∞ one has asymptotics proportional to

$$\sum_{\sigma \in S_N} \frac{C(x_\sigma)}{C(x)} \exp(2irn \cdot x_\sigma).$$

(If so, the commuting dual dynamics yield a factorized S-matrix.)
- For the cases $b = a_+$ and $b = a_-$ the joint eigenfunction ONB amounts to 'free fermions' ($\sim$ Schur polynomials), the AΔO-eigenvalues are obvious, and the eigenvalues for a modified HS family are explicitly known too (S. R., 2009).

3. The BC_1 case

- Here the 'initial' kernel identity reads

$$A_\delta(\gamma; x)\mathcal{S}(\sigma(\gamma); x, y) = A_\delta(\gamma'; y)\mathcal{S}(\sigma(\gamma); x, y), \quad \delta = +, -,$$

where

$$\gamma' \equiv -J\gamma,$$

$$\sigma(\gamma) \equiv -\frac{1}{4}\sum_{\mu=0}^{7}\gamma_\mu = -\frac{1}{4}\langle\zeta, \gamma\rangle, \quad \zeta \equiv (1, \ldots, 1),$$

and J can be viewed as the reflection associated with the highest E_8 root $\zeta/2$, i. e.,

$$J \equiv \mathbf{1}_8 - \frac{1}{4}\zeta \otimes \zeta.$$

- The BC_1 kernel function is given by

$$\mathcal{S}(t; x, y) \equiv \prod_{\delta_1, \delta_2 = +, -} G(\delta_1 x + \delta_2 y - ia + it),$$

with $G(z)$ the elliptic gamma function.

- For the AΔOs $A_{\pm}(\gamma; x)$ the relevant Hilbert space is the weighted L^2 space
$$\mathcal{H}_w \equiv L^2([0, \pi/2r], w_e(\gamma; x)dx).$$
- It is crucial to switch from this AΔO pair to the D_8-invariant AΔOs
$$\mathcal{A}_\delta(\gamma; x) = c_e(\gamma; x)^{-1} A_\delta(\gamma; x) c_e(\gamma; x),$$
which are formally self-adjoint on
$$\mathcal{H} = L^2([0, \pi/2r], dx),$$
for suitable γ (in particular for $\gamma \in \mathbb{R}^8$).
- They satisfy the kernel identity
$$\mathcal{A}_\delta(\gamma; x)\mathcal{K}(\gamma; x, y) = \mathcal{A}_\delta(\gamma'; -y)\mathcal{K}(\gamma; x, y),$$
with
$$\mathcal{K}(\gamma; x, y) \equiv \frac{\mathcal{S}(\sigma(\gamma); x, y)}{c_e(\gamma; x) c_e(\gamma'; -y)}.$$

- With further restrictions on γ, the kernel function $\mathcal{K}(\gamma; x, y)$ yields a complete HS integral operator $\mathcal{I}(\gamma)$ on $\mathcal{H}$. Requiring $\gamma \in \mathbb{R}^8$ from now on, it suffices to further restrict γ by

$$\gamma_\mu, \ \gamma'_\mu \in (-a, a), \ \ \sigma(\gamma) \in (0, a).$$

- With this restriction, we can show that the resulting eigenvector $\mathcal{H}$-ONB $f_n(\gamma)$, $n = 0, 1, 2, \ldots$, for the self-adjoint trace class operator $\mathcal{I}(\gamma)\mathcal{I}(\gamma)^*$ has the following features:
  - $f_n(\gamma)$ is the restriction to $[0, \pi/2r]$ of a meromorphic function $f_n(\gamma; x)$ with known pole locations depending only on γ;
  - Setting

$$a_s \equiv \min(a_+, a_-), \ a_l \equiv \max(a_+, a_-),$$

and assuming a_l is not a multiple of a_s, the functions $f_n(\gamma; x)$ are joint eigenfunctions of $\mathcal{A}_\pm(\gamma; x)$ with real eigenvalues.

- Consequence: With the above restrictions on $a_\pm$ and γ understood, the AΔOs give rise to commuting self-adjoint operators $\hat{\mathcal{A}}_\pm(\gamma)$ on $\mathcal{H}$ with discrete spectra.
- Further results include:
  • The definition of $\hat{\mathcal{A}}_\pm(\gamma)$ implies that the operators are invariant under D_8-transformations of γ.
  • For γ in the ball $\|\gamma\|_2 < a$ (with the origin deleted), the operators are isospectral under E_8-transformations. Generically, this yields 135 (=$|W(E_8)/W(D_8)|$) distinct isospectral operators.
  • For generic γ, we also get 64 distinct commuting HS operators.
  • The asymptotic behavior as $n \to \infty$ of the eigenfunctions $f_n(\gamma; x)$ is the same as that of an $\mathcal{H}$-ONB of functions $P_n(\gamma; x)/c_P(\gamma; x)$, with $P_n(\gamma; x)$ orthonormal polynomials; this relation also leads to detailed information on eigenvalue asymptotics.

Recent references re HS approach:

- Hilbert-Schmidt-operators vs. integrable systems of elliptic Calogero-Moser type. II. The A_{N-1} case: First steps, Comm. Math. Phys. **286** (2009), 659–680
- Hilbert-Schmidt-operators vs. integrable systems of elliptic Calogero-Moser type. III. The Heun case, SIGMA **5** (2009), 049, 21 pages
- On positive Hilbert-Schmidt operators, Integr. Equ. Oper. Theory, **75** (2013), 393–407
- Hilbert-Schmidt-operators vs. integrable systems of elliptic Calogero-Moser type. IV. The relativistic Heun (van Diejen) case, SIGMA **11** (2015), 004, 78 pages