On the spectra of complex Lamé operators

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Lamé operators

Let $\mathcal{E} = \mathbb{C}/\mathcal{L}$ be a general elliptic curve, with period lattice

$$\mathcal{L} = 2\mathbb{Z}\,\omega_1 + 2\mathbb{Z}\,\omega_3, \quad \mathrm{Im}\,\omega_3/\omega_1 > 0,$$

and let $\wp(z)$ be the corresponding Weierstrass' elliptic function,

$$\wp(z+\Omega) = \wp(z), \quad \Omega \in \mathcal{L},$$

satisfying

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

We study *complex Lamé operators* in $L^2(\mathbb{R})$ of the form

$$L = -\frac{d^2}{dx^2} + m(m+1)\omega^2\wp(\omega x + z_0), \tag{1}$$

with

$$m \in \mathbb{N}, \quad 2\omega \in \mathcal{L},$$

and $z_0 \in \mathbb{C}$ chosen such that

$$z = \omega x + z_0 \notin \mathcal{L}, \quad x \in \mathbb{R}.$$

Note that the potential $m(m+1)\omega^2\wp(\omega x + z_0)$ is regular and periodic with period 2, but in general complex-valued.

Viewed as an equation in $\mathbb{C}$, the solutions of the Lamé equation

$$-\frac{d^2\psi}{dz^2} + m(m+1)\wp(z)\psi = \lambda\psi$$

were described explicitly by Hermite and Halphen.

Solutions and spectrum for $m = 1$

For the $m = 1$ Lamé equation

$$-\frac{d^2\psi}{dz^2} + 2\wp(z)\psi = \lambda\psi, \quad \lambda = -\wp(k),$$

the solutions are given by

$$\psi(z,k) = \frac{\sigma(z+k)}{\sigma(z)\sigma(k)}\exp(-\zeta(k)z),$$

with $k \in \mathbb{C}$. (Here $\sigma(z)$ and $\zeta(z)$ are the Weierstrass σ- and ζ-function.) Due to the Floquet property

$$\psi(z+2,k) = \exp\left(2\eta k - 2\zeta(k)\omega\right)\psi(z,k),$$

with $\eta = \zeta(\omega)$, they remain bounded on the line $z = \omega x + z_0$, $x \in \mathbb{R}$, if and only if

$$u(k) := \mathrm{Re}[\eta k - \zeta(k)\omega] = 0.$$

It follows (from a result by Rofe-Beketov) that the corresponding values of $\lambda = -\omega^2\wp(k)$ constitute the spectrum of the $m = 1$ Lamé operator

$$L = -\frac{d^2}{dx^2} + 2\omega^2\wp(\omega x + z_0).$$

The problem is thus to study the zero level set of the real analytic function $u(k)$, $k \in \mathcal{E}^\times \equiv \mathcal{E} \setminus 0$.

Main results

Assuming the *non-degeneracy* conditions

$$\eta + \omega e_j \neq 0, \quad j = 1,2,3 \tag{2}$$

$u(k)$ is a Morse function on $\mathcal{E}^\times$. Assuming, in addition, that the level set $u(k) = 0$ is *non-singular*, i.e.

$$u(k^*) \neq 0, \quad k^* \text{ a critical point of } u(k), \tag{3}$$

we use Morse theory arguments to prove the following result.

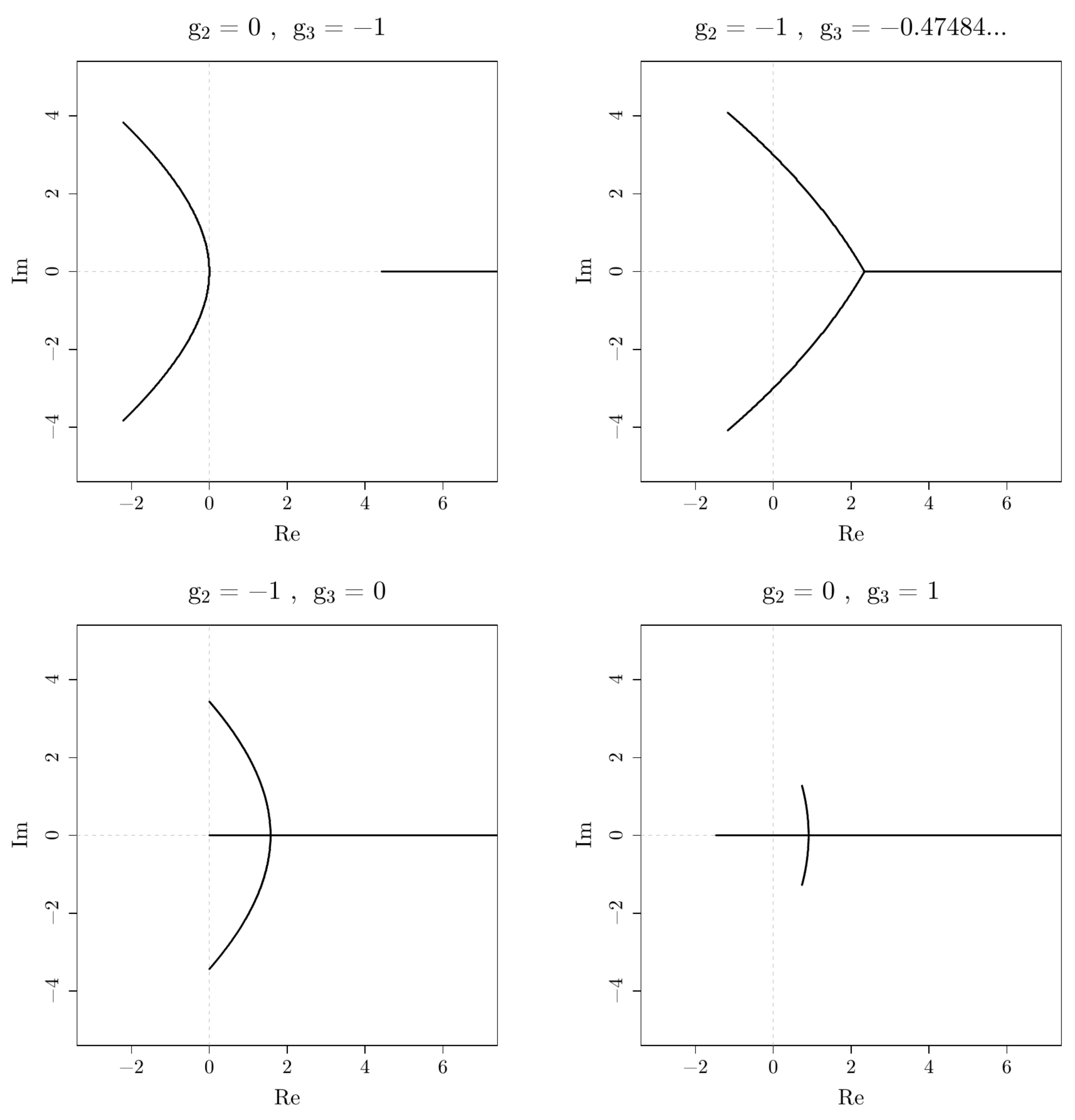
Theorem: *Under our non-degeneracy and non-singularity assumptions* (2) *and* (3)*, the spectrum of the $m = 1$ complex Lamé operator* (1) *consists of two regular analytic arcs. Precisely one arc extends to infinity and the remaining endpoints are $-\omega^2 e_j$, $j = 1,2,3$.*

Examples w/ rhombic period lattices and $m = 1$

We pay particular attention to rhombic period lattices:

$$\omega_1 \in (0,\infty), \quad \mathrm{Re}\,\omega_3 = \frac{\omega_1}{2}, \quad \mathrm{Im}\,\omega_3 \in (0,\infty).$$

Using the software *R*, we have plotted the spectrum of the complex Lamé operator, with $m = 1$ and $\omega = \omega_1 \in (0,\infty)$, in four rhombic cases, which exemplify the different types of spectra that occur.



In the lower left plot, we have the pseudo-lemniscatic case; and the upper left and lower right plots correspond to the two real forms of the equianharmonic curve. The top right plot displays the unique *exceptional ellitptic curve $\mathcal{E}^*$* for which the non-degeneracy conditions (2) are violated, with the value of the j-invariant $j^* \approx 243.797$.

Main results (cont.)

By analysing the non-degeneracy conditions (2) and non-singularity condition (3), we obtain the following result.

Theorem: *In the rhombic case, with $m = 1$ and $\omega = \omega_1$, the non-degeneracy conditions* (2) *are violated for exactly one exceptional elliptic curve $\mathcal{E}^*$, uniquely determined by the condition*

$$\eta_1 + \omega_1 e_1 = 0.$$

The corresponding spectrum has a tripod structure with three simple analytic arcs joined at $2\pi/3$ angles. The same curve is the bifurcation point for the non-singularity condition (3)*, separating the cases with intersecting and non-intersecting spectral arcs.*

Reference

Further details, including proofs of the above results and references to earlier literature on the subject, can be found in the following preprint:

W. H.-H., M. H. and A. V, *On the spectra of real and complex Lamé operators*, arXiv:1609.06247.