ESI Erwin Schrödinger International Institute for Mathematics and Physics

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Workshop on

Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics

March 20–24, 2017

ABSTRACTS OF TALKS

Fokko van de Bult (TU Delft)

Hypergeometric functions and integrals

ABSTRACT: This talk aims to give an introduction to elliptic hypergeometric functions. We will discuss the basic definitions and notations. We will consider the connections to ordinary (and basic) hypergeometric functions. And then we will discuss several identities satisfied by elliptic hypergeometric functions, such as summation formulas, integral evaluations, transformations and contiguous relations.

Dmitry Chicherin (Johannes Gutenberg University of Mainz)

Elliptic solutions of the Yang–Baxter equation

ABSTRACT: I will review the construction of the most general solution of the Yang–Baxter equation with a rank one symmetry algebra. It is an integral operator with the elliptic hypergeometric kernel. Then I will explain how to obtain many other solutions of the Yang–Baxter equation from it. I will consider the generating function of finite-dimensional elliptic solutions and the generating function of solutions which are finite-difference operators with elliptic coefficients. The construction is based on the representation theory of the elliptic modular double and it is closely related to lattice models with continuous spin variables.

Sergey Derkachov (Steklov Mathematical Institute, St. Petersburg)

Spin chains and Gustafson's integrals

ABSTRACT: The Gustafson's integrals are the multidimensional generalization of the classical Mellin–Barnes integrals. We show that some of these integrals arise from relations between matrix elements in the Sklyanin's representation of Separated Variables in the spin chain models.

Ilmar Gahramanov (Max Planck Institute for Gravitational Physics)

Integrable lattice spin models from supersymmetric gauge theories

ABSTRACT: Recently, there has been observed several connections of integrable models to supersymmetric gauge theories and special functions of hypergeometric type. One of such connections is a correspondence between supersymmetric quiver gauge theories and integrable lattice models such that the integrability emerges as a manifestation of supersymmetric dualities. Particularly, partition functions of supersymmetric quiver gauge theories with four supercharges on different manifolds can be identified with partition functions of two-dimensional exactly solvable statistical models. This relationship has led to construction of new exactly solvable models of statistical mechanics, namely the Yang–Baxter equation was solved in terms of new special functions. In the talk I will present some new solutions of the Yang–Baxter equation and related basic and hyperbolic hypergeometric integral identities.

Masahiko Ito (Tokyo Denki University)

A determinant formula associated with the elliptic hypergeometric integrals of type BC_n

ABSTRACT: In the paper [Aomoto–Ito, Adv. Math. 221(2009)], it was shown that a determinant, whose entries are the Jackson integrals of type BC_n (the BC_n type q-Selberg integrals), is expressed explicitly as a product of q-gamma functions. This determinant is regarded as a Wronskian, which gives a criterion for the linearly independence of the solutions of the q-deference system that the Jackson integrals satisfy. From the product expression it is understandable that the determinant is non-degenerate. In this talk, I will present an elliptic analog of the determinant formula of the above situation. This means that the determinant whose entries are the BC_n type elliptic Selberg integrals is expressed explicitly as a product of elliptic gamma functions. For this purpose I will show the elliptic Lagrange interpolation functions, which play important roles in both definition of the determinant and proof of the product formula. (This is a joint work with M. Noumi.)

Makoto Katori (Chuo University)

Elliptic Dyson models

ABSTRACT: Stochastic analysis on interacting particle systems is important to provide useful models describing equilibrium and non-equilibrium phenomena studied in statistical physics. Determinantal process is a stochastic system of interacting particles which is integrable in the sense that all spatio-temporal correlation functions are given by determinants controlled by a single function called the spatio-temporal correlation kernel. Since the generating function of correlation functions is given by the Laplace transform of probability density, the stochastic integrability of time-evolution system is proved by showing that the Laplace transform of any multi-time joint probability density is expressed by a spatio-temporal Fredholm determinant associated with the correlation kernel. The purpose of this talk is to present a new class of determinantal processes in which the interactions between particles are described by elliptic functions. A classical example of determinantal processes is Dyson's Brownian motion model with parameter $\beta = 2$, which is a dynamical version of the eigenvalue statistics of random matrices in the Gaussian unitary ensemble (GUE), and it has been simply called the Dyson model. We extend the Dyson model to the elliptic-function-level in this talk. We use the notion of martingales in probability theory and the elliptic determinant evaluations of the Macdonald denominators of reduced affine root systems given by Rosengren and Schlosser (2006).

Hee-Cheol Kim (Perimeter Institute)

Duality domain walls in 5d supersymmetric gauge theories

ABSTRACT: Supersymmetric gauge theories in 5-dimensions enjoy interesting dualities inherited from the symmetries of CFTs at UV fixed point. I will propose a description of duality domain walls interpolating different dual gauge theories at low energy. I will show that the partition function of the system with the duality domain wall is an elliptic Fourier transform which converts a Nekrasov instanton partition function of a gauge theory into that of its dual gauge theory. In fact, the Nekrasov partition function turns out to be an eigenfunction of an elliptic integral equation.

Hitoshi Konno (Tokyo University of Marine Sciences and Technology)

Elliptic Weight Functions and Finite Dimensional Representations of Elliptic Quantum Groups

ABSTRACT: In the first half we present a simple rule of deriving the elliptic weight functions associated with $\widehat{\mathfrak{sl}}_N$, which play an important role in the integral solutions to the face type (i.e. dynamical) elliptic q-KZ equations. The $\widehat{\mathfrak{sl}}_2$ case coincides with the one obtained by Felder, Tarasov and Varchenko. Our derivation is based on the infinite-dimensional representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. In the second half we discuss finite-dimensional representations of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ on the Gelfand–Tsetlin basis following the idea by Gorbounov–Rimányi–Tarasov–Varchenko. The change of basis matrix from the standard one to the Gelfand-Tsetlin basis is given by a specialization of the elliptic weight functions. The action of the elliptic currents of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ is described in a combinatorial way associated with a coordinate of the partial flag variety. The trigonometric limit is quite similar to the one appearing in the geometric representations of the quantum affine algebras on the equivariant K-theory obtained by Nakajima and Vasserot. These results suggests that the elliptic weight functions should be identified with the elliptic stable envelopes constructed by Aganagic and Okounkov, and furthermore the finite-dimensional representations could be interpreted as possible geometric representations on the equivariant elliptic cohomology.

Masatoshi Noumi (University of Kobe)

Discrete Painlevé equations and special functions

ABSTRACT: This lecture will be a survey on the role of affine Weyl groups in the theory of discrete Painlevé equations. The main focus will be on the discrete Painlevé equation of type $E8$ with elliptic coefficients, and its special solutions expressible in terms of elliptic hypergeometric functions.

Andrei Okounkov (Columbia University)

Elliptic stable envelopes

ABSTRACT: This will be an introduction to the concept and the applications of elliptic stable envelopes introduced in joint work with Mina Aganagic.

Eric Rains (California Institute of Technology)

The monodromy of an elliptic difference equation

ABSTRACT: One particularly important tool for understanding Painlevé-type equations is the Riemann–Roch correspondence, i.e., the functor taking a Fuchsian differential equation to its monodromy. I'll discuss an elliptic analogue of this correspondence, as well as some of its consequences for elliptic special functions.

Shlomo M. Razamat (Rutgers University)

$\mathcal{N} = 1$ SCFTs, dualities, and integrable models

ABSTRACT: We will consider $\mathcal{N} = 1$ SCFTs in four dimensions obtained from compactifications of six dimensional models. We will observe that the supersymmetric index of the SCFTs is expected to be naturally expressed in terms of eigenfunctions of generalizations of Ruijsenaars–Schneider models. We will comment on properties of these eigenfunctions and on some open problems.

Hjalmar Rosengren (Chalmers University of Technology)

Basic hypergeometry and biorthogonal functions related to supersymmetric dualities

ABSTRACT: Dolan and Osborn discovered that elliptic hypergeometric integrals can be constructed from four-dimensional supersymmetric quantum field theories. There are similar relations between basic hypergeometric functions and three-dimensional theories. We will discuss some recent work on basic hypergeometric integrals and biorthogonal rational functions that was motivated by connections to quantum field theory. The talk is partly based on joint work with Ilmar Gahramanov.

Simon Ruijsenaars (University of Leeds)

Quantum integrable systems of elliptic Calogero–Moser type

ABSTRACT: We present an introduction to the non-relativistic (differential operator) and relativistic (difference operator) N-particle elliptic Calogero–Moser systems, restricting attention to the systems associated with A_{N-1} and BC_N. They have three distinct avatars, connected by similarities involving a generalised Harish–Chandra c-function. Each of the resulting three families of commuting operators has its pros and cons. The c-function is a product of elliptic gamma functions in the relativistic case and of elliptic theta functions in the non-relativistic case. A brief sketch of the state of the art regarding eigenfunctions and so-called kernel functions is included.

Simon Ruijsenaars (University of Leeds)

Hilbert–Schmidt integral operators vs. systems of elliptic Calogero–Moser type

ABSTRACT: We explain how kernel functions can be used as a tool to associate well-defined Hilbert space operators to the commuting differential/difference operators arising in the N-particle systems of elliptic non-relativistic/relativistic Calogero–Moser type. We also sketch some results for the relativistic Heun (aka BC_1 or van Diejen) system, including a spectral invariance under the E_8 Weyl group.

Michael J. Schlosser (University of Vienna)

Elliptic hypergeometric combinatorics

ABSTRACT: In this introductory lecture I will exhibit some connections between elliptic hypergeometric series and combinatorics. A central theme here is combinatorial enumeration using elliptic weight functions. For instance, (suitable) "elliptic enumeration" of lattice paths leads to a closed form elliptic generalization of binomial coefficients. Convolution of these elliptic binomial coefiicients yields the celebrated Frenkel–Turaev ${}_{10}V_9$ summation, an identity which is fundamental in the theory of elliptic hypergeometric series. Similarly, by suitably introducing elliptic weights in the respective classical models (such as rook theory), one can obtain elliptic generalizations of various special combinatorial numbers (sequences), including the Stirling numbers of the first and second kinds. As time allows my talk will also touch on closely connected topics (which are all useful tools in *Elliptic Combinatorial Analysis*) such as elliptic determinants, elliptic commuting variables, and elliptic Taylor expansions.

Junichi Shiraishi (University of Tokyo)

Some conjectures about the Macdonald polynomials of type C

ABSTRACT: Some explicit formulas for the Macdonald polynomials of type C are given. The first is a four-fold summation formula for the Macdonald polynomials of type C_2 with general two row partitions. The second is a sixfold summation formula for the Macdonald polynomials of type C_3 for the rectangle diagrams with three rows. The last is an expansion formula for the Macdonald polynomials of type A_5 with some kind of folding in terms of the Macdonald polynomials of type C_3.

Vyacheslav P. Spiridonov (Joint Institute for Nuclear Research & NRU HSE)

Applications of the elliptic hypergeometric integrals

ABSTRACT: We outline the most important known applications of the elliptic hypergeometric integrals. First, we describe an integral analogue of the Bailey chains techniques using the elliptic beta integral. The corresponding Bailey lemma generates the star-triangle relation, which can be interpreted as a Coxeter relation for a permutation group generators realized by integral operators. Second, we show how this result can be used for building integral operator solutions of the Yang–Baxter equation. Finally, we discuss an interpretation of elliptic hypergeometric integrals as superconformal indices of four-dimensional field theories and try to summarize an impact of this interrelation both on physical and mathematical questions.

Jasper Stokman (University of Amsterdam)

Elliptic integrable structures arising from harmonic analysis on affine symmetric spaces

ABSTRACT: In the past two decades interesting classes of elliptic integrable structures and elliptic special functions have been obtained by combining ideas from conformal field theory with ideas from representation theory of loop groups G. The integrable structures are Knizhnik–Zamolodchikov–Bernard (KZB) equations and KZB heat equations. The KZB equations are explicitly described in terms of Felder's universal elliptic solution of the dynamical Yang–Baxter equation. An important special case of the KZB heat equation is the non-stationary Schrödinger equation of the elliptic Calogero–Moser system.

From the physics perspective, the KZB equations can be interpreted as the compatibility conditions for certain correlation functions when (quasi-)periodic boundary conditions are imposed. From the mathematics perspective, representation theory of G can be thought of as harmonic analysis on the affine symmetric space $G \times G/\operatorname{diag}(G)$, with $\operatorname{diag}(G)$ the diagonal embedding of G in $G \times G$. In this talk I will explain how reflecting integrable boundary conditions naturally arise by considering harmonic analysis on the affine symmetric space G/H, with H the fixed point group of the Chevalley involution on G.

I will explain how this perspective leads to boundary versions of the KZB equations. These can be described in terms of explicit universal elliptic solutions of the dynamical reflection equation. I will also give a boundary version of the KZB heat equation. It naturally encompasses the non-stationary Schrödinger equation of Inozemtsev's integrable system.

This is based on unpublished joint work with Nicolai Reshetikhin.

Yi Sun (Columbia University)

Affine Macdonald conjectures and special values of Felder–Varchenko functions

ABSTRACT: I will explain how to refine the statement of the denominator and evaluation conjectures for affine Macdonald polynomials proposed by Etingof–Kirillov Jr. and to prove the first non-trivial cases of these conjectures. Our method applies recent work of the speaker to relate these conjectures for $U_q(\widehat{\mathfrak{sl}}_2)$ to evaluations of certain theta hypergeometric integrals defined by Felder–Varchenko. We then evaluate the resulting integrals, which may be of independent interest, by well-chosen applications of the elliptic beta integral of Spiridonov.

These results are joint work with E. Rains and A. Varchenko.

Takashi Takebe (NRU Higher School of Economics)

Q-operators for higher spin eight vertex models

ABSTRACT: We construct the Q-operator for generalised eight vertex models associated to higher spin representations of the Sklyanin algebra, following Baxter's original method. As an application, we prove the sum rule for the Bethe roots.

Kouichi Takemura (University of Leeds & Chuo University)

Degenerations of Ruijsenaars–van Diejen operator, q-Painlevé equations and q-Heun equations

ABSTRACT: It is known that the Painlevé VI is obtained by monodromy preserving deformation of some linear differential equations, and the Heun equation is obtained by a specialization of the linear differential equations. We investigate degenerations of the Ruijsenaars–van Diejen difference opearators and show difference analogues of the Painlevé–Heun correspondence. Here q-Painlevé equations of types $E_7^{(1)}$, $E_6^{(1)}$ and $D_5^{(1)}$ (symmetry) appear. The eigenvalue problems of each degenerated Ruijsenaars–van Diejen difference operator may be regarded as q-Heun equations. This talk is mainly based on `arXiv:1608.07265`.

S. Ole Warnaar (University of Queensland)

Known and unknown conjectures from the theory of elliptic hypergeometric functions

ABSTRACT: I will discuss some known conjectures from the theory of elliptic hypergeometric functions, as well as some future ones.

Junya Yagi (University of Warsaw)

String theory and integrable lattice models

ABSTRACT: I discuss a string theoretic approach to integrable lattice models. This approach provides a unified perspective on various important notions, such as the Belavin model, the Jimbo–Miwa–Okado model, the Bazhanov–Sergeev model, Felder's elliptic quantum groups, the vertex-face correspondence, the elliptic modular double, and an elliptic lift of the relation between the chiral Potts model and the six-vertex model. Furthermore, it relates these notions to four-dimensional supersymmetric field theories and their surface operators.

Yashiko Yamada (Kobe University)

q-Garnier system and its autonomous limit

ABSTRACT: We will study the q-Garnier system from geometric points of view. The autonomous limit of the q-Garnier system will be identified with certain generalization of the QRT (Quispel–Roberts–Thompson) integrable mapping to hyperelliptic curves.

Meesue Yoo (Sungkyunkwan University)

Elliptic rook and file numbers

ABSTRACT: In this talk, we construct elliptic analogues of the rook numbers and file numbers by attaching elliptic weights to the cells in a board. We show that our elliptic rook and file numbers satisfy elliptic extensions of corresponding factorization theorems which in the classical case was established by Goldman, Joichi and White and by Garsia and Remmel in the file number case. This factorization theorem can be used to define elliptic analogues of various kinds of Stirling numbers of the first and second kind, and Abel numbers. We also give analogous results for matchings of graphs, elliptically extending the result of Haglund and Remmel.

This is joint work with Michael Schlosser.

Andrei Zotov (Steklov Mathematical Institute, Moscow)

Quantum and classical counterparts of quantum-classical correspondence in integrable systems

ABSTRACT: We review the quantum-classical correspondence between inhomogeneous quantum spin chains (Gaudin models) and classical integrable many-body systems of Ruijsenaars (Calogero) type. Then the classical-classical and quantum-quantum analogues of the correspondence are discussed.

Wadim Zudilin (University of Newcastle)

Elliptic dilogarithm and Mahler measures

ABSTRACT: The main theme of the talk is a (crucial) involvement of the elliptic dilogarithm in a (conjectural!) evaluation of the Mahler measure of 6-variable polynomial $x_0 + x_1 + \cdots + x_5$. The intermediate topics (including short random walks) and intermediate Mahler measures will be outlined. The talk is based on joint work in progress with Armin Straub.
