Quantum and classical counterparts of quantum-classical correspondence in integrable systems

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ESI, Vienna, March 20–24

Dualities in integrable systems

Ruijsenaars duality (or p-q or bispectral or action-angle): interchanges coordinates of particles and action variables in Calogero-Ruijsenaars models

Spectral duality (or rank-size or $(\mathfrak{gl}_N,\mathfrak{gl}_M)$): roughly speaking interchanges rank of group and number of sites (marked points) in spin chains and/or Gaudin models.

Elliptic versions of these dualities are studied but there are no good final answers.

QC - relates classical many-body systems of Ruijsenaars (Calogero) type and quantum spin chains (Gaudin models).

Quantum-classical duality:

On the quantum side, consider the inhomogeneous $GL(N)$-based generalized spin chain of XXX type with a formal Planck's constant $\hbar$ on n sites with inhomogeneity parameters q_i and vector representations at each site. Let us impose twisted boundary conditions with the twist matrix $\mathbf{g} = \mathrm{diag}(g_1, \ldots, g_N)$, with the generating function of commuting integrals of motion (the transfer matrix) depending on the spectral parameter z being of the form

$$\mathbf{T}(z) = \mathrm{tr}_0\Big(\mathbf{g}^{(0)}\widetilde{\mathbf{R}}_{0n}(z-q_n)\ldots\widetilde{\mathbf{R}}_{02}(z-q_2)\widetilde{\mathbf{R}}_{01}(z-q_1)\Big) = \mathrm{tr}\,\mathbf{g}\cdot\mathbf{I}+\sum_{j=1}^{n}\frac{\eta\mathbf{H}_j}{z-q_j}\,.$$

$$\widetilde{\mathbf{R}}_{ij}(z) = \mathbf{I} + \frac{\eta}{z}\mathbf{P}_{ij}\,, \quad \mathbf{P}_{ij} = \sum_{a,b} e_{ab}^{(i)} e_{ba}^{(j)}\,.$$

The residues $\mathbf{H}_j$ are (non-local) Hamiltonians of the spin chain. Their eigenvalues

$$\frac{1}{\eta} H_i = g_1 \prod_{k=1}^{n} \frac{q_i - q_k + \eta}{q_i - q_k} \prod_{\gamma=1}^{n_1} \frac{q_i - \mu_\gamma^1 - \eta}{q_i - \mu_\gamma^1}$$

depend on a solution $\Big\{ \{\mu_i^1\}_{n_1}, \ldots, \{\mu_i^{N-1}\}_{n_{N-1}} \Big\}$ of Bethe equations:

$$BE_1: \quad g_1 \prod_{k=1}^{n} \frac{\mu_\beta^1 - q_k + \eta}{\mu_\beta^1 - q_k} = g_2 \prod_{\gamma \neq \beta}^{n_1} \frac{\mu_\beta^1 - \mu_\gamma^1 + \eta}{\mu_\beta^1 - \mu_\gamma^1 - \eta} \prod_{\gamma=1}^{n_2} \frac{\mu_\beta^1 - \mu_\gamma^2 - \eta}{\mu_\beta^1 - \mu_\gamma^2}$$

$$BE_b: \quad g_b \prod_{\gamma=1}^{n_{b-1}} \frac{\mu_\beta^b - \mu_\gamma^{b-1} + \eta}{\mu_\beta^b - \mu_\gamma^{b-1}} = g_{b+1} \prod_{\gamma \neq \beta}^{n_b} \frac{\mu_\beta^b - \mu_\gamma^b + \eta}{\mu_\beta^b - \mu_\gamma^b - \eta} \prod_{\gamma=1}^{n_{b+1}} \frac{\mu_\beta^b - \mu_\gamma^{b+1} - \eta}{\mu_\beta^b - \mu_\gamma^{b+1}}$$

where n_a denotes the number of Bethe roots at the a-th level

On the classical side, consider the RS model with coupling constant η and the number of particles, n, equal to the number of sites of the $GL(N)$ spin chain. The Lax matrix of the model is $n \times n$:

$$L^{\mathsf{RS}}_{ij}(\{\dot{q}_i\}_n, \{q_i\}_n, \hbar) = \frac{\eta\varepsilon\, e^{\varepsilon p_j}}{q_i - q_j + \eta\varepsilon} \prod_{k\neq j}^{n} \frac{q_j - q_k + \eta\varepsilon}{q_j - q_k} = \frac{\eta\, \dot{q}_j}{q_i - q_j + \eta\varepsilon}\,, \quad (1)$$

where $\{q_i\}_n$, $\{\dot{q}_i\}_n$ are coordinates and velocities, and ε – inverse light speed.

$$H^{\mathsf{RS}} = \operatorname{tr} L^{\mathsf{RS}} = \sum_{j=1}^{n} e^{\varepsilon p_j} \prod_{k\neq j}^{n} \frac{q_j - q_k + \eta\varepsilon}{q_j - q_k}$$

$$\dot{q}_j = \frac{\partial H}{\partial p_i}^{\mathsf{RS}} = \varepsilon e^{\varepsilon p_j} \prod_{k\neq j}^{N} \frac{q_j - q_k + \eta\varepsilon}{q_j - q_k}$$

The claim is that under the substitution

$$\dot{q}_j = \frac{1}{\eta} H_j\Big(\{q_i\}_n; \{\mu_i^1\}_{n_1}, \dots, \{\mu_i^{N-1}\}_{n_{N-1}}\Big)\,, \quad j = 1\,, \dots, n\,, \tag{2}$$

where the set of μ_i^a's is any solution of the nested BE for the spin chain, the eigenvalues of the Lax matrix are

$$\Big(\underbrace{g_1\,, \cdots\,, g_1}_{n-n_1}, \underbrace{g_2\,, \cdots\,, g_2}_{n_1-n_2}, \cdots\,, \underbrace{g_{N-1}\,, \cdots\,, g_{N-1}}_{n_{N-2}-n_{N-1}}, \underbrace{g_N\,, \cdots\,, g_N}_{n_{N-1}}\Big). \tag{3}$$

i.e.

$$\det\left[L^{\mathrm{RS}}\left(\frac{1}{\eta}\left\{H_j\right\}_n, \left\{q_j\right\}_n, \eta\varepsilon\right)\Big|_{BE} - \lambda\right] = \prod_{a=1}^{N}(g_a - \lambda)^{M_a}, \tag{4}$$

where $M_1 = n - n_1$, $M_a = n_{a-1} - n_a$ $(2 \leq a \leq N)$

classical Calogero – quantum Gaudin case: $\varepsilon \to 0$

$$L_{ij}^{\mathsf{CM}} = \lim_{\varepsilon\to 0} \frac{L_{ij}^{\mathsf{RS}} - \delta_{ij}}{\varepsilon} = \delta_{ij}\Big(p_i + \eta \sum_{k\neq i} \frac{1}{q_i - q_k}\Big) + \eta \frac{1-\delta_{ij}}{q_i - q_j}$$

$$\mathbf{H}_i^{\mathsf{G}} = \sum_a v_a e_{aa}^{(i)} + \sum_{j\neq i}^{n} \frac{\eta}{q_i - q_j} \sum_{ab}^{N} e_{ab}^{(i)} e_{ba}^{(j)} = \lim_{\varepsilon\to 0} \frac{\mathbf{H}_i(\{q_i\}_n, \mathbf{g} = e^{\varepsilon v}, \varepsilon\eta) - \varepsilon\eta C_1^{(i)}}{\eta\varepsilon^2}$$

where $C_1^{(i)} = \sum_a e_{aa}^{(i)}$.

Idea of the proof (with A.Gorsky and A.Zabrodin):

let us introduce the following pair of matrices:

$$\mathcal{L}_{ij}(\{x_i\}_N, \{y_i\}_M, g) = \frac{g\,\hbar}{x_i - x_j + \hbar} \prod_{k\neq j}^{N} \frac{x_j - x_k + \hbar}{x_j - x_k} \prod_{\gamma=1}^{M} \frac{x_j - y_\gamma}{x_j - y_\gamma + \hbar}\,, \quad i\,, j = 1\,, \ldots\,, N$$

and

$$\widetilde{\mathcal{L}}_{\alpha\beta}(\{y_i\}_M, \{x_i\}_N, g) = \frac{g\,\hbar}{y_\alpha - y_\beta + \hbar} \prod_{\gamma\neq\beta}^{M} \frac{y_\beta - y_\gamma - \hbar}{y_\beta - y_\gamma} \prod_{k=1}^{N} \frac{y_\beta - x_k}{y_\beta - x_k - \hbar}\,, \;\; \alpha, \beta = 1\,, \ldots\,, M\,,$$

QC duality is based on algebraic relation between $\mathcal{L}$ and $\widetilde{\mathcal{L}}$:

$$\det_{N\times N}\Big(\mathcal{L}\,(\{x_i\}_N, \{y_i\}_M, g) - \lambda\Big) = (g-\lambda)^{N-M} \det_{M\times M}\Big(\widetilde{\mathcal{L}}\,(\{y_i\}_M, \{x_i\}_N, g) - \lambda\Big)$$

It provides an alternative (to the nested Bethe ansatz) method for computation of spectra of the spin chains. Namely, the spectrum of the quantum transfer matrix for the inhomogeneous $\mathfrak{gl}_n$-invariant XXX spin chain on N sites with twisted boundary conditions can be found in terms of velocities of particles in the rational N-body Ruijsenaars-Schneider model. The possible values of the velocities are to be found from intersection points of two Lagrangian submanifolds in the phase space of the classical model. One of them is the Lagrangian hyperplane corresponding to fixed coordinates of all N particles and the other one is an N-dimensional Lagrangian submanifold obtained by fixing levels of N classical Hamiltonians in involution. The latter are determined by eigenvalues of the twist matrix.

To find the spectrum of chain we need to find velocities of RS particles for a fixed set of eigenvalues of L^{RS}.

The simplest examples show that there are more solutions then Bethe ansatz provides.

It happens because the statement of QC-duality is valid for all $N+1$ SUSY chains with groups $GL(a|b)$, $a+b=N$ (with Z.Tsuboi and A.Zabrodin):

$$GL(N|0)\,, GL(N-1|1)\,, \ldots, GL(0|N)$$

quantum-classical

? quantum-quantum

? classical-classical

The QQ case is the Matsuo-Cherednik type correspondence between the quantum Knizhnik-Zamolodchikov equations associated with $GL(N)$ and n-particle quantum Ruijsenaars model, with n being not necessarily equal to N.

qKZ-Ruijsenaars correspondence:

The quantum Knizhnik-Zamolodchikov (qKZ) equations

$$e^{\eta\hbar\partial_{x_i}}\Big|\Phi\Big\rangle = \mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle, \qquad i = 1, \ldots, n$$

$$\mathbf{K}_i^{(\hbar)} = \mathbf{R}_{i\,i-1}(x_i - x_{i-1} + \eta\hbar)\ldots\mathbf{R}_{i1}(x_i - x_1 + \eta\hbar)\mathbf{g}^{(i)}\mathbf{R}_{in}(x_i - x_n)\ldots\mathbf{R}_{i\,i+1}(x_i - x_{i+1})$$

$$\mathbf{R}_{ij}(x) = \frac{x\mathbf{I} + \eta\mathbf{P}_{ij}}{x + \eta}, \quad \mathbf{R}_{ij}(x)\mathbf{R}_{ji}(-x) = \mathrm{id}$$

Solutions can be found in the form

$$\Big|\Phi\Big\rangle = \sum_{\sigma\in S_n}\Phi_\sigma\Big|e_\sigma\Big\rangle, \qquad \Big|e_\sigma\Big\rangle = e_{\sigma(1)}\otimes e_{\sigma(2)}\otimes\ldots\otimes e_{\sigma(n)},$$

where e_a are standard basis vectors in $V = \mathbb{C}^N = \bigoplus_{a=1}^N \mathbb{C}e_a$ and S_n is the symmetric group.

There is a natural weight decomposition of the Hilbert space of the spin chain:

$$\mathcal{V} = V^{\otimes n} = \bigoplus_{M_1,\ldots,M_N} \mathcal{V}(\{M_a\})$$

defined by operators

$$\mathbf{M}_a = \sum_{l=1}^{n} e_{aa}^{(l)}\,, \qquad [\mathbf{M}_a, \mathbf{M}_b] = [\mathbf{H}_i, \mathbf{M}_a] = 0$$

The basis vectors in $\mathcal{V}(\{M_a\})$ are $\Big|J\Big\rangle = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n}$, where the number of indices j_k such that $j_k = a$ is equal to M_a for all $a = 1, \ldots, N$. Important property:

$$\sum_{i=1}^{n} \mathbf{H}_i = \sum_{i=1}^{n} \mathbf{g}^{(i)} = \sum_{a=1}^{N} g_a \mathbf{M}_a$$

Statement: Let $\Big|\Phi\Big\rangle = \sum_J \Phi_J \Big|J\Big\rangle$ be a solution of qKZ in weight subspace $\mathcal{V}(\{M_a\})$. Then for $E = \sum_{a=1}^{N} M_a g_a$ and

$$\Psi = \sum_J \Phi_J = \Big\langle \Omega \Big| \Phi \Big\rangle, \quad \Big\langle \Omega \Big| = \sum_J \Big\langle J \Big|$$

we have

$$\sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \Psi(x_1, \ldots, x_i + \eta\hbar, \ldots, x_n) = E\Psi(x_1, \ldots, x_n).$$

The proof is based on the property of $\mathbf{M_a}$, relation $\widetilde{\mathbf{R}}(x) = \frac{x+\eta}{x} \mathbf{R}(x) = \mathbf{I} + \frac{\eta}{x} \mathbf{P}$ and $\Big\langle \Omega \Big| \mathbf{P}_{ij} = \Big\langle \Omega \Big|$. Therefore, $\Big\langle \Omega \Big| \mathbf{R}_{ij}(x) = \Big\langle \Omega \Big|$ and $\Big\langle \Omega \Big| \mathbf{K}_i^{(\hbar)} = \Big\langle \Omega \Big| \mathbf{K}_i^{(0)}$.

$$e^{\eta\hbar\partial_{x_i}}\left\langle\Omega\middle|\Phi\right\rangle = e^{\eta\hbar\partial_{x_i}}\Psi = \left\langle\Omega\middle|\mathbf{K}_i^{(\hbar)}\middle|\Phi\right\rangle = \left\langle\Omega\middle|\mathbf{K}_i^{(0)}\middle|\Phi\right\rangle.$$

Therefore, multiplying by $\prod_{j\neq i}^{n}\frac{x_i-x_j+\eta}{x_i-x_j}$ and summing over i, we get:

$$\sum_{i=1}^{n}\left(\prod_{j\neq i}^{n}\frac{x_i-x_j+\eta}{x_i-x_j}\right)e^{\eta\hbar\partial_{x_i}}\Psi \;=\; \sum_{i=1}^{n}\prod_{j\neq i}^{n}\frac{x_i-x_j+\eta}{x_i-x_j}\left\langle\Omega\middle|\mathbf{K}_i^{(0)}\middle|\Phi\right\rangle$$

$$= \sum_{i=1}^{n}\left\langle\Omega\middle|\mathbf{H}_i\middle|\Phi\right\rangle \;=\; \sum_{i=1}^{n}\left\langle\Omega\middle|\mathbf{g}^{(i)}\middle|\Phi\right\rangle \;=\; \sum_{a=1}^{N}g_a\left\langle\Omega\middle|\mathbf{M}_a\middle|\Phi\right\rangle \;=\; \left(\sum_{a=1}^{N}g_aM_a\right)\Psi.$$

A natural conjecture is that Ψ is the common eigenfunction for all higher Ruijsenaars Hamiltonians $\widehat{\mathcal{H}}_k$ with the eigenvalues $E_k = \sum_a M_a g_a^k$.

Classical-classical version

What is the classical analogue of the (q)KZ equations?

How it is related to Lax equations?

Which constructions work in elliptic case?

1. R-matrix valued Lax pairs (classical KZ equations with spectral parameter)

2. Chen-Lee-Pereira like linear problems for Calogero-Ruijsenaars models

3. Gaudin-Schlesinger form of Calogero-Moser model

R-matrix valued Lax pair for CM model (with A.Levin and M.Olshanetsky):

$$\mathcal{L} = \sum_{a,b=1}^{n} E_{ab} \otimes \mathcal{L}_{ab}\,, \quad \mathcal{L}_{ab} = \delta_{ab} p_a\, 1_a \otimes 1_b + \nu(1-\delta_{ab}) R^{\hbar}_{ab}\,, \quad R^{\hbar}_{ab} = R^{\hbar}_{ab}(q_a - q_b)$$

$$\mathcal{M}_{ab} = \nu \delta_{ab} d_a + \nu(1-\delta_{ab}) F^{\hbar}_{ab} + \nu \delta_{ab}\, \mathcal{F}^0\,, \quad F^{\hbar}_{ab} = \partial_{q_a} R^{\hbar}_{ab}(q_a - q_b)\,,$$

where $d_a = -\sum\limits_{c:\, c\neq a}^{n} F^0_{ac}\,, \quad F^0_{ac} = F^{\hbar}_{ac}\,|_{\hbar=0}\,, \quad \mathcal{F}^0 = \sum\limits_{b,c:\, b>c}^{n} F^0_{bc}\,.$

When $N=1$ it is the Krichever's Lax pair with spectral parameter.

$\mathcal{F}^0$ term in case $R^z_{ij}(q) = P_{ij}\phi(z, q_i - q_j)$ and $q_i = i/n$ is the Inozemtsev-Haldane-Shastry Hamiltonian $\sum_{i,j} P_{ij} \wp(\frac{i-j}{n})$.

Painlevé equations: $\partial_\tau \mathcal{L} - \partial_\hbar \mathcal{M} = [\mathcal{L}, \mathcal{M}]$ due to $2\pi\imath \partial_\tau R^{\hbar}_{12}(z) = \partial_z \partial_\hbar R^{\hbar}_{12}(z)$.

Matrix generalization of elliptic functions

$$\phi(\eta, z) = \begin{cases} 1/\eta + 1/z\,, \\ \coth(\eta) + \coth(z)\,, \\ \dfrac{\vartheta_1'(0)\vartheta_1(\eta+z)}{\vartheta_1(\eta)\vartheta_1(z)}\,. \end{cases} \qquad E_1(z) = \begin{cases} 1/z\,, \\ \coth(z)\,, \\ \dfrac{\vartheta_1'(z)}{\vartheta_1(z)} \end{cases}$$

$$\phi(\hbar, q_{12})\phi(\eta, q_{23}) = \phi(\hbar-\eta, q_{12})\phi(\eta, q_{13}) + \phi(\eta-\hbar, q_{23})\phi(\hbar, q_{13})\,,$$

$$R^{\hbar}_{12}(q_{12})R^{\eta}_{23}(q_{23}) = R^{\eta}_{13}(q_{13})R^{\hbar-\eta}_{12}(q_{12}) + R^{\eta-\hbar}_{23}(q_{23})R^{\hbar}_{13}(q_{13})$$

$$\phi(x, q_{ab})f(x, q_{bc}) - f(x, q_{ab})\phi(x, q_{bc}) = \phi(x, q_{ac})(\wp(q_{ab}) - \wp(q_{bc}))\,,$$

$$R^{\hbar}_{ab}F^{\hbar}_{bc} - F^{\hbar}_{ab}R^{\hbar}_{bc} = F^{0}_{bc}R^{\hbar}_{ac} - R^{\hbar}_{ac}F^{0}_{ab}$$

Chen-Lee-Pereira linear problems (1979)

$$\dot{q}_i = -\nu \sum_{j\neq i}^{n} \frac{1}{q_i - q_j} + \nu \sum_{\alpha}^{N} \frac{1}{q_i - \mu_\alpha}\,, \qquad i = 1...n$$
$$\dot{\mu}_\alpha = \nu \sum_{\beta\neq\alpha}^{N} \frac{1}{\mu_\alpha - \mu_\beta} - \nu \sum_{i}^{n} \frac{1}{\mu_\alpha - q_i}\,, \qquad \alpha = 1...N$$

In elliptic case $1/x \to E_1(x)$ (G.Bonelli, A.Sciarappa, A.Tanzini, P.Vasko 2014)

Statement for RS models (with A.Zabrodin)

$$\dot{q}_i = \prod_{k\neq i}^{n} \frac{\vartheta_1(q_i - q_k + \eta)}{\vartheta_1(q_i - q_k)} \prod_{\gamma=1}^{n} \frac{\vartheta_1(q_i - \mu_\gamma - \eta)}{\vartheta_1(q_i - \mu_\gamma)}\,,$$
$$\dot{\mu}_\alpha = \prod_{\beta\neq\alpha}^{n} \frac{\vartheta_1(\mu_\alpha - \mu_\beta - \eta)}{\vartheta_1(\mu_\alpha - \mu_\beta)} \prod_{j=1}^{n} \frac{\vartheta_1(\mu_\alpha - q_j + \eta)}{\vartheta_1(\mu_\alpha - q_j)}\,.$$

Gaudin-Schlesinger form of Calogero-Moser model:

$$L_{ij} = \delta_{ij} p_j + (1 - \delta_{ij}) \tfrac{\nu}{q_i - q_j}\,, \qquad L\Psi = \Psi\Lambda\,, \quad \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)\,,$$

where Ψ is the matrix of eigenvectors. Introduce a "factitious"spectral parameter z by the following gauge transformation:

$$L'(z) = (z - Q)L(z - Q)^{-1} = L + [L, Q](z - Q)^{-1} = L - \mathcal{O}(z - Q)^{-1},$$

where $Q = \mathrm{diag}(q_1, ..., q_n)$ and $\mathcal{O} = [Q, L]$: $\mathcal{O}_{ij} = \nu(1 - \delta_{ij})$.

$$L'(z) = L - \sum_{a=1}^{n} \frac{\mathcal{O}^a}{z - q_a}\,, \quad \mathcal{O}^a_{ij} = \nu(1 - \delta_{ij})\delta_{aj}\,.$$

$$L''(z) = \Psi^{-1} L'(z) \Psi = \Lambda - \sum_{a=1}^{n} \frac{\Psi^{-1}\mathcal{O}^a\Psi}{z - q_a}$$

In Schlesinger version $L \to \partial_z + L$ and Lax equations are replaced by $\partial_t L - \partial_z M = [L, M]$. After gauge transformations:

$$\partial_z + L''(z) = \partial_z + \Psi^{-1} L'(z) \Psi = \Lambda - \nu \sum_{a=1}^{n} \frac{\Psi^{-1} \tilde{\mathcal{O}}^a \Psi}{z - q_a}, \qquad \tilde{\mathcal{O}}^a_{ij} = \nu(1 - \delta_{ij})\delta_{aj} + \delta_{ij}\delta_{aj}.$$

Hamiltonians:

$$H_a = \mathrm{tr}(L\tilde{\mathcal{O}}^a) - \sum_{c \neq a} \frac{\mathrm{tr}(\tilde{\mathcal{O}}^a \tilde{\mathcal{O}}^c)}{q_a - q_c}$$

Since $\mathrm{tr}(L\tilde{\mathcal{O}}^a) = p_a$ and $\mathrm{tr}(\tilde{\mathcal{O}}^a \tilde{\mathcal{O}}^c) = \delta_{ac} + \nu(1 - \delta_{ac})$ we have

$$H_a = p_a - \sum_{c \neq a} \frac{\nu}{q_a - q_c}$$

Symplectic form:

$$\sum_i dH_i \wedge dt_i = \sum_i dp_i \wedge dq_i$$

Based on

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