Degenerations of Ruijsenaars-van Diejen operator, q-Painleve equations and q-Heun equations

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It is known that the Painleve VI is obtained by monodromy preserving deformation of some linear differential equations, and the Heun equation is obtained by a specialization of the linear differential equations. We inverstigate degenerations of the Ruijsenaars-van Diejen difference opearators and show difference analogues of the Painleve-Heun correspondence. The eigenvalue problems of each degenerated Ruijsenaars-van Diejen difference opearator may be regarded as q-Heun equations.

Heun's differential equation

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0.$$

Four singularities $\{0,1,t,\infty\}$; they are **regular**. We impose the condition $\gamma+\delta+\epsilon=\alpha+\beta+1$ so that the exponents at $z=\infty$ are α and β.

Heun's differential equation is a standard form of Fuchsian differential equation with **four** singularities.

The parameter q is called an accessory parameter.

Hypergeometric equation: Fuchsian differential equation with three singularities $\{0,1,\infty\}$.
no accessory parameter

q-Heun equation

Motivated by degeneration of Ruijsenaars-van Diejen system and specialisation of linear q-difference equation related with q-Painlevé VI (explain later), we define q-Heun equation as follows:

$$
\begin{aligned}
&(x-h_1q^{1/2})(x-h_2q^{1/2})g(x/q)+l_3l_4(x-l_1q^{-1/2})(x-l_2q^{-1/2})g(xq) \\
&-\{(l_3+l_4)x^2+Ex+(l_1l_2l_3l_4h_1h_2)^{1/2}(h_3^{1/2}+h_3^{-1/2})\}g(x)=0.
\end{aligned}
\tag{1}
$$

$($degree 2 in $x)g(x/q)-($degree 2 in $x)g(x)+($degree 2 in $x)g(xq)=0.$

It has a limit to the Heun equation as $q\to 1$.

Basic hypergeometric equation (q-hypergeometric equation)

$$(x-q)f(x/q)-((a+b)x-q-c)f(x)+(abx-c)f(qx)=0.$$

Variants of q-Heun equation

$$\prod_{n=1}^{3}(x-h_n q^{1/2})g(x/q)+\prod_{n=1}^{3}(x-l_n q^{-1/2})g(qx)$$
$$+\Big\{-(q^{1/2}+q^{-1/2})x^3+\sum_{n=1}^{3}(h_n+l_n)x^2$$
$$-Ex+(l_1l_2l_3h_1h_2h_3)^{1/2}(h_4^{-1/2}+h_4^{1/2})\Big\}g(x)=0. \quad (2)$$

$$\prod_{n=1}^{4}(x-h_n q^{1/2})g(x/q)+\prod_{n=1}^{4}(x-l_n q^{-1/2})g(qx)$$
$$+\Big\{-(q^{1/2}+q^{-1/2})x^4+\sum_{n=1}^{4}(h_n+l_n)x^3+Ex^2$$
$$+\prod_{n=1}^{4}h_n^{1/2}l_n^{1/2}\cdot[\sum_{n=1}^{4}(h_n^{-1}+l_n^{-1})x-(q^{1/2}+q^{-1/2})]\Big\}g(x)=0. \quad (3)$$

We obtain q-Heun equations (1), (2), (3) by two methods.

One is **degenerations of Ruijsenaars-van Diejen operators**. The other is **specialisations of linear q-difference equation related with q-Painlevé equations**.

In other words, degenerations of Ruijsenaars-van Diejen operators are related with specialisations of linear q-difference equation related with q-Painlevé equations.

Ruijsenaars-van Diejen system (**Ruijsenaars system of type** BC_N) is a difference (relativistic) analogue of Inozemtsev system.

Inozemtsev system:

Quantum mechanical system with N-particles whose Hamiltonian is given by

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2l(l+1) \sum_{1\le j<k\le N} (\wp(x_j - x_k) + \wp(x_j + x_k))$$
$$+ \sum_{j=1}^{N} \sum_{i=0}^{3} l_i(l_i+1)\wp(x_j + \omega_i),$$

Inozemtsev model of type BC_N is quantum Liouville integrable, i.e.,
$\exists\ H_k = \sum_{j=1}^{N} \left(\frac{\partial}{\partial x_j}\right)^{2k} +$ (lower terms) $(k = 2, \dots, N)$
s.t. $[H, H_k] = 0$ and $[H_{k_1}, H_{k_2}] = 0$ $(k, k_1, k_2 = 2, \dots, N)$.

Inozemtsev system with one variable $(N=1)$:

$$H=-\frac{d^2}{dx^2}+\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)$$

For $E\in\mathbb{C}$, $Hf(x)=Ef(x)$ is equivalent to Heun's equation, i.e. $\left(-\frac{d^2}{dx^2}+\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)-E\right)f(x) = 0$ is an elliptic representation of Heun's equation.

Eigenvalue $E \Leftrightarrow$ Accessory parameter q.

Quantum Inozemtsev model is a generalization of elliptic form of Heun's equation

Ruijsenaars-van Diejen operator for one particle is an elliptic difference analogue of Heun's equation

Ruijsenaars-van Diejen operator for one particle

$a_+, a_- \in \mathbb{R}_{>0} + \sqrt{-1}\mathbb{R}$

$R_\pm(z)$: modified versions of theta function defined by

$$R_\pm(z) = \prod_{k=1}^{\infty}(1 - q_\pm^{2k-1}e^{2\pi iz})(1 - q_\pm^{2k-1}e^{-2\pi iz}),\ q_\pm = e^{-\pi a_\pm}. \qquad (4)$$

The Ruijsenaars-van Diejen operator of one variable is given by

$$A_+(h;z) = V_+(h;z)\exp(-ia_-\partial_z) + V_+(h;-z)\exp(ia_-\partial_z) + U_+(h;z),$$

$$V_+(h;z) = \frac{\prod_{n=1}^{8} R_+(z_j - h_n - ia_-/2)}{R_+(2z_j + ia_+/2)R_+(2z_j - ia_- + ia_+/2)},$$

$$U_+(h;z) = \frac{\sum_{t=0}^{3} p_{t,+}(h)[\mathcal{E}_{t,+}(\mu;z) - \mathcal{E}_{t,+}(\mu;\omega_{t,+})]}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)},$$

parameters: $h_1, \ldots, h_8, a_-$,

$\exp(\pm ia_-\partial_z)f(z) = f(z \pm ia_-)$

$$\omega_{0,+}=0,\ \ \omega_{1,+}=1/2,\ \ \omega_{2,+}=ia_+/2,\ \ \omega_{3,+}=-1/2-ia_+/2,$$

$$p_{0,+}(h)=\prod_{n=1}^{8}R_+(h_n),\ p_{1,+}(h)=\prod_{n=1}^{8}R_+(h_n-1/2),$$

$$p_{2,+}(h)=e^{-2\pi a_+}\prod_{n=1}^{8}e^{-i\pi h_n}R_+(h_n-ia_+/2),$$

$$p_{3,+}(h)=e^{-2\pi a_+}\prod_{n=1}^{8}e^{i\pi h_n}R_+(h_n+1/2+ia_+/2),$$

$$\mathcal{E}_{t,+}(\mu;z)=$$

$$\frac{R_+(z+\mu-ia_+/2-ia_-/2-\omega_{t,+})R_+(z-\mu+ia_+/2+ia_-/2-\omega_{t,+})}{R_+(z-ia_+/2-ia_-/2-\omega_{t,+})R_+(z+ia_+/2+ia_-/2-\omega_{t,+})},$$

$$(t=0,1,2,3).$$

We obtain **four degenerated operators from Ruijsenaars-van Diejen operator** with N-particles.

For simplicity, we observe degenerations of Ruijsenaars-van Diejen operator with one-particle, which is related with q-Painlevé equations.

Trigonometric limit

The function $R_+(z)$ satisfies

$$R_+(z \mp ia_+) = -e^{\pi a_+} e^{\pm 2\pi iz} R_+(z)$$

and we have the following expansion as $q_+ \to 0$ (or $a_+ \to +\infty$):

$$R_+(z) = 1 - (e^{2\pi iz} + e^{-2\pi iz})q_+ + q_+^2 + O(q_+^3),$$
$$R_+(z \pm ia_+/2) = (1 - e^{\mp 2\pi iz})(1 - (e^{2\pi iz} + e^{-2\pi iz})q_+^2 + O(q_+^4)).$$

We set $h_n = \tilde{h}_n - ia_+/2$. As $q_+ \to 0$, we have

$$V_+(h;z) = \frac{\prod_{n=1}^{8} R_+(z - \tilde{h}_n - ia_-/2 + ia_+/2)}{R_+(2z + ia_+/2)R_+(2z - ia_- + ia_+/2)}$$
$$\to V^{\langle 1\rangle}(h;z) = \frac{\prod_{n=1}^{8}(1 - e^{-2\pi iz}e^{2\pi i\tilde{h}_n}e^{-\pi a_-})}{(1 - e^{-4\pi iz})(1 - e^{-4\pi iz}e^{-2\pi a_-})}.$$

First degeneration

Proposition 1. *Let* $A(h, q_+; z)$ *be the Ruijsenaars-van Diejen operator. As* $q_+ \to 0$, *we have*

$$\left(A(h, q_+; z) + \frac{\prod_{n=1}^{8} e^{\pi i \tilde{h}_n}}{(1 - e^{\pi a_-})^2} q_+^{-2} + C \right) f(z) \to A^{\langle 1 \rangle}(h; z) f(z)$$

for any $f(z)$, *where* C *is a certain constant,*

$$\begin{aligned} & A^{\langle 1 \rangle}(h; z) \\ & = V^{\langle 1 \rangle}(h; z) \exp(-ia_- \partial_z) + V^{\langle 1 \rangle}(h; -z) \exp(ia_- \partial_z)) + U^{\langle 1 \rangle}(h; z), \end{aligned}$$

where

$$V^{\langle 1\rangle}(h;z) = \frac{\prod_{n=1}^{8}(1-e^{-2\pi iz}e^{2\pi i\tilde{h}_n}e^{-\pi a_-})}{(1-e^{-4\pi iz})(1-e^{-4\pi iz}e^{-2\pi a_-})},$$

and

$$\begin{aligned} U^{\langle 1\rangle}(h;z) =& \frac{\prod_{n=1}^{8}(e^{2\pi i\tilde{h}_n}-1)}{2(1-e^{2\pi iz}e^{\pi a_-})(1-e^{-2\pi iz}e^{\pi a_-})} \\ &+ \frac{\prod_{n=1}^{8}(e^{2\pi i\tilde{h}_n}+1)}{2(1+e^{2\pi iz}e^{\pi a_-})(1+e^{-2\pi iz}e^{\pi a_-})} \\ &+ e^{-\pi a_-}\prod_{n=1}^{8}e^{\pi i\tilde{h}_n}\cdot\Big[(e^{2\pi iz}+e^{-2\pi iz})\sum_{n=1}^{8}(e^{2\pi i\tilde{h}_n}+e^{-2\pi i\tilde{h}_n}) \\ &\qquad - (e^{\pi a_-}+e^{-\pi a_-})(e^{4\pi iz}+e^{-4\pi iz})\Big]. \end{aligned}$$

By the gauge transformation

$$\tilde{A}^{\langle 1\rangle}(h,z) = R_-(z)^{-2} \circ A^{\langle 1\rangle}(h,z) \circ R_-(z)^2,$$

we have the operator:

$$\tilde{A}^{\langle 1\rangle}(h;z) = \tilde{V}^{\langle 1\rangle}(h;z)\exp(-ia_-\partial_z)+\tilde{W}^{\langle 1\rangle}(h;z)\exp(ia_-\partial_z)+U^{\langle 1\rangle}(h;z), \tag{5}$$

$$\tilde{V}^{\langle 1\rangle}(h;z) = \frac{\prod_{n=1}^{8}(1-e^{-2\pi iz}e^{2\pi i\tilde{h}_n}e^{-\pi a_-})}{e^{-2\pi a_-}e^{-4\pi iz}(1-e^{-4\pi iz})(1-e^{-4\pi iz}e^{-2\pi a_-})},$$

$$\tilde{W}^{\langle 1\rangle}(h;z) = \frac{\prod_{n=1}^{8}(1-e^{2\pi iz}e^{2\pi i\tilde{h}_n}e^{-\pi a_-})}{e^{-2\pi a_-}e^{4\pi iz}(1-e^{4\pi iz})(1-e^{4\pi iz}e^{-2\pi a_-})}.$$

This operator was essentially obtained by van Diejen in the multivariable case.

Second degeneration

Proposition 2. *In Eq.(5), we replace z by $z+iR$, $\tilde{h}_n$ $(n=1,2,3,4)$ by h_n+iR, $\tilde{h}_n$ $(n=5,6,7,8)$ by h_n-iR and take the limit $R\to+\infty$. Then we arrive at the operator*

$$A^{\langle 2\rangle}(h;z)=V^{\langle 2\rangle}(h;z)\exp(-ia_-\partial_z)+W^{\langle 2\rangle}(h;z)\exp(ia_-\partial_z)+U^{\langle 2\rangle}(h;z),$$

$$V^{\langle 2\rangle}(h;z)=e^{4\pi iz}\prod_{n=1}^{4}(1-e^{-2\pi iz}e^{2\pi ih_n}e^{-\pi a_-})\prod_{n=5}^{8}e^{2\pi ih_n},$$

$$W^{\langle 2\rangle}(h;z)=e^{2\pi a_-}e^{-4\pi iz}\prod_{n=5}^{8}(1-e^{2\pi iz}e^{2\pi ih_n}e^{-\pi a_-}),$$

$$U^{\langle 2\rangle}(h;z)=\prod_{n=5}^{8}e^{2\pi ih_n}\Big[\Big(\sum_{n=1}^{4}e^{2\pi ih_n}+\sum_{n=5}^{8}e^{-2\pi ih_n}\Big)e^{-\pi a_-}e^{2\pi iz}-(1+e^{-2\pi a_-})e^{4\pi iz}\Big]$$
$$+\prod_{n=1}^{8}e^{\pi ih_n}\Big[\Big(\sum_{n=1}^{4}e^{-2\pi ih_n}+\sum_{n=5}^{8}e^{2\pi ih_n}\Big)e^{-\pi a_-}e^{-2\pi iz}-(1+e^{-2\pi a_-})e^{-4\pi iz}\Big].$$

Namely we have

$$e^{-4\pi R}\tilde{A}^{\langle 1\rangle}(h+iRv;z+iR)f(z)\to A^{\langle 2\rangle}(h;z)f(z)$$

as $R\to+\infty$ *for any* $f(z)$*, where* $v=(1,1,1,1,-1,-1,-1,-1)$*.*

Set $l_n = -h_{n+4}$ $(n = 1, 2, 3, 4)$. By the multiplication and the gauge transformation given by

$$\tilde{A}^{\langle 2\rangle}(h, l; z) = e^{\pi a_-} \prod_{n=5}^{8} e^{-2\pi i h_n} \cdot e^{\pi i z} \circ A^{\langle 2\rangle}(h, z) \circ e^{-\pi i z},$$

we have the operator

$$\begin{aligned} &\tilde{A}^{\langle 2\rangle}(h, l; z) \qquad (6)\\ &= \tilde{V}^{\langle 2\rangle}(h; z) \exp(-i a_- \partial_z) + \tilde{W}^{\langle 2\rangle}(l; z) \exp(i a_- \partial_z) + \tilde{U}^{\langle 2\rangle}(h, l; z),\end{aligned}$$

where

$$\tilde{V}^{\langle 2\rangle}(h;z) = e^{4\pi iz}\prod_{n=1}^{4}(1 - e^{2\pi ih_n}e^{-\pi a_-}e^{-2\pi iz}),$$

$$\tilde{W}^{\langle 2\rangle}(l;z) = e^{4\pi iz}\prod_{n=1}^{4}(1 - e^{2\pi il_n}e^{\pi a_-}e^{-2\pi iz}),$$

$$\tilde{U}^{\langle 2\rangle}(h,l;z) = \Big(\sum_{n=1}^{4}e^{2\pi ih_n} + \sum_{n=1}^{4}e^{2\pi il_n}\Big)e^{2\pi iz} - (e^{\pi a_-} + e^{-\pi a_-})e^{4\pi iz}$$

$$+ \prod_{n=1}^{4}e^{\pi i(h_n+l_n)}\Big[\Big(\sum_{n=1}^{4}e^{-2\pi ih_n} + \sum_{n=1}^{4}e^{-2\pi il_n}\Big)e^{-2\pi iz} - (e^{\pi a_-} + e^{-\pi a_-})e^{-4\pi iz}\Big].$$

This operator was also essentially obtained by van Diejen.

Set $x=e^{2\pi iz}$, $q=e^{-2\pi a_-}$, and replace $e^{2\pi ih_n}$ and $e^{2\pi il_n}$ by h_n and l_n. Then the difference operator $\tilde{A}^{\langle 2\rangle}(h,l;z)$ is written as

$$
\begin{aligned}
&A^{\langle 2\rangle}(x)g(x)\\
&=x^{-2}\prod_{n=1}^{4}(x-h_nq^{1/2})g(x/q)+x^{-2}\prod_{n=1}^{4}(x-l_nq^{-1/2})g(qx)+U(x)g(x),\\
&U(x)=-(q^{1/2}+q^{-1/2})x^2+\sum_{n=1}^{4}(h_n+l_n)x\\
&\qquad+\prod_{n=1}^{4}h_n^{1/2}l_n^{1/2}\cdot[-(q^{1/2}+q^{-1/2})x^{-2}+\sum_{n=1}^{4}(h_n^{-1}+l_n^{-1})x^{-1}].
\end{aligned}
$$

The equation $A^{\langle 2\rangle}(x)g(x)=Eg(x)$ is a variant of q-Heun equation in (3).

Third degeneration

Proposition 3. *In Eq.(6), we replace z by $z - iR$, h_n $(n = 1, 2)$ by $h_n - iR$, h_n $(n = 3, 4)$ by $h_n + iR$, l_n $(n = 1, 2, 3, 4)$ by $l_n - iR$ and take the limit $R \to +\infty$. Then we arrive at the operator*

$$A^{\langle 3 \rangle}(h, l; z) = V^{\langle 3 \rangle}(h; z) \exp(-ia_- \partial_z) + W^{\langle 3 \rangle}(l; z) \exp(ia_- \partial_z) + U^{\langle 3 \rangle}(h, l; z),$$

$$V^{\langle 3 \rangle}(h; z) = e^{4\pi iz} \prod_{n=1}^{2} (1 - e^{2\pi ih_n} e^{-\pi a_-} e^{-2\pi iz}),$$

$$W^{\langle 3 \rangle}(l; z) = e^{4\pi iz} \prod_{n=1}^{4} (1 - e^{2\pi il_n} e^{\pi a_-} e^{-2\pi iz}),$$

$$U^{\langle 3 \rangle}(h, l; z) = \Big(\sum_{n=1}^{2} e^{2\pi ih_n} + \sum_{n=1}^{4} e^{2\pi il_n} \Big) e^{2\pi iz} - (e^{\pi a_-} + e^{-\pi a_-}) e^{4\pi iz}$$

$$+ e^{\pi ih_1} e^{\pi ih_2} (e^{\pi i(h_3 - h_4)} + e^{\pi i(h_4 - h_3)}) \prod_{n=1}^{4} e^{\pi il_n} \cdot e^{-2\pi iz}.$$

Set $x=e^{2\pi i z}$, $q=e^{-2\pi a_-}$, and replace $e^{2\pi i h_n}$ and $e^{2\pi i l_n}$ by h_n and l_n. Then the difference operator $A^{\langle 3\rangle}(h,l;z)$ is written as

$$\begin{aligned}
&A^{\langle 3\rangle}(x)g(x)\\
&=\prod_{n=1}^{2}(x-h_n q^{1/2})g(x/q)+x^{-2}\prod_{n=1}^{4}(x-l_n q^{-1/2})g(qx)+U(x)g(x),
\end{aligned}$$

$$\begin{aligned}
U(x)=&\Big(\sum_{n=1}^{2}h_n+\sum_{n=1}^{4}l_n\Big)x-(q^{1/2}+q^{-1/2})x^2\\
&+(l_1l_2l_3l_4h_1h_2)^{1/2}(h_3^{1/2}h_4^{-1/2}+h_3^{-1/2}h_4^{1/2})x^{-1}.
\end{aligned}$$

Let E be a constant. By applying an appropriate gauge transformation and replacing the parameters, the equation $A^{\langle 3\rangle}(x)g(x)=Eg(x)$ is equivalent to a variant of q-Heun equation in (2).

To obtain the fourth degeneration, we apply the gauge transformation

$$\tilde{A}^{\langle 3\rangle}(h,l;z) = R_-(z)^2 \circ A^{\langle 3\rangle}(h,l;z) \circ R_-(z)^{-2}.$$

Then we have

$$\begin{aligned} &\tilde{A}^{\langle 3\rangle}(h,l;z) \\ &= \tilde{V}^{\langle 3\rangle}(h;z)\exp(-ia_-\partial_z) + \tilde{W}^{\langle 3\rangle}(l;z)\exp(ia_-\partial_z) + U^{\langle 3\rangle}(h,l;z), \end{aligned} \tag{7}$$

where

$$\tilde{V}^{\langle 3\rangle}(h;z) = e^{-2\pi a_-}\prod_{n=1}^{2}(1 - e^{2\pi i h_n}e^{-\pi a_-}e^{-2\pi i z}),$$

$$\tilde{W}^{\langle 3\rangle}(l;z) = e^{-2\pi a_-}e^{8\pi i z}\prod_{n=1}^{4}(1 - e^{2\pi i l_n}e^{\pi a_-}e^{-2\pi i z}).$$

Fourth degeneration

Proposition 4. *In Eq.(7), we replace z by $z+iR$, h_n $(n=1,2,3,4)$ by h_n+iR, l_n $(n=1,2)$ by l_n+iR, l_n $(n=3,4)$ by l_n-iR and take the limit $R\to+\infty$. Then we arrive at the operator*

$$
\begin{aligned}
&A^{\langle 4\rangle}(h,l;z)\\
&=V^{\langle 4\rangle}(h;z)\exp(-ia_-\partial_z)+W^{\langle 4\rangle}(l;z)\exp(ia_-\partial_z)+U^{\langle 4\rangle}(h,l;z),
\end{aligned}
$$

$$
\begin{aligned}
V^{\langle 4\rangle}(h;z)&=e^{-2\pi a_-}\prod_{n=1}^{2}(1-e^{2\pi ih_n}e^{-\pi a_-}e^{-2\pi iz}),\\
W^{\langle 4\rangle}(l;z)&=e^{4\pi iz_j}\prod_{n=1}^{2}(1-e^{2\pi il_n}e^{\pi a_-}e^{-2\pi iz})\prod_{n=3}^{4}e^{2\pi il_n},\\
U^{\langle 4\rangle}(h,l;z)&=(e^{2\pi il_3}+e^{2\pi il_4})e^{2\pi iz}\\
&+e^{\pi ih_1}e^{\pi ih_2}(e^{\pi i(h_3-h_4)}+e^{\pi i(h_4-h_3)})\prod_{n=1}^{4}e^{\pi il_n}\cdot e^{-2\pi iz}.
\end{aligned}
$$

By the multiplication and the gauge transformation given by

$$\tilde{A}^{\langle 4\rangle}(h,l;z) = -R_-(z)^{-1}e^{-\pi iz} \circ A^{\langle 4\rangle}(h,l;z) \circ R_-(z)e^{\pi iz},$$

we have

$$\begin{aligned}
&\tilde{A}^{\langle 4\rangle}(h,l;z) \\
&= \tilde{V}^{\langle 4\rangle}(h;z)\exp(-ia_-\partial_z) + \tilde{W}_j^{\langle 4\rangle}(l;z)\exp(ia_-\partial_z) - U^{\langle 4\rangle}(h,l;z),
\end{aligned}$$

where

$$\tilde{V}^{\langle 4\rangle}(h;z) = e^{2\pi iz}\prod_{n=1}^{2}(1 - e^{2\pi ih_n}e^{-\pi a_-}e^{-2\pi iz}),$$

$$\tilde{W}^{\langle 4\rangle}(l;z) = e^{2\pi il_3}e^{2\pi il_4}e^{2\pi iz}\prod_{n=1}^{2}(1 - e^{2\pi il_n}e^{\pi a_-}e^{-2\pi iz}).$$

Set $x = e^{2\pi i z}$, $q = e^{-2\pi a_-}$, and replace $e^{2\pi i h_n}$ and $e^{2\pi i l_n}$ by h_n and l_n. Then the difference operator is written as

$$
\begin{aligned}
&A^{\langle 4\rangle}(x)g(x) \\
&= x^{-1}(x - h_1 q^{1/2})(x - h_2 q^{1/2})g(x/q) \\
&\qquad + x^{-1} l_3 l_4 (x - l_1 q^{-1/2})(x - l_2 q^{-1/2})g(qx) \\
&\qquad - \{(l_3 + l_4)x + (l_1 l_2 l_3 l_4 h_1 h_2)^{1/2}(h_3^{1/2} h_4^{-1/2} + h_3^{-1/2} h_4^{1/2})x^{-1}\}g(x).
\end{aligned}
$$

Let E be a constant. The equation $A^{\langle 4\rangle}(x)g(x) = Eg(x)$ is q-Heun equation in (1).

Ruijsenaars-van Diejen operator

($N = 1$: **elliptic difference Heun equation**)

$\Downarrow$ trigonometric limit $q_+ \to 0$

Firstly degenerated Ruijsenaars-van Diejen operator

$\Downarrow$ $z \Rightarrow z + iR$, $\tilde{h}_n \Rightarrow h_n \pm iR$, $R \to +\infty$

Secondly degenerated Ruijsenaars-van Diejen operator

($N = 1$: a variant of q-Heun equation)

$\Downarrow$ $z \Rightarrow z - iR$, $h_n \Rightarrow h_n \mp iR$, $l_n \Rightarrow l_n - iR$, $R \to +\infty$

Thirdly degenerated Ruijsenaars-van Diejen operator

($N = 1$: a variant of q-Heun equation)

$\Downarrow$ $z \Rightarrow z + iR$, $h_n \Rightarrow h_n + iR$, $l_n \Rightarrow l_n \pm iR$, $R \to +\infty$

Fourthly degenerated Ruijsenaars-van Diejen operator

($N = 1$: **q-Heun equation**)

Painlevé-Heun correspondence

We obtain a correspondence between q**-Painlev’e equations** and **degenerate Ruijsenaars-van Diejen operators**, which is q-deformation of Painlevé-Heun correspondence.

We explain **Painlevé-Heun correspondence** in the setting of differential equations.

Painlevé VI system (canonical formalism in analytical mechanics)

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}$$

$$H = \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t)$$
$$+ (\theta_t - 1)\lambda(\lambda-1)\}\mu + \kappa_1(\kappa_2+1)(\lambda-t)].$$

By eliminating μ, we obtain Painlevé VI equation for the variable λ

Painlevé property

Painlevé VI is a non-linear ordinary differential equation whose solutions do not have movable singularities other than poles.

Isomonodromic deformation

Painlevé VI is obtained by monodromy preserving deformation of the Fuchsian system with four singularities $\{0, 1, t, \infty\}$.

$\mathbf{2 \times 2}$ Fuchsian system with sing. $\{\mathbf{0}, \mathbf{1}, \mathbf{t}, \infty\}$

$$\frac{dY}{dz} = A(z)Y,\ A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t},\ Y = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}. \qquad (8)$$

By eliminating $y_2(z)$, we have

$$\frac{d^2y_1}{dz^2} + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda}\right)\frac{dy_1}{dz}$$
$$+ \left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} - \frac{t(t-1)H}{z(z-1)(z-t)}\right)y_1 = 0,$$
$$H = \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t)$$
$$+ \theta_1\lambda(\lambda-t) + (\theta_t-1)\lambda(\lambda-1)\}\mu + \kappa_1(\kappa_2+1)(\lambda-t)].$$

The condition for isomonodromic deformation (Painlevé VI) is written as

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}.$$

Fuchsian system and Heun equation

Recall that the function $y_1(z)$ satisfies

$$\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{1-\theta_t}{z-t}-\frac{1}{z-\lambda}\right)\frac{dy_1}{dz}$$
$$+\left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)}+\frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}-\frac{t(t-1)H}{z(z-1)(z-t)}\right)y_1=0.$$

$z=\lambda$: apparent (non-log.) singularity with exponents 0 and 2.

By the suitable limits $\lambda\to 0,1,t,\infty$, we may obtain Heun equation. For example we have

$$\frac{d^2y_1}{dz^2}+\left(\frac{1-\theta_0}{z}+\frac{1-\theta_1}{z-1}+\frac{-\theta_t}{z-t}\right)\frac{dy_1}{dz}$$
$$+\frac{\kappa_1(\kappa_2+1)(z-t)+\theta_t t(t-1)\mu}{z(z-1)(z-t)}y_1=0$$

as $\lambda\to t$.

Painlevé VI equation for variable t

$\uparrow$ isomonodromic deformation w.r.t. t

$$\boxed{\begin{array}{c}\textbf{Fuchsian system with 4 singularities}\\ \dfrac{dY}{dz} = \left(\dfrac{A_0}{z} + \dfrac{A_1}{z-1} + \dfrac{A_t}{z-t}\right) Y\end{array}}$$

$\downarrow$ specialisation

Heun equation

Difference Painlevé equations

Difference analogue of Painlevé equations which have nice properties

Singularity confinement,

Algebraic entropy $= 0$: (polynomial growth of degrees of solutions)

Symmetry of affine Weyl group,

Geometric construction from algebraic surfaces (Sakai's table)

Elliptic difference	$E_8^{(1)}$
q-difference	$E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_5^{(1)}, A_4^{(1)}, (A_2 + A_1)^{(1)}, \ldots$
additive-difference	$E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_4^{(1)}, A_3^{(1)}, \ldots$

Our results are related with q-difference equations with the type $E_7^{(1)}$, $E_6^{(1)}$ and $D_5^{(1)} = q$-PVI.

q-Painlevé VI (Jimbo, Sakai (1996))

$$\frac{\mu(t)\mu(qt)}{b_3b_4} = \frac{(\lambda(t)-ta_1)(\lambda(t)-ta_2)}{(\lambda(t)-a_3)(\lambda(t)-a_4)},$$
$$\frac{\lambda(t)\lambda(qt)}{a_3a_4} = \frac{(\mu(qt)-tb_1)(\mu(qt)-tb_2)}{(\mu(qt)-b_3)(\mu(qt)-b_4)},$$

under the condition $b_1b_2a_3a_4 = qa_1a_2b_3b_4$.
It arises from the condition of preserving the connection matrix of linear q-difference equations;

$$Y(qx) = (A_0 + A_1x + A_2x^2)Y(x), \quad Y(x) = \begin{pmatrix} Y_1(x) \\ Y_2(x) \end{pmatrix}.$$

By the limit $q \to 1$, we have the usual Painlevé VI and Fuchsian differential equation with the singularities $\{0, 1, t, \infty\}$.

Linear system of q-differential equations

$$Y(qx,t) = A(x,t)Y(x,t). \tag{9}$$

Connection matrix $P(x,t) = Y_0(x,t)^{-1}Y_\infty(x,t)$

$Y_0(x,t)$ (resp. $Y_\infty(x,t)$): local solution about $x = 0$ (resp. $x = \infty$)

Then we have $P(qx,t) = P(x,t)$.

Connection preserving deformation

We deform $A(x,t)$ in t to preserve $P(x,qt) = P(x,t)$

Connection preserving deformation is described by compatibility condition of Eq.(9) with a deformation equation

$$Y(x,qt) = B(x,t)Y(x,t) \tag{10}$$

for some $B(x,t)$, i.e. $A(x,qt)B(x,t) = B(qx,t)A(x,t)$.

We call Eqs.(9, 10) a **Lax form** of q-Painlevé VI.

We focus on Eq.(9). The 2×2 matrix $A(x,t)$ is taken in the form

$$A(x,t) = A_0(t) + A_1(t)x + A_2x^2 = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad A_0(t) \text{ has eigenvalues } t\theta_1, t\theta_2,$$

$$\det A(x,t) = \kappa_1\kappa_2(x - ta_1)(x - ta_2)(x - a_3)(x - a_4).$$

Note that we have the relation $\kappa_1\kappa_2 a_1 a_2 a_3 a_4 = \theta_1\theta_2$. Define λ, μ_1, μ_2 by

$$a_{12}(\lambda) = 0,\ \mu_1 = a_{11}(\lambda)/\kappa_1,\ \mu_2 = a_{22}(\lambda)/\kappa_2$$

so that $\mu_1\mu_2 = (\lambda - ta_1)(\lambda - ta_2)(\lambda - a_3)(\lambda - a_4)$ and introduce μ by $\mu = (\lambda - ta_1)(\lambda - ta_2)/(q\kappa_1\mu_1)$. Then the matrix elements can be parametrized by these variables and the gauge freedom w.

$$Y(qx) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} Y(x), \quad Y(x) = \begin{pmatrix} Y_1(x) \\ Y_2(x) \end{pmatrix}.$$

$Y_1(x)$ satisfies the following equation:

$$Y_1(q^2x) - \left(a_{11}(qx) + \frac{a_{12}(qx)}{a_{12}(x)} a_{22}(x) \right) Y_1(qx) \\ + \frac{a_{12}(qx)}{a_{12}(x)} (a_{11}(x)a_{22}(x) - a_{12}(x)a_{21}(x)) Y_1(x) = 0.$$

In our parametrization, we have

$$
\begin{aligned}
&\frac{a_{12}(qx)}{a_{12}(x)}(a_{11}(x)a_{22}(x) - a_{12}(x)a_{21}(x)) \\
&= \frac{qx-\lambda}{x-\lambda}\kappa_1\kappa_2(x-ta_1)(x-ta_2)(x-a_3)(x-a_4), \\
&a_{11}(qx) + \frac{a_{12}(qx)}{a_{12}(x)}a_{22}(x) = \frac{q(q\kappa_1+\kappa_2)x^3 + c_2x^2 + c_1x - \lambda t(\theta_1+\theta_2)}{x-\lambda}, \\
&c_2 = \frac{q^2\kappa_1\kappa_2(\lambda-a_3)(\lambda-a_4)\mu}{\lambda} - (q+1)(q\kappa_1+\kappa_2)\lambda \\
&\qquad - \frac{qt(\theta_1+\theta_2)}{\lambda} + \frac{(\lambda-a_1t)(\lambda-a_2t)}{\lambda\mu}, \\
&c_1 = -q\kappa_1\kappa_2(\lambda-a_3)(\lambda-a_4)\mu + (q\kappa_1+\kappa_2)\lambda^2 \\
&\qquad + (q+1)t(\theta_1+\theta_2) - \frac{(\lambda-a_1t)(\lambda-a_2t)}{\mu}.
\end{aligned}
$$

Note that there are two accessory parameters λ and μ, which play the role of dependent variables in the q-Painlevé VI equation.

We impose a restriction on the accessory parameters as $\lambda = a_3$.

$$
\begin{aligned}
&Y_1(q^2x) - \{q(q\kappa_1+\kappa_2)x^2 + d_1x + t(\theta_1+\theta_2)\}Y_1(qx) \\
&\qquad + \kappa_1\kappa_2(qx-a_3)(x-ta_1)(x-ta_2)(x-a_4)Y_1(x) = 0,
\end{aligned}
$$

where

$$
d_1 = \frac{(a_1t-a_3)(a_2t-a_3)}{a_3\mu} - a_3(q\kappa_1+\kappa_2) - \frac{qt(\theta_1+\theta_2)}{a_3}.
$$

Let $u(x)$ be the function which satisfies $u(qx) = (x - ta_1)(x - ta_2)u(x)$. Then the function $f(x) = Y_1(qx)/u(qx)$ satisfies

$$(x - ta_1)(x - ta_2)f(qx) + (\kappa_1\kappa_2/q)(x - a_3)(x - qa_4)f(x/q) \quad (11)$$
$$- \{((q\kappa_1 + \kappa_2)/q)x^2 + (d_1/q)x + t(\theta_1 + \theta_2)\}f(x) = 0.$$

Note that there is the relation $\kappa_1\kappa_2 a_1 a_2 a_3 a_4 = \theta_1\theta_2$. In Eq.(1), we set

$$l_1 = a_1 t q^{1/2},\ l_2 = a_2 t q^{1/2},\ h_1 = a_3 q^{-1/2},\ h_2 = a_4 q^{1/2},$$
$$l_3 = 1/\kappa_1,\ l_4 = q/\kappa_2,\ (h_3/h_4)^{1/2} = \theta_1(a_1 a_2 a_3 a_4 \kappa_1 \kappa_2)^{-1/2},$$
$$E = d_1/(\kappa_1\kappa_2).$$

Then we have $(h_3/h_4)^{-1/2} = \theta_2(a_1 a_2 a_3 a_4 \kappa_1 \kappa_2)^{-1/2}$ and Eq.(11).

Hence Eq.(11) is obtained by the fourth degeneration of the Ruijsenaars-van Diejen operator. We may regard d_1 as an accessory parameter.

Jimbo and Sakai (1996) obtained q-Painlevé VI (or the q-Painlevé equation of type $D_5^{(1)}$) by finding ”Lax forms”.

Yamada and Rains discovered Lax forms for the elliptic difference Painlevé equation independently.

Yamada also found Lax forms of q-difference Painlevé equations of types $D_5^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ explicitly.

We observe that degenerete operators of Ruijsenaars-van Diejen appear by restricting the parameters in the linear q-differential equations related to q-Painlevé equations of types $E_6^{(1)}$, $E_7^{(1)}$.

Yamada (2011, IMRN) derived a q-difference Painlevé equation of type $E_6^{(1)}$ by Lax formalism, i.e. the compatibility condition for two linear q-difference equations. One of the linear difference equations is written as

$$\frac{(b_1q-z)(b_2q-z)(b_3q-z)(b_4q-z)t^2}{q(fq-z)z^4}\left[y(z/q)-\frac{gz}{t^2(gz-q)}y(z)\right] \qquad (12)$$
$$+\frac{(b_5t-z)(b_6t-z)}{(f-z)z^2t^2}\left[y(qz)-\frac{(gz-1)t^2}{gz}y(z)\right]$$
$$+\left[\frac{(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)t^2}{g(fg-1)z^2(gz-q)}-\frac{b_5b_6(b_7g-t)(b_8g-t)}{fgz^3}\right]y(z)=0.$$

We may regard f, g as accessory parameters. The other linear equation (deformation equation) contains the difference on the variable t as well as the variable x. By compatibility condition for two linear difference equations, we have q-Painlevé equation of type $E_6^{(1)}$ for the dependent variables f and g and independent variable t.

We specialize the parameters to $f = b_1$ in Eq.(12). Then we obtain

$$\frac{(b_2q-z)(b_3q-z)(b_4q-z)t^2}{qz^4}y(z/q)+\frac{c(z)}{z^2q^{1/2}(b_1-z)}y(z) \tag{13}$$
$$+\frac{(z-b_5t)(z-b_6t)}{(b_1-z)z^2t^2}y(qz)=0,$$
$$c(z)=-(q^{1/2}+q^{-1/2})z^2$$
$$+(b_1q^{-1/2}+b_2q^{1/2}+b_3q^{1/2}+b_4q^{1/2}+b_5tq^{1/2}+b_6tq^{1/2})z$$
$$+c_1+\frac{q^{1/2}tb_5b_6(b_7+b_8)}{z},$$

and the term c_1 contains the accessory parameter g.

By applying a gauge transformation, we have

$$
\begin{aligned}
&\frac{(z-b_1)(z-b_2q)(z-b_3q)(z-b_4q)}{z^2}\tilde{y}(z/q)+c(z)\tilde{y}(z) \\
&+(z-b_5t)(z-b_6t)\tilde{y}(qz)=0, \\
&c(z)=-(q^{1/2}+q^{-1/2})z^2 \\
&\qquad +(b_1q^{-1/2}+b_2q^{1/2}+b_3q^{1/2}+b_4q^{1/2}+b_5tq^{1/2}+b_6tq^{1/2})z \\
&\qquad +c_1+\frac{q^{1/2}tb_5b_6(b_7+b_8)}{z},
\end{aligned}
\tag{14}
$$

By replacing q to q^{-1}, Eq.(14) is equivalent to $A^{\langle 3\rangle}(x)g(x)=Eg(x)$, where $A^{\langle 3\rangle}(x)$ is the third degenerated Ruijsenaars-van Diejen operator. The eigenvalue E essentially corresponds to the accessory parameter g in c_1.

Yamada (2011, IMRN) also derived a q-difference Painlevé equation of type $E_7^{(1)}$ by Lax formalism, i.e. the compatibility condition for two linear q-difference equations. Set

$$\begin{aligned} B_1(z) &= (1-b_1z)(1-b_2z)(1-b_3z)(1-b_4z), \\ B_2(z) &= (1-b_5z)(1-b_6z)(1-b_7z)(1-b_8z). \end{aligned}$$

Then one of the q-difference equations is written as

$$\begin{aligned} &\frac{B_2(t/z)}{t^2(f-z)}\left[y(qz)-\frac{t^2(1-gz)}{t^2-gz}y(z)\right] \\ &+\frac{t^2B_1(q/z)}{q(fq-z)}\left[y(z/q)-\frac{qt^2-gz}{t^2(q-gz)}y(z)\right] \\ &+\frac{(1-t^2)}{gz^2}\left[\frac{qB_1(g)}{(fg-1)(gz-q)}-\frac{t^4B_2(g/t)}{(fg-t^2)(gz-t^2)}\right]y(z)=0. \end{aligned} \tag{15}$$

We also obtain the q-Painlevé equation of type $E_7^{(1)}$ by a compatibility condition of Eq.(15) with the deformation equation.

By specializing to $f = b_1$ in Eq.(15) and applying a gauge transformation, we have

$$\begin{aligned}&(z-b_5t)(z-b_6t)(z-b_7t)(z-b_8t)y(qz)-c(z)y(z)\\&\quad+(z-b_1)(z-b_2q)(z-b_3q)(z-b_4q)y(z/q)=0,\end{aligned} \tag{16}$$

where

$$\begin{aligned}c(z)&=q^{-1/2}\{(1+q)z^4+c_3z^3+c_2z^2+c_1z+(b_5b_6b_7b_8+q^2b_1b_2b_3b_4)t^2q\},\\c_3&=-(b_1+b_2q+b_3q+b_4q+b_5tq+b_6tq+b_7tq+b_8tq),\\c_1&=-q(b_2b_3b_4t^2q^2+b_1b_2b_4t^2q+b_1b_3b_4t^2q+b_1b_2b_3t^2q\\&\qquad+b_6b_7b_8t+b_5b_7b_8t+b_5b_6b_8t+b_5b_6b_7t),\end{aligned}$$

and the term c_2 contains the accessory parameter g. There is a relation $qb_1b_2b_3b_4 = b_5b_6b_7b_8$ in Yamada's paper.

Eq.(16) is equivalent to $A^{\langle 2\rangle}(x)g(x) = Eg(x)$ by setting

$$h_1 = b_1 q^{-1/2},\ h_2 = b_2 q^{1/2},\ h_3 = b_3 q^{1/2},\ h_4 = b_4 q^{1/2},$$
$$l_1 = b_5 q^{1/2} t,\ l_2 = b_6 q^{1/2} t,\ l_3 = b_7 q^{1/2} t,\ l_4 = b_8 q^{1/2} t,$$

where $A^{\langle 2\rangle}(x)$ is the second degenerated Ruijsenaars-van Diejen operator.

Summary

- Heun equation: linear differential equation with 4 regular singularities
- q-Heun equations: linear q-difference equations, which admits 3 types
- Ruijsenaars-van Diejen system: difference version of Inozemtsev system
- Four degenerations of Ruijsenaars-van Diejen operator
- In one particle case, we obtain q-Heun equations
- Painlevé-Heun correspondence via Fuchsian system of equations
- q-Painlevé VI is obtained by connection preserving deformation of linear difference equations (Jimbo, Sakai)
- q-Heun equation is obtained by specialisation of the linear difference equations
- Yamada's Lax pair for q-Painlevé equation
- Variants of q-Heun equations are obtained by specialisation of linear q-difference equations related with q-Painlevé of types $D_5^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$

Problems

Extended our results to the case of q-Painlevé equation of type $E_8^{(1)}$ and the elliptic-difference Painlevé equation.

Yamada and his collaborators found Lax pairs of the q-Painlevé equations of type $E_8^{(1)}$ and the elliptic-difference Painlevé equation. Rains and Ormerod also found that. It might be necessary to find other realisation of Lax pairs.

Commuting operators for multivariable degenerate operators

Komori and Hikami proved existence of the commuting operators for the multivariable Ruijsenaars-van Diejen operator [3].

Symmetry and Kernel functions of degenerate operators

Ruijsenaars-van Diejen operator of one variable admits E_8 symmetry [7] and a kernel function plays important roles.

Further degenerations of Ruijsenaars-van Diejen operator

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