elliptic stable envelopes (ESE)

an informal discussion

(Mina Aganagic + A.O.)

ESEs are a bit like classes of Schubert varieties in elliptic cohomology

not an accurate comparison

Schubert cells

Schubert varieties = closures of attracting manifolds for the action of

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

on $Gr(k, n)$

$$|a_1| \gg |a_2| \gg \dots \gg |a_n|$$

Grassmannian of k-dim subspaces $L \subset \mathbb{C}^n$

Analogous cycles are important in many other contexts

What does it mean to give a formula for a cycle ?

$H^\bullet(Gr(k,n))$ is generated by Chern classes of the tautological bundle

$$\{L \subset \mathbb{C}^n\}$$

$$\mathrm{Taut} = \mathbb{C}^k \times \mathrm{Hom}(\mathbb{C}^k \to \mathbb{C}^n)_{\text{full rank}} / GL(k)$$

$$\downarrow$$

$$Gr = \mathrm{Hom}(\mathbb{C}^k \to \mathbb{C}^n)_{\text{full rank}} / GL(k)$$

Hence $H^\bullet(Gr) = \text{symm poly}(\xi_1, \ldots, \xi_k) / \text{relations}$

meaning $\operatorname{Spec} H^\bullet(Gr) \hookrightarrow \left(\operatorname{Lie} GL(k)\right)/\mathrm{Ad}$

It may be easier to imagine A - equivariant K-theory of Gr, generated by equivariant vector bundles $\mathcal{V} \to Gr$ with operations $\oplus$, $\otimes$, and modulo extensions

Now the variables and

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in A \qquad \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} \in GL(k)$$

play an almost symmetric role and

$$\operatorname{Spec} K_A(Gr) \subset A \times \underset{\text{of } GL(k)}{\text{max torus}} \Big/\!\!\Big/ \underset{\text{of } GL(k)}{\text{Weyl group}}$$

coordinate subspaces

The natural map

$$K_A(\mathrm{Gr}) \longrightarrow K_A(\mathrm{Gr}^A)$$

is a normalization and it exhibits

arrangement of tori

$$\operatorname{Spec} K_A(\mathrm{Gr}) = \bigcup_{\substack{k\text{-element} \\ \text{subsets } S \subset \{1,\dots,n\}}} \{x_i = a_{s_i}\}_{i=1}$$

may look familiar as poles in various integration formulas related to $\mathfrak{sl}_2$

To write an element of $K_A(Gr)$ is to write a function on this arrangement of tori

Similarly, in equivariant elliptic cohomology there is a mod $q^{\mathbb{Z}}$ reduction of this picture, that is, a rank r equivariant vector bundle gives a map

$$Ell_A(X) \longrightarrow \mathcal{E}_A \times S^r E \qquad E = \mathbb{C}^\times / q^{\mathbb{Z}}$$

now a scheme, U of $\mathcal{E}_A$

where $\mathcal{E}_A = A / q^{\operatorname{cochar} A} \simeq E^{\operatorname{rk} A}$

where $|q| < 1$ is the modulus of the base curve E

We will be working with manifolds for which cohomology is generated by vector bundles in the sense the map

$$Ell_A(X) \longrightarrow \mathcal{E}_A \times \prod S^{r_i} E$$

is, modulo technicalities, an embedding

It is silly to look for functions on this, but it makes sense to study line bundles

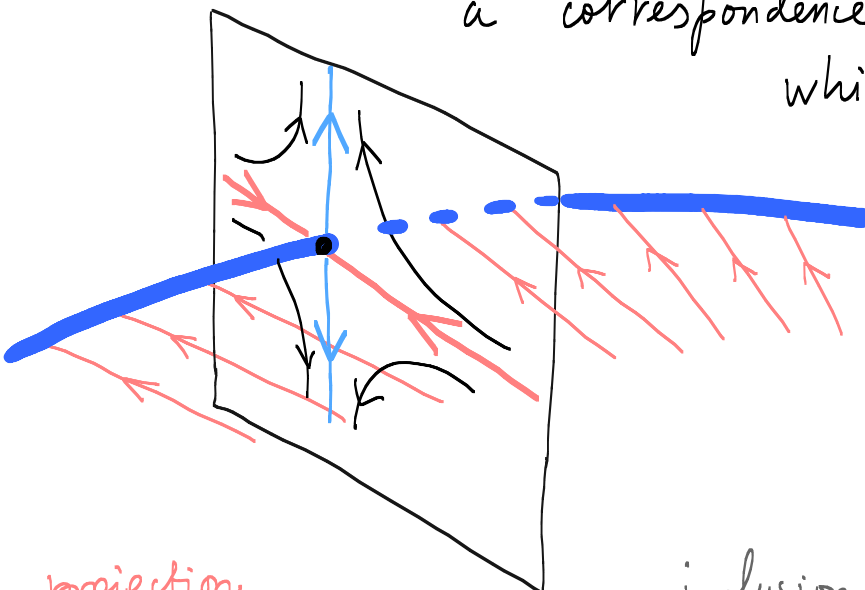
Geometrically defined line bundles on $\mathrm{Ell}\,X$

- for any vector bundle V with Chern roots $x_1, \dots, x_{\mathrm{rank}}$ we have $\Theta(V)$ with section $\prod \vartheta(x_i)$
- elements of $\mathrm{Pic}_0(\mathrm{Ell}\,X)$ come from line bundles $\mathcal{L}$ and we have the Poincaré bundle $\mathcal{U}$ with rational section

$$\frac{\vartheta(x z)}{\vartheta(x)\,\vartheta(z)} \qquad x = \text{chern root of } \mathcal{L}$$

$$\mathcal{L} \otimes z \in \mathcal{E}_{\mathrm{Pic}} = \mathrm{Pic}(X) \otimes E$$

elliptic stable envelopes will be sections of some combinations of these line bundles now back to Schubert classes and more general attracting manifolds: one is looking for a correspondence $X^A \to X$ which looks like



fixed locus of A

$$X^A \xleftarrow{\text{projection}} \text{Attracting manifold} \xhookrightarrow{\text{inclusion}} X$$

defining a good correspondence is delicate (closures are bad) especially in elliptic cohomology which is very sensitive to normal bundles etc.

If we assume that X = holomorphic symplectic, and A preserves sympl form ω then, at least

$$\mathrm{Normal}(X^A) = \text{Attracting} \oplus \hbar^{-1} \otimes \text{Attracting}^*$$

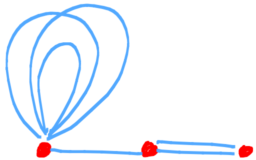
equivariant weight of ω

upgrade: $Gr \rightsquigarrow T^* Gr$, an extra $\mathbb{C}^\times_\hbar$ scales T^*

Schubert cells $\rightsquigarrow$ conormal to Schubert cells

$T = \mathbb{C}^\times_\hbar \times A$

T^*Gr is an example of Nakajima quiver variety

quiver =  $\rightsquigarrow$ Cartan matrix

$$\begin{pmatrix} -4 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

Nakajima quiver varieties are indexed by a pair $v, w \in \mathbb{N}^{\text{vertices}}$

for $\mathfrak{g} = \mathfrak{sl}_2$, $C = (2)$, quiver $= \{\bullet\}$

one gets $X = T^*Gr(v, w)$

have natural $\frac{1}{2}$'s of the tangent bundle, called polarizations of X and denoted $T^{1/2}$

General definition of ESE: this is a map

Stab: $\Theta(T^{1/2}X^A) \otimes \text{shift}\ \mathcal{U} \longrightarrow \Theta(T^{1/2}X) \otimes \mathcal{U} \otimes \dots$

as sheaves on $\mathcal{B} = \mathcal{E}_T \otimes \mathcal{E}_{\mathrm{Pic}}$ which is supported on the full attracting set of X^A and equals

$$\text{Stab} = \pm\, \text{inclusion}_* \ \text{projection}^* \quad \text{near } X^A \qquad (\star)$$

here both shift and .... = pull-back of a line bundle from $\mathcal{B}/\mathcal{E}_A$

are uniquely determined by $(\star)$

Concretely for $T^*Gr(k,n)$ this means that for every subset $S \subset \{1,..,n\}$, $|S| = k$ $\rightsquigarrow$ coordinate subspace in Gr

$$\|$$

a point of X^A

we should find a section of $\Theta(T^{1/2}X)$ with triangular support, prescribed automorphy w.r.t. $a_i \mapsto q\, a_i$, and given normalization on the corr. component of

$$\bigsqcup_{|S|=k} \mathcal{E}_T = \mathrm{Ell}(X^A) \to \mathrm{Ell}(X)$$

ESE for $T^*\mathbb{P}^{n-1}$

The answer for T^*Gr may be written down explicitly

Define:

$$f_s(x,z) = \prod_{i<s} \vartheta(x a_i) \frac{\vartheta(x z a_i \hbar^{s-n})}{\vartheta(z \hbar^{s-n})} \prod_{i>s} \vartheta(x a_i \hbar)$$

$$\mathrm{Stab}_S = \mathrm{Symm} \frac{\prod_{s_i \in S} f_{s_i}(x_i, z\hbar^{2\rho_i})}{\prod_{i<j} \vartheta(x_i/x_j)\, \vartheta(x_j/x_i/\hbar)}$$

elliptic version of the interpolation Schur function

restricts to a regular function on $\mathrm{Ell}\, X$

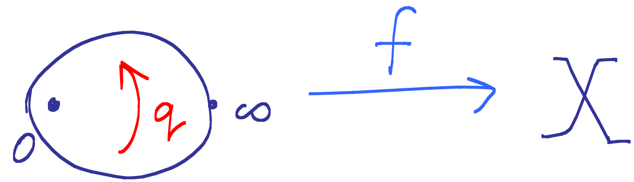
an instance of abelianization

General theorems [Aganagic – A.O

- ESE are always unique
- ESE exists for Nakajima varieties
- for Nakajima varieties, ESE give Gauß factorization of elliptic solutions to dynamical YB

Main application: monodromy of geometric q-difference equations

quantum K-theory of X $\approx$ K-theory of moduli spaces of rational curves in X



a very important role is played by quantum connection

– a flat q difference connection with fiber $K(X)$

over $A \times \underbrace{\left(\mathrm{Pic}(X) \otimes \mathbb{C}^{\times}\right)}$ ← "Kähler torus" $\mathfrak{Z}$ with coordinates z

it is a complicated Theorem [Smirnov-O] that these are the

$$\text{qKZ equations} + \text{dynamical equations}$$

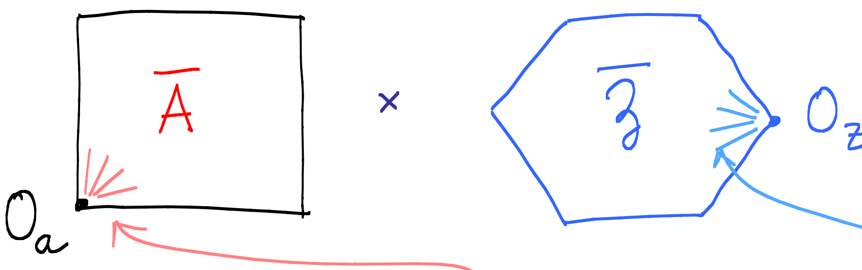
for a certain $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ constructed in the style of [Maulik-O]

the very definition of dynamical equations is not obvious because the W_{aff} = affine Weyl group of $\hat{\mathfrak{g}}$ is typically much too small this connection is separately regular in

equivariant variables a and Kähler variables z

but is not regular jointly (like hypegeom eq in z +
contiguity relations in a)

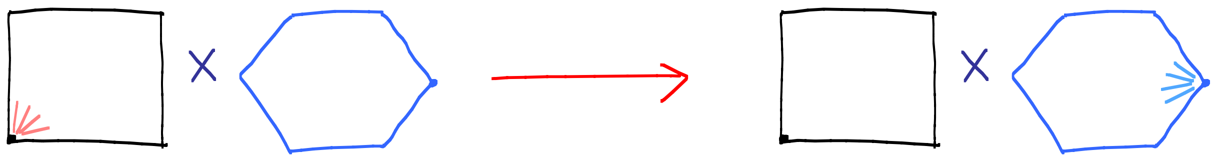
so near a point $(a, z) = (0_a, 0_z) \in \overline{A} \times \overline{\mathcal{Z}}$



we have an a-solution, holo here, and a z-solution holo here

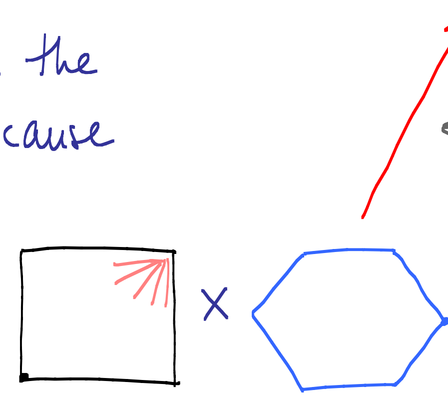
in the other variable $\rightsquigarrow$ accumulation of poles

Therefore we have an elliptic pole-subtraction matrix



a - solutions $\qquad\qquad$ z - solutions

these really constrain the monodromy because



another pole-subtraction matrix, just like the first one

a - solutions

near a <u>different</u> end of $\bar{A}$

A geometric argument proves the following

Theorem [Aganagic - O]

(with suitable normalization)

pole subtraction matrix = elliptic stable envelopes for $X^A \hookrightarrow X$

the point $0_a \in \overline{A}$ = choice of attracting / repelling directions

the point $0_z \in \overline{Z}$ = choice of one X among all flops