$\mathcal{N} = 1$ CFTs, dualities, integrable models

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Disclaimers

- I am a physicist, I am sorry
- The talk will lack proofs
- It will have consistency checks

General Idea of Generating Results

- Compute the same thing in different ways
- The result is the same, the computations might differ vastly
- Calculate with the symmetry of the problem not manifest
- The result should exhibit the symmetry

Outline

- How we know what we know?
- What we think we know?
- What we know we do not know?

Models

Dimensional Reductions

- Reduce physics in D dimension to D' $(< D)$ dimensions on compact manifold

- Choice of theory in D dimensions and the choices made during compactification will determine the model in D' dimensions

- Same compactifications (even if looking differently) lead to same theories, and many properties such as symmetries are implied from the compactification

- In our discussion $D' = 4$ and $D = 6$

Six dimensions

- We start with a model in six dimensions, $\mathcal{T}^{6d}$

- Vast number of possibilities

- A nice set of models labeled by a pair of ADE groups, $T^{6d}_{ADE_1,ADE_2}$

- *ADE_1 type of M5 branes probing ADE_2 type singularity*

- T^{6d}_{ADE,A_0} widely studied (Gaiotto), we will discuss $T^{6d}_{A_{N-1},A_{k-1}}$

Six dimensions, symmetry

- $T^{6d}_{ADE_1,ADE_2}$ are strongly coupled SCFTs

- We know they have flavor symmetry G

- $ADE_1 = A_{N-1}$ and $ADE_2 = A_{k-1}$

$$\text{general } N,\, k: \qquad G = su(k)su(k)u(1)$$

$$N = 2, \text{ general } k: \qquad G = su(2k)$$

$$\text{general } N,\, k = 2: \qquad G = su(2)su(2)su(2)$$

$$N = 2 \text{ and } k = 2: \qquad G = so(7)$$

Details of compactification

- Consider T^{6d} on a compact Riemann surface $\mathcal{C}_{g,s}$

- Upon compactification have discrete choices of
  - genus g
  - punctures s
  - (information at the punctures)
  - Flux for G through $\mathcal{C}_{g,s}$; vector of ints $\mathcal{F} = (n_1, \cdots, n_{rank(G)})$

Details of compactifications II

- The flux $\mathcal{F}$ breaks G to G_{max} (containing L abelian factors)
- Upon compactification have continuous choices of

  • complex structure moduli ($3g-3$ complex parameters)

  • flat connections for G_{max} ($(g-1)dimG_{max}+L$ complex parameters)

Four dimensional models

- The low energy limit of $\mathcal{T}^{6d}$ on the Riemann surface with fluxes is described by a four dimensional theory, $\mathcal{T}^{4d|ADE_1,ADE_2}_{g,s;\mathcal{F}}$

- different discrete choices $g, s, \mathcal{F}$ give different theories

- different continuous choices complex structure and flat connections give different parameters (couplings) for a given theory

- different choices give different properties in four dimensions, different number of parameters and different flavor symmetries

Example

- Take $\mathcal{T}^{6d}_{A_1,A_1}$ ($N = 2$ and $k = 2$), $G = so(7)$
- Different choices of $\mathcal{F} = (a, b, c)$ give theories in four dimensions with different symmetries

G_{max}	$u(1)^3$	$su(2)_{diag}u(1)^2$	$su(2)u(1)^2$	$su(2)su(2)u(1)$
$\mathcal{F}$	(a, b, c)	$(b, \pm b, c)$	$(a, 0, b)$	$(b, 0, 0)$

G_{max}	$su(3)u(1)$	$so(5)u(1)$	$so(7)$
$\mathcal{F}$	$(b, 0, \pm b)$	$(0, 0, b)$	$(0, 0, 0)$

Four dimensional theories, dualities

- Note that to specify the compactification we need to specify a point on a moduli space (complex structure and flat connections)

- There are different ways to present the same compactification (different pairs of pants decompositions for a surface)

- The action of the mapping class group of the Riemann surface is identified with a duality group o field theory (for example $\tau \to 1+\tau$ and $-\frac{1}{\tau}$ for a torus)

- This is the source of duality in four dimensions

Summary

- Upon compactification of six dimensional theories we can obtain a vast variety of four dimensional theories
- One six dimensional theory can produce a plethora of four dimensional models depending on choices made upon compactification
- Some compactifications are equivalent though are constructed differently, giving duality in four dimensions
- What are the theories in four dimensions? (We do not know in general!! This is the number one motivation to work on the subject)

Computations

Supersymmetric Partition Functions

- What can we compute for the four dimensional theories?
- Even when the theories are known they are complicated QFTs and computations are difficult
- For some special computations the integration over the space of fields localizes to finite dimensional sums and integrals; supersymmetric localization
- We can compute the partition functions explicitly using this localization when the Lagrangians are known
- Usually these are given in terms of integrals and sums over special functions. Well defined mathematical expressions which encode PHYSICS in numerous intricate ways.

Partition functions

- Supersymmetric index, partition function on $\mathbb{S}^3 \times \mathbb{S}^1$ (integrals over elliptic gamma functions)
- Lens index, partition function on $\mathbb{S}^3/\mathbb{Z}_r \times \mathbb{S}^1$ (sums of integrals over elliptic gamma functions)
- "Twisted" partition functions, partition functions over $\mathcal{S} \times \mathbb{S}^1$ where $\mathcal{S}$ is Seifert manifold (simpler special functions, complicated contours (JK))
- The above do not depend on continuous parameters
- partition function on $\mathbb{S}^4$. Depends on couplings.

Supersymmetric Index

- Typical computation for theory with a Lagrangian (SQCD with N_f flavors of $SU(N)$)

$$\mathcal{I}(t_l, r_l) =$$

$$\frac{((q;q)(p;p))^{N-1}}{N!}\oint\prod_{j=1}^{N-1}\frac{dz_j}{2\pi i z_j}\frac{1}{\prod_{k\neq j}\Gamma_e(z_j/z_k)}\prod_{j=1}^{N}\prod_{l=1}^{N_f}\Gamma_e(t_l z_j)\Gamma_e(r_l\frac{1}{z_j})$$

- The parameters t_l and r_l label the maximal torus of the flavor symmetry $(SU(N_f)^2 \times U(1))$
- The index can be thought of as a thermal partition function of the model with $(-1)^F$

Index from reductions

- In the construction of the four dimensional theories from reductions we thus can compute functions

$$\mathcal{I}^{ADE_1,ADE_2}_{g,s,\mathcal{F}}(t_i)$$

where t_i label the maximal torus of flavor symmetry (G_{max})

- These functions should be invariant under dualities

- These functions should exhibit all the expected properties

Physics from index

- Index can be thought as sum over characters of the superconformal group
- It encodes non trivial physics
- For example, $\mathcal{I} = 1 + \cdots + (M - J)pq + \dots$
- M are marginal operators in the theory and J are the conserved currents. J has to be in the adjoint of the flavor.

Results

Dualities

- Many non trivial identities follow from dualities
- Dualities correspond to different pair of pants decompositions of a given Riemann surface
- In general we do not know how to write explicitly indices of theories corresponding to a compactification
- In some cases ($\mathcal{T}^{6d}_{A_1,A_0}$, $\mathcal{T}^{6d}_{A_2,A_0}$, and $\mathcal{T}^{6d}_{A_1,A_1}$) we know all the indices and can write all the non trivial relations

Notations

- $\Gamma_e(u) = \prod_{\kappa,\lambda=0}^{\infty} \frac{1-\frac{pq}{u}p^\kappa q^\lambda}{1-up^\kappa q^\lambda}$

- $\prod_{i=1}^k \beta_i = \prod_{j=1}^k \gamma_j = 1$ (Cartans of $su(k)su(k)$)

- $\prod_{j=1}^N z_j = 1$

Simple identities

- For general case of $\mathcal{T}^{6d}_{A_{N-1},A_k}$ we know the indices for certain four punctured spheres

$$\mathcal{I}_{N,k}(\mathbf{h},a,\mathbf{w},b|p,q,t,\beta,\gamma) = \left(\frac{((q;q)(p;p))^{N-1}}{N!}\right)^k$$

$$\oint \prod_{i=1}^{k}\prod_{j=1}^{N-1} \frac{dz_{i;j}}{2\pi i z_{i;j}} \frac{\prod_{i=1}^k \prod_{j,l=1}^N \Gamma_e(\frac{pq}{t}\beta_i\gamma_i z_{i,j} z_{i+1,l}^{-1})}{\prod_{i=1}^k \prod_{j\neq l}^N \Gamma_e(z_{i,j} z_{i,l}^{-1})}$$

$$\prod_{i=1}^{k}\prod_{j,l=1}^{N} \Gamma_e(t^{\frac{1}{2}}\beta_i^{-1} a z_{i;j} h_{i;l}^{-1})\Gamma_e(t^{\frac{1}{2}}\beta_i^{-1} b w_{i;j} z_{i-1;l}^{-1})$$

$$\Gamma_e(t^{\frac{1}{2}}\gamma_i^{-1} a^{-1} h_{i;j} z_{i-1;l}^{-1})\Gamma_e(t^{\frac{1}{2}}\gamma_i^{-1} b^{-1} w_{i;j}^{-1} z_{i;l})$$

- This is invariant under exchange of $a \longleftrightarrow b$
- $k=1$ and $N=2$ proven by Van de Bult.

Simpler identity

- The identity on previous slide follows from $k = 1$ case and identities proven by Rains

- The k being one identity is

$$\oint \prod_{l=1}^{N-1} \frac{du_l}{2\pi i u_l} \prod_{m\neq s} \frac{\Gamma_e(\frac{pq}{t} u_s/u_m)}{\Gamma_e(u_m/u_s)} \prod_{l,s=1}^{N} \Gamma_e(t^{\frac{1}{2}}(a h_s u_l)^{\pm 1})\Gamma_e(t^{\frac{1}{2}}(b v_s u_l^{-1})^{\pm 1})$$

- is invariant under exchange of a and b

Non obvious symmetry

- The index of the three punctured sphere of $\mathcal{T}^{6d}_{A_2,A_0}$ is

$$\mathcal{I}((w,r),\mathbf{u},\mathbf{v}) = \frac{(q,q)(p,p)}{2}$$
$$\oint \frac{dz}{2\pi i z}\frac{\Gamma_e(t^{-\frac{1}{2}}z^{\pm1}w^{\pm1})}{\Gamma_e(z^{\pm2})\Gamma_e(\frac{pq}{t}w^{\pm2})\Gamma_e(t^{-1})}\mathcal{I}_{3,1}(\mathbf{u},s^{\frac{1}{3}}r,\mathbf{v},s^{-\frac{1}{3}}r)$$

- This is explicitly invariant under Weyl of $su(3)_{\mathbf{u}} \times su(3)_{\mathbf{v}} \times su(2)_w$
- Physically we can argue that $su(2)_w u(1)_r$ enhances to $su(3)$ and then the three $su(3)s$ enhance to E_6.
- To derive the above we used Spiridonov-Warnaar inversion formula.
- We will soon mention that this function is a Kernel function for A_2 RS integrable system.

Trinion A of $\mathcal{T}^{4d}_{A_1,A_1}$

- For $\mathcal{T}^{6d}_{A_1,A_1}$ we can compute indices of some three punctured spheres

$$\mathcal{I}_{T_A^+}(\mathbf{v},\mathbf{w},\mathbf{c}) = \Gamma_e(t(\frac{\gamma}{\beta}v_2)^{\pm1}v_1^{\pm1})\Gamma_e(p\,q\frac{1}{\beta^2\gamma^2})(p;p)(q;q)$$

$$\oint \frac{dz}{4\pi i z}\frac{\Gamma_e(\frac{pq}{t\gamma\beta}(\frac{\beta}{\gamma v_2})^{\pm1}z^{\pm1})}{\Gamma_e(z^{\pm2})}\Gamma_e(\gamma\beta z^{\pm1}v_1^{\pm1})\mathcal{I}_{2,2}(\mathbf{c},\sqrt{zv_2},\mathbf{w},\sqrt{\frac{v_2}{z}})$$

- This expression is again not explicitly invariant under the exchange of the three factors of $su(2)^2$ but physics tells us it is.

- The theory corresponds to $\mathcal{F}=(\frac{1}{4},\frac{1}{4},1)$

- The index of T_A^- with $\mathcal{F}=(-\frac{1}{4},-\frac{1}{4},-1)$ obtained by inverting some of the parameters ($t\to pq/t$, $\gamma\to 1/\gamma$ and so on)

Trinion B of $\mathcal{T}^{4d}_{A_1,A_1}$

- We can compute index of another trinion

$$\mathcal{I}_{T_B^+}(\mathbf{v};\mathbf{w},\mathbf{c}) = \Gamma_e(t(\beta\gamma v_2)^{\pm1}v_1^{\pm1})\Gamma_e(p\,q\frac{\beta^2}{\gamma^2})(p;p)(q;q)$$
$$\oint\frac{dz}{4\pi i z}\frac{\Gamma_e(\frac{pq\beta}{t\gamma}(\beta\gamma v_2)^{\pm1}z^{\pm1})}{\Gamma_e(z^{\pm2})}\Gamma_e(\frac{\gamma}{\beta}z^{\pm1}v_1^{\pm1})\mathcal{I}_{2,2}(\mathbf{c},\sqrt{zv_2},\mathbf{w},\sqrt{\frac{v_2}{z}})$$

- In this trinion the first $su(2)^2$ factor is not the same as the two others.

- The theory corresponds to $\mathcal{F} = (-\frac{1}{4},\frac{1}{4},1)$

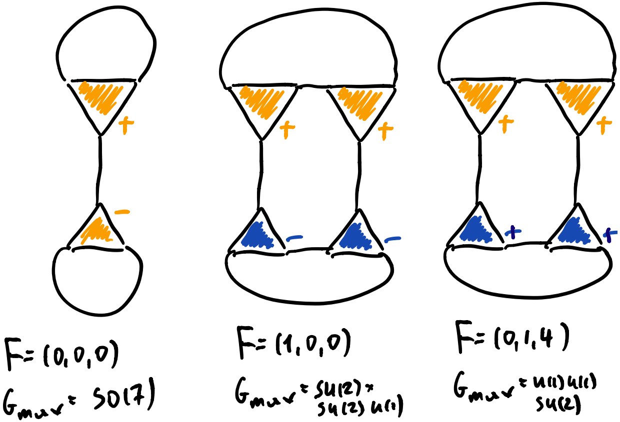
- The index of T_B^- with $\mathcal{F} = (\frac{1}{4},-\frac{1}{4},-1)$ obtained by inverting some of the parameters ($t\to pq/t$, $\gamma\to 1/\gamma$ and so on)

Gluing

- We can take indices of two theories corresponding to Riemann surface of genus $g_{1,2}$ and having $s_{1,2}$ punctures and fluxes $\mathcal{F}_{1,2}$ to construct the index of theory corresponding to model with flux $\mathcal{F}_1+\mathcal{F}_2$, with s_2+s_1-2 punctures and genus g_1+g_2

$$\mathcal{I}=\frac{(q;q)^2(p;p)^2}{4}\oint\frac{da_1}{2\pi i a_1}\frac{da_2}{2\pi i a_2}\frac{\Gamma_e(\frac{pq}{t}(\beta/\gamma)^{\pm1}a_1^{\pm1}a_2^{\pm1})}{\Gamma_e(a_2^{\pm2})\Gamma_e(a_1^{\pm2})}\mathcal{I}_1(\mathbf{a})\mathcal{I}_2(\bar{\mathbf{a}})$$

- With this gluing we can combine the trinions to form closed Riemann surfaces with a variety of fluxes and having any genus



- Remember that the theories are guaranteed to have symmetry G_{max} (with β, γ, and t parametrizing the Cartan)
- This is highly non trivial as the ingredients do not have these symmetries, but the combined indices indeed exhibit it!!

Index and moduli spaces

- Remember that at order pq the index gives marginal operators minus the conserved currents. These are precisely (in a general situation) the couplings of the models

- Computing the index of general models we obtain

$$\mathcal{I} = 1 + \cdots + ((g-1)\chi_{Adj.}(G_{max}) + 3g - 3 + L - L)pq + \cdots$$

- This gives us exactly $3g - 3$ complex structure moduli, L conserved currents of symmetries unbroken on the general point of the conformal manifold, and $(g-1)dimG_{max} + L$ corresponding to flat connections

- The index encodes simple invariants of the underlying geometry!!

Residues of the index and surface defects

- The index has many poles in the parameters

- The residues of the poles have a physical meaning

- These are indices of theories obtained in the IR of an RG flow triggered by turning on vacuum expectation values to some operators

- Certain type of poles depending on p and q correspond to turning on space-time dependent vacuum expectation values and lead to models with surface defects

Residues and difference operators

- Interestingly when one computes the residues corresponding to insertion of defects one obtains the following relation

$$Res_{a\to a^*}\mathcal{I}_{g,s+1}^{ADE_1,ADE_2}(a,z,y,\dots)=$$
$$\mathcal{D}_{ADE_1,ADE_2}(z)\mathcal{I}_{g,s}^{ADE_1,ADE_2}(z,y,\dots)=$$
$$\mathcal{D}_{ADE_1,ADE_2}(y)\mathcal{I}_{g,s}^{ADE_1,ADE_2}(z,y,\dots)$$

- Here $\mathcal{D}_{ADE_1,ADE_2}$ are certain difference operators
- Because of the duality properties of the index it does not matter on which set of parameters they act. The indices are kernel functions of these.
- In case of $ADE_2=A_0$ the operators are Ruijsenaars-Schneider elliptic Hamiltonians corresponding to ADE_1!!

Operators for $\mathcal{T}_{A_1,A_1}$ case

- We can compute the operators for the $ADE_1 = A_{N-1}$ and $ADE_2 = A_{k-1}$ case and here is example of the simplest operator for the $N = 2$ $k = 2$ case

$$\mathcal{T}(v_1, v_2; \beta, \gamma, t) = \frac{\theta(\frac{tv_1^{-1}v_2^{-1}}{q}(\frac{\gamma}{\beta})^{\pm 1}; p)\theta(\frac{t\beta v_1}{\gamma v_2}; p)\theta(\frac{t\beta^3\gamma v_2}{v_1}; p)}{\theta(v_1^2; p)\theta(v_2^2; p)}$$

$$\mathcal{D} \cdot f(v_1, v_2) = \sum_{a,b=\pm 1} \mathcal{T}(v_1^a, v_2^b; \beta, \gamma, t) f(q^{\frac{a}{2}} v_1, q^{\frac{b}{2}} v_2)$$

- Computing different residues a commuting set of operators can be constructed

Eigenfunctions

- We can ask what are the eigenfunctions of these operators, at least in special limits ($p=0$). These should be orthonormal under the measure of gluing Taking for simplicity also $q=0$ and β, γ to one, we obtain that these are given by $\frac{1}{(1-t\, z_1^{\pm 1} z_2^{\pm 1})^2}$ times

$$\hat{\psi}_{(0)} = 1 \qquad \hat{\psi}_{(1)_\pm} = ((z_1 + z_1^{-1}) \pm (z_2 + z_2^{-1})),$$

$$\hat{\psi}_{(2)_0} = ((z_1^2 + z_1^{-2}) - (z_2^2 + z_2^{-2})),$$

$$\hat{\psi}_{(2)_\pm} = \left(-\frac{\left(\pm\sqrt{2-t^2}+t\right)\left(z_1^4+1\right)}{2\left(t^2-1\right) {z_1}^2} - \frac{\left(\pm\sqrt{2-t^2}+t\right)\left({z_2}^4+1\right)}{2\left(t^2-1\right) {z_2}^2}\right.$$

$$\left. \pm\sqrt{2-t^2} - t + \left(z_1 + \frac{1}{z_1}\right)\left(z_2 + \frac{1}{z_2}\right)\right) .$$

$$k = 1 \ (\mathcal{N} = 2) \qquad \to \qquad k > 1 \ (\mathcal{N} = 1)$$

rational $\qquad \to \qquad$ algebraic

Open problems

Proofs

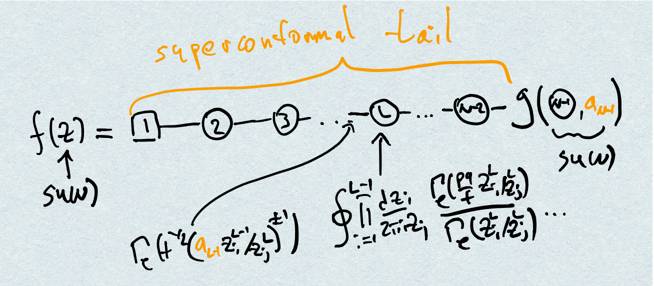
- It should be nice to have proofs of the different statements
- From physics point of view a proof of some identity will be yet another consistency check
- However, hopefully the proofs will also give new insight to why the physics is correct

Eigenfunctions

- Having understanding of the eigenfunctions, even in limits, will be extremely helpful
- The eigenfunctions can give us a computational tool to study models about which otherwise we know little
- Some limits of the eigenfunctions, when we know them, have a physical meaning by themselves in three dimensions. Knowing these can teach us about the dimensional reductions of the theories

Inversions

- The Spiridonov-Warnaar inversions are extremely useful in the field theoretic constructions.
- They have physical meaning and allow to extract information about non trivial theories from duality
- However we need more inversions



Thank You !!