Elliptic hypergeometric functions

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Overview

- Hypergeometric series
- Elliptic hypergeometric series
- The families of hypergeometric functions
- Identities for elliptic hypergeometric functions

Geometric series

Definition

A geometric series is a series $\sum_n c_n$ for which the ratio $r = \frac{c_{n+1}}{c_n}$ is a constant.

Corollary

Thus $c_n = rc_{n-1} = \cdots = r^n c_0$. *The series becomes* $\sum_n r^n c_0 = \frac{c_0}{1-r}$.

Hypergeometric series

Definition

A hypergeometric series is a series $\sum_n c_n$ for which the ratio $r(n) = \frac{c_{n+1}}{c_n}$ is a rational function of n.

Example

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}$$

Factoring rational functions

Theorem

Any rational function can be written as

$$r(n) = \frac{(n+a_1)(n+a_2)\cdots(n+a_r)}{(n+1)(n+b_1)\cdots(n+b_s)}z$$

Corollary

The coefficients c_n in a hypergeometric series $\sum c_n$ are

$$c_n = \frac{((n-1)+a_1)\cdots}{\cdots}\frac{((n-2)+a_1)\cdots}{\cdots}\cdots\frac{(0+a_1)\cdots}{\cdots}c_0$$

Pochhammer symbol

Definition

The Pochhammer symbol is defined as

$$(a)_n = a(a+1)\cdots(a+n-1) = \prod_{j=0}^{n-1}(a+j).$$

We write

$$(a_1,\ldots,a_r)_n = (a_1)_n\cdots(a_r)_n \qquad (\pm a)_n = (a)_n(-a)_n$$

Observe

$$(1)_n = n!$$

Gamma function

Definition

We set

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx = \frac{1}{z} \prod_{n=1}^{\infty} \frac{1}{(n+z)} n(1+\frac{1}{n})^z$$

Lemma

$$\Gamma(z+1) = z\Gamma(z), \qquad \frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n$$

${}_rF_s$

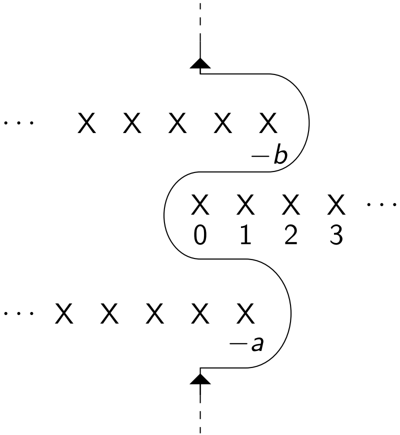
Definition

$$ {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{n!(b_1, b_2, \ldots, b_s)_n} z^n $$

Integral/series correspondence

Theorem

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s, b+s, -s)}{\Gamma(c+s)} (-z)^s \frac{ds}{2\pi i}$$
$$= \sum_{n=0}^{\infty} Res\left(\frac{\Gamma(a+s, b+s, -s)}{\Gamma(c+s)} (-z)^s, s=n\right)$$
$$= \frac{\Gamma(a,b)}{\Gamma(c)} {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; z\right]$$



Overview

- Hypergeometric series
- Elliptic hypergeometric series
- The families of hypergeometric functions
- Identities for elliptic hypergeometric functions

Elliptic hypergeometric series

Definition

An elliptic hypergeometric series is a series $\sum_n c_n$ for which the ratio $r(n) = \frac{c_{n+1}}{c_n}$ is an elliptic function of n.

Definition

An elliptic function $f : \mathbb{C} \to \mathbb{C}$ is a meromorphic function which is periodic in two directions:

$$f(z + \omega_1) = f(z), \qquad f(z + \omega_2) = f(z)$$

for $\omega_1/\omega_2 \notin \mathbb{R}$.

p-ellipticity

Remark

For a $q \in \mathbb{C}$ with $|q| < 1$ we use $r(n) = s(q^n)$.

- One period is given: $r(n + 2\pi i/\log(q)) = r(n)$.
- The second period means $s(q^n) = s(q^{n+\omega_2})$, so with $p = q^{\omega_2}$ we get $s(z) = s(pz)$.

Definition

A function $f : \mathbb{C}^* \to \mathbb{C}^*$ is called p-elliptic if $f(pz) = f(z)$.

Theta function

Definition

For $|p| < 1$

$$\theta(x;p) = \prod_{r=0}^{\infty} (1 - p^r x)(1 - p^{r+1}/x).$$

Lemma

- *$\theta(x;p)$ is a holomorphic function of x for $x \in \mathbb{C} \setminus \{0\}$, with zeros at $p^{\mathbb{Z}}$.*
- *Symmetries:* $\theta(px;p) = \theta(1/x;p) = -\dfrac{1}{x}\theta(x;p)$

Factoring elliptic functions

Theorem

Any p-elliptic function can be factored as

$$f(z) = \frac{\theta(a_1 z, a_2 z, \ldots, a_r z; p)}{\theta(qz, b_1 z, b_2 z, \ldots, b_{r-1} z; p)} x$$

under the balancing condition

$$a_1 a_2 \cdots a_r = q b_1 b_2 \cdots b_{r-1}$$

Standard form c_n

Corollary

If

$$\frac{c_{n+1}}{c_n} = \frac{\theta(a_1 q^n, a_2 q^n, \ldots, a_r q^n; p)}{\theta(q^{n+1}, b_1 q^n, b_2 q^n, \ldots, b_{r-1} q^n; p)} x$$

then

$$c_n = \frac{\theta(a_1, a_2, \ldots, a_r; p)_n}{\theta(q, b_1, b_2, \ldots, b_{r-1}; p)_n} x^n c_0$$

where

$$\theta(x; p)_n = \prod_{r=0}^{n-1} \theta(xq^r; p)$$

Elliptic gamma functions

Definition

$$\Gamma_e(z;p,q) = \Gamma_e(z) = \prod_{r,s\geq 0} \frac{1-p^{r+1}q^{s+1}/z}{1-p^r q^s z}$$

Corollary

$$\Gamma_e(pz) = \theta(z;q)\Gamma_e(z) \qquad \Gamma_e(qz) = \theta(z;p)\Gamma_e(z)$$

and thus

$$\theta(x;p)_n = \frac{\Gamma_e(q^n x)}{\Gamma_e(x)}$$

${}_rE_{r-1}$

Definition

Any elliptic hypergeometric series can be expressed as

$$ {}_rE_{r-1}\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{r-1} \end{matrix} ;z\right] = \sum_{k=0}^{\infty} \frac{\theta(a_1,\ldots,a_r;p)_k}{\theta(q,b_1,\ldots,b_{r-1};p)_k} z^k $$

under the balancing condition $a_1 a_2 \cdots a_r = q b_1 b_2 \cdots b_{r-1}$.

${}_rV_{r-1}$

Definition

Assuming r is even and the balancing condition $b_1 b_2 \cdots b_{r-6} = a^{\frac{r}{2}-3} q^{\frac{r}{2}+n-4}$ holds, the terminating very-well-poised series is given by

$$
\begin{aligned}
{}_rV_{r-1}&(a; b_1, \ldots, b_{r-6}, q^{-n}) \\
&= {}_rE_{r-1}\left[\begin{matrix} a, \pm q\sqrt{a}, \pm q\sqrt{ap}, b_1, \ldots, b_{r-6}, q^{-n} \\ \pm\sqrt{a}, \pm\sqrt{ap}, \frac{aq}{b_1}, \ldots, \frac{aq}{b_{r-6}}, aq^{n+1} \end{matrix} ; q \right] \\
&= \sum_{k=0}^{n} \frac{\theta(aq^{2k}; p)}{\theta(a; p)} \frac{\theta(a, b_1, \ldots, b_{r-6}, q^{-n}; p)_k}{\theta(q, \frac{aq}{b_1}, \ldots, \frac{aq}{b_{r-6}}, aq^{n+1}; p)_k} q^k
\end{aligned}
$$

p-ellipticity of series

Lemma

As long as the balancing condition $b_1 b_2 \cdots b_{r-6} = a^{\frac{r}{2}-3} q^{\frac{r}{2}+n-4}$ remains valid, we have for $\alpha, \beta_j \in \mathbb{Z}$ we have

$$ {}_rV_{r-1}(ap^{\alpha}; b_1 p^{\beta_1}, \ldots, b_{r-6} p^{\beta_{r-6}}, q^{-n}) = {}_rV_{r-1}(a; b_1, \ldots, b_{r-6}, q^{-n}) $$

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3 types of series

Definition

The series $\sum c_n$ with $c_{n+1}/c_n = r(n)$ is

- hypergeometric if $r(n)$ is a rational function of n.
- basic or q-hypergeometric if $r(n)$ is a rational function of q^n
- elliptic hypergeometric if $r(n)$ is an elliptic function of n

Limits between the types

Idea

- *Taking the limit $p \to 0$ sends elliptic hypergeometric to basic hypergeometric*
- *Taking the limit $q \to 1$ sends basic hypergeometric to hypergeometric*

Lemma

$$\lim_{p\to 0} {}_rE_{r-1}\left[\begin{matrix} a_1,\dots,a_{r-1},q^{-n} \\ b_1,\dots,b_{r-1}\end{matrix};z\right] = {}_r\phi_{r-1}\left[\begin{matrix} a_1,\dots,a_{r-1},q^{-n} \\ b_1,\dots,b_{r-1}\end{matrix};q,z\right]$$

$$\lim_{q\to 1} {}_r\phi_{r-1}\left[\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_{r-1}}\end{matrix};q,z\right] = {}_rF_{r-1}\left[\begin{matrix} \alpha_1,\dots,\alpha_r \\ \beta_1,\dots,\beta_{r-1}\end{matrix};z\right]$$

Limits between the types

Idea

- *Taking the limit $p \to 0$ sends elliptic hypergeometric to basic hypergeometric*
- *Taking the limit $q \to 1$ sends basic hypergeometric to hypergeometric*

Lemma

$$\lim_{p\to 0} {}_rV_{r-1}(a; b_1, \dots, b_{r-6}, q^{-n}) = {}_{r-2}W_{r-3}(a; b_1, \dots, b_{r-6}, q^{-n}; q, q)$$

$$\lim_{q\to 1} {}_rW_{r-1}(q^{\alpha}; q^{\beta_1}, \dots, q^{\beta_{r-3}}; q, z)$$

$$= {}_{r-1}F_{r-2}\left[\begin{matrix} \alpha, 1+\frac{1}{2}\alpha, \beta_1, \dots, \beta_{r-3} \\ \frac{1}{2}\alpha, 1+\alpha-\beta_1, \dots, 1+\alpha-\beta_{r-3} \end{matrix}; z\right]$$

Limits of Gamma functions

Definition

$$\Gamma_q = (1-q)^{1-z}\frac{(q;q)}{(q^z;q)} = (1-q)^{1-z}\prod_{r=0}^{\infty}\frac{1-q^{r+1}}{1-q^{r+z}}$$

	quasi-period		period	
Γ_e	2	$\Gamma_e(qz) = \theta(z;p)\Gamma_e(z)$	"1"	$\Gamma_e(ze^{2\pi i}) = \Gamma_e(z)$
Γ_q	1	$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z)$	"1"	$\Gamma_q(z+\frac{2\pi i}{\log(q)}) = \text{cst}\ \Gamma_q(z)$
Γ	1	$\Gamma(z+1) = z\Gamma(z)$	0	

Theorem

$$\lim_{p\to 0}\Gamma_e(q^z) = \frac{1}{(q^z;q)} = \Gamma_q(z)\frac{(1-q)^{z-1}}{(q;q)} \qquad \lim_{q\to 1}\Gamma_q(z) = \Gamma(z)$$

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Frenkel-Turaev summation formula

Theorem

We have

$$
{}_{10}V_9\left(a; b_1, b_2, b_3, \frac{q^{n+1}a^2}{b_1 b_2 b_3}, q^{-n}\right) = \frac{\theta\left(\frac{aq}{b_1 b_2}, \frac{aq}{b_1 b_3}, \frac{aq}{b_2 b_3}, aq; p\right)_n}{\theta\left(\frac{aq}{b_1}, \frac{aq}{b_2}, \frac{aq}{b_3}, \frac{aq}{b_1 b_2 b_3}; p\right)_n}
$$

Limits of Frenkel-Turaev summation

$$_{10}V_9\left(a;b_1,b_2,b_3,\frac{q^{n+1}a^2}{b_1b_2b_3},q^{-n}\right)=\frac{\theta(\frac{aq}{b_1b_2},\frac{aq}{b_1b_3},\frac{aq}{b_2b_3},aq;p)_n}{\theta(\frac{aq}{b_1},\frac{aq}{b_2},\frac{aq}{b_3},\frac{aq}{b_1b_2b_3};p)_n}$$

Idea

Substitute $a=ap^\alpha$, $b_i=b_ip^{\beta_i}$, *and take limit* $p\to 0$.
$\alpha=\beta_i=0$ *gives*

$$_{8}W_7\left(a;b_1,b_2,b_3,\frac{q^{n+1}a^2}{b_1b_2b_3},q^{-n};q,q\right)=\frac{(\frac{aq}{b_1b_2},\frac{aq}{b_1b_3},\frac{aq}{b_2b_3},aq;q)_n}{(\frac{aq}{b_1},\frac{aq}{b_2},\frac{aq}{b_3},\frac{aq}{b_1b_2b_3};q)_n}$$

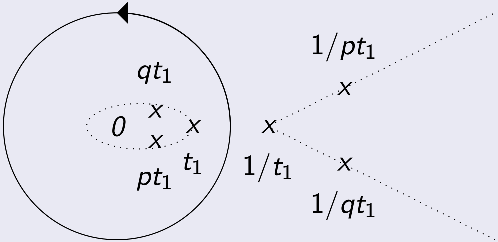
We obtain 7 out of 10 terminating evaluations from [App II, G & R]

Elliptic beta integral evaluation

Theorem

Assuming $\prod_{r=1}^{6} t_r = pq$ *we have*

$$\frac{(p;p)(q;q)}{2}\int_{\mathcal{C}} \frac{\prod_{r=1}^{6}\Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})}\frac{dz}{2\pi i z} = \prod_{1\le r<s\le 6}\Gamma_e(t_r t_s).$$



Elliptic hypergeometric integrals

Idea

An integral

$$\int \Delta(z) \frac{dz}{2\pi iz}$$

is elliptic hypergeometric if $\Delta(z)$ *satisfies*

$$\frac{\Delta(qz)}{\Delta(z)} = \frac{\Delta(pqz)}{\Delta(pz)}.$$

Then a series $\sum Res(\frac{\Delta(z)}{z}, z = aq^k)$ *becomes elliptic hypergeometric.*

Elliptic integral to series

Lemma

For parameters satifying $\prod_{r=1}^{6} t_r = pq$, *we have at* $t_1 t_2 = q^{-n}$

$$\frac{(p;p)(q;q)}{2\prod_{1\le r<s\le 6}\Gamma_e(t_r t_s)} \int_{\mathcal{C}} \frac{\prod_{r=1}^{6}\Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})} \frac{dz}{2\pi i z}$$
$$= \frac{\prod_{r=2}^{6}\Gamma_e(\frac{t_r}{t_1})}{\Gamma_e(1/t_1^2)\prod_{2\le r<s\le 6}\Gamma_e(t_r t_s)}$$
$$\times {}_{10}V_9(t_1^2; t_1 t_3, t_1 t_4, t_1 t_5, \frac{q^{n+1} t_1}{t_3 t_4 t_5}, q^{-n})$$

Limits of elliptic beta integral evaluation

Theorem

Several limits are

$$\frac{(q;q)}{2}\int_{\mathcal{C}} \frac{(z^{\pm 2}, \frac{q}{t_6} z^{\pm 1}; q)}{(t_1 z^{\pm 1}, \ldots, t_5 z^{\pm 1}; q)} \frac{dz}{2\pi i z} = \frac{\prod_{r=1}^{5} (q/t_r t_6; q)}{\prod_{1 \leq r < s \leq 5} (t_r t_s; q)}$$

$${}_6W_5(bal) = \cdots, \quad {}_3\phi_2\left(\ bal\ ; q, q\right) + {}_3\phi_2\left(\ bal\ ; q, q\right) = \cdots$$

$${}_6\psi_6\left(\textit{very-well-poised}\right) = \cdots$$

At least 16 out of 27 from [Appendix II, Gasper and Rahman].

Transformation of elliptic hypergeometric series

Theorem

Assume the balancing condition $a^3 q^{n+2} = b_1 b_2 b_3 b_4 b_5 b_6$ *holds. Then*

$$ {}_{12}V_{11}(a; b_1, b_2, b_3, b_4, b_5, b_6, q^{-n}) = \frac{\theta(aq, \frac{aq}{b_4 b_5}, \frac{aq}{b_4 b_6}, \frac{aq}{b_5 b_6}; p)_n}{\theta(\frac{aq}{b_4}, \frac{aq}{b_5}, \frac{aq}{b_6}, \frac{aq}{b_4 b_5 b_6}; p)_n} $$
$$ \times {}_{12}V_{11}(\frac{a^2 q}{b_1 b_2 b_3}; \frac{aq}{b_2 b_3}, \frac{aq}{b_1 b_3}, \frac{aq}{b_1 b_2}, b_4, b_5, b_6, q^{-n}) $$

Limits of the transformation formula

Theorem

Limits of the transformation formula include

- *terminating very-well-poised* ${}_{10}W_9$ *is similar*
- *terminating very-well poised* ${}_8W_7$ *is terminating balanced* ${}_4\phi_3$

At least 11 out of 16 terminating transformations in [App III, G&R].

Contiguous relations

Lemma

$$\frac{1}{y}\theta(wx^{\pm 1}, yz^{\pm 1}; p) + \frac{1}{z}\theta(wy^{\pm 1}, zx^{\pm 1}; p) + \frac{1}{x}\theta(wz^{\pm 1}; xy^{\pm 1}; p) = 0$$

Theorem

$$\begin{aligned} &{}_{12}V_{11}(a; b_1, b_2, b_3, b_4, b_5, b_6, q^{-n}) \\ &= \frac{\theta(b_2, \frac{b_2}{a}, \frac{b_1}{b_3 q}, \frac{b_1 b_3}{aq}; p)}{\theta(\frac{b_1}{q}, \frac{b_1}{aq}, \frac{b_2}{b_3}, \frac{b_2 b_3}{a}; p)} {}_{12}V_{11}(a; \frac{b_1}{q}, b_2 q, b_3, b_4, b_5, b_6, q^{-n}) \\ &\qquad + (b_2 \leftrightarrow b_3) \end{aligned}$$

Contiguous relations

Theorem

For any three ${}_{12}V_{11}$'s whose parameters differ by integer powers of q, but preserving the balancing condition and the terminating property there exists an equation

$$\begin{aligned} &c_1\ {}_{12}V_{11}(aq^{\alpha}; b_1 q^{\beta_1}, \ldots, b_6 q^{\beta_6}, q^{-n+\gamma}; p, q) \\ &\quad + c_2\ {}_{12}V_{11}(aq^{\hat{\alpha}}; b_1 q^{\hat{\beta}_1}, \ldots, b_6 q^{\hat{\beta}_6}, q^{-n+\hat{\gamma}}; p, q) \\ &\quad + c_3\ {}_{12}V_{11}(aq^{\tilde{\alpha}}; b_1 q^{\tilde{\beta}_1}, \ldots, b_6 q^{\tilde{\beta}_6}, q^{-n+\tilde{\gamma}}; p, q) = 0 \end{aligned}$$

Beta integral transformation

Theorem

For parameters $\prod_{r=1}^{8} t_r = (pq)^2$ define the integral

$$I(t_1,\ldots,t_8) = \frac{(p;p)(q;q)}{2}\int_{\mathcal{C}} \frac{\prod_{r=1}^{8}\Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})}\frac{dz}{2\pi i z}$$

with the usual contour. Then we have

$$I(t_1,\ldots,t_8) = \prod_{1\leq r<s\leq 4}\Gamma_e(t_r t_s)\prod_{5\leq r<s\leq 8}\Gamma_e(t_r t_s) \times I\left(\frac{t_1}{\sigma},\ldots,\frac{t_4}{\sigma}, t_5\sigma,\ldots,t_8\sigma\right)$$

where $\sigma^2 = \frac{t_1t_2t_3t_4}{pq} = \frac{pq}{t_5t_6t_7t_8}$.

Limits of the transformation formula

Theorem

Limits of the transformation formula include

- *Bailey's 4-term relation: sum of two* ${}_{10}W_9$*'s equals sum of two* ${}_{10}W_9$*'s.*
- ${}_8W_7$ *is a sum of two balanced* ${}_4\phi_3$*'s.*

At least 11 out of 22 non-terminating transformations in [App III, G&R].

$W(E_7)$ symmetry

Theorem

Define $\rho = \langle \frac{1}{2}, \ldots, \frac{1}{2} \rangle \in \mathbb{R}^8$. *Let* $W = W(E_7)$ *be the Weyl group of type* E_7 *with roots*

$$R(E_7) = \{\mathbf{v} \in \mathbb{Z}^8 \cup (\mathbb{Z}^8 + \rho) \mid \mathbf{v} \cdot \mathbf{v} = 2, \mathbf{v} \cdot \rho = 0\}.$$

It acts on $\{\mathbf{t} \in \mathbb{C}^8 \mid \prod_i t_i = p^2 q^2\}$ *by the usual action on* $\langle \ln(t_1), \ldots, \ln(t_8) \rangle$. *Define*

$$E(t_1, \ldots, t_8; p, q) = \frac{(p;p)(q;q)}{2 \prod_{1 \le r < s \le 8} (t_r t_s; p, q)^{-1}} \int_{\mathcal{C}} \frac{\prod_{r=1}^{8} \Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})} \frac{dz}{2\pi i z}$$

then

$$E(w(t)) = E(t), \qquad \forall w \in W.$$

Contiguous relations

Lemma

$$\frac{1}{y}\theta(wx^{\pm1}, yz^{\pm1}; p) + \frac{1}{z}\theta(wy^{\pm1}, zx^{\pm1}; p) + \frac{1}{x}\theta(wz^{\pm1}; xy^{\pm1}; p) = 0$$

Theorem

Writing

$$I(t_1, \ldots, t_8) = \int \frac{\prod_{r=1}^{8} \Gamma_e(t_r z^{\pm1})}{\Gamma_e(z^{\pm2})} \frac{dz}{2\pi i z}$$

we have

$$I(t) = \frac{\theta(\frac{t_1 t_3}{q}, \frac{t_1}{q t_3}; p)}{\theta(t_2 t_3, \frac{t_2}{t_3}; p)} I(\frac{t_1}{q}, q t_2) + \frac{\theta(\frac{t_1 t_2}{q}, \frac{t_1}{q t_2}; p)}{\theta(t_2 t_3, \frac{t_3}{t_2}; p)} I(\frac{t_1}{q}, q t_3)$$

E_7-contiguity

Theorem

Recall that $\rho = \langle \frac{1}{2}, \ldots, \frac{1}{2} \rangle \in \mathbb{R}^8$ *and that*

$$R(E_7) = \{\mathbf{v} \in \mathbb{Z}^8 \cup (\mathbb{Z}^8 + \rho) \mid \mathbf{v} \cdot \mathbf{v} = 2, \mathbf{v} \cdot \rho = 0\}.$$

Writing $tq^\alpha = \langle t_1 q^{\alpha_1}, \ldots, t_8 q^{\alpha_8} \rangle$ *there exists for any triple* α, β *and* γ *in the root lattice* $\mathbb{Z}^8 \cup (\mathbb{Z}^8 + \rho)$ *coefficients such that*

$$c_1 I(tq^\alpha) + c_2 I(tq^\beta) + c_3 I(tq^\gamma) = 0.$$

Summary

${}_{10}V_9$	Evaluation	
$\int \frac{\prod_{r=1}^{6} \Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})} \frac{dz}{2\pi i z}$	Evaluation	
${}_{12}V_{11}$	Transformation	Contiguous rel. (q)
$\int \frac{\prod_{r=1}^{8} \Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})} \frac{dz}{2\pi i z}$	Transformation	Contiguous rel. (p and q)

Biorthogonality

Definition

Suppose $t_1t_2t_3t_4u_1u_2 = pq$ then we define

$$R_n(z; t_1 : t_2, t_3, t_4; u_1, u_2)$$
$$= {}_{12}V_{11}\left(\frac{t_1}{u_1}; \frac{pq^n}{u_1u_2}, q^{-n}, t_1z^{\pm 1}, \frac{q}{u_1t_2}, \frac{q}{u_1t_3}, \frac{q}{u_1t_4}\right)$$

Continuous Biorthogonality

Definition

We define a bilinear form for $t_1 t_2 t_3 t_4 t_5 t_6 = pq$ as

$$\langle f, g\rangle_{t_1,\ldots,t_6} = \frac{(p;p)(q;q)}{2 \prod_{1\leq r<s\leq 6} \Gamma_e(t_r t_s)} \int_{\mathcal{C}} f(z)g(z) \frac{\prod_{r=1}^{6} \Gamma_e(t_r z^{\pm 1})}{\Gamma_e(z^{\pm 2})} \frac{dz}{2\pi i z}$$

Theorem

$$\langle R_n(\cdot; t_1 : t_2, t_3, t_4; u_1, u_2), R_m(\cdot; t_1 : t_2, t_3, t_4; u_2, u_1)\rangle_{t_1,t_2,t_3,t_4,u_1,u_2}$$
$$= \delta_{n,m} \frac{\theta(\frac{p}{u_1 u_2}; p)_{2n} \theta(q, t_2 t_3, t_2 t_4, t_3 t_4, \frac{q t_1}{u_1}, \frac{p q t_1}{u_2}; p)_n}{\theta(\frac{pq}{u_1 u_2}; p)_{2n} \theta(\frac{p}{u_1 u_2}, t_1 t_2, t_1 t_3, t_1 t_4, \frac{p}{t_1 u_2}, \frac{1}{t_1 u_1}; p)_n} q^{-n}.$$

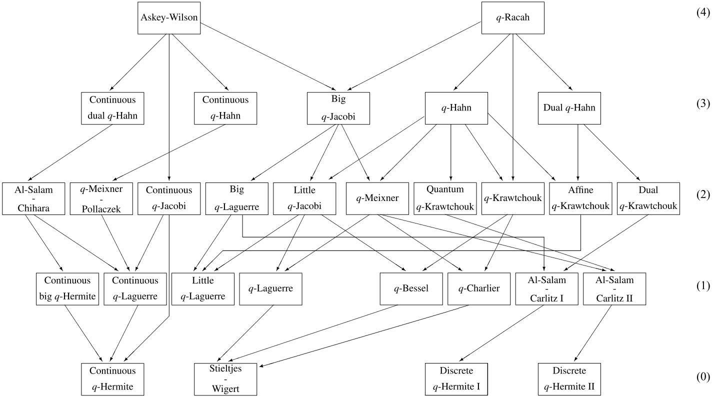
Discrete Biorthogonality

Definition

Assuming $t_1 t_2 = q^{-N}$ and $t_3 t_4 t_5 t_6 = pq^{N+1}$ we define

$$\langle f, g \rangle_{t_1,\ldots,t_6} = \sum_{k=0}^{N} f(t_1 q^k) g(t_1 q^k) \frac{\theta(q t_1^2; q; p)_{2k}}{\theta(t_1^2; q; p)_{2k}} \times \frac{\theta(t_1^2, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5, \frac{t_1 t_6}{p}; q; p)_k}{\theta(q, \frac{q t_1}{t_2}, \frac{q t_1}{t_3}, \frac{q t_1}{t_4}, \frac{q t_1}{t_5}, \frac{p q t_1}{t_6}; q; p)_k} q^k \times \frac{\theta(\frac{q t_1}{t_5}, t_2 t_3, t_2 t_4, \frac{t_2 t_6}{p}; q; p)_N}{\theta(\frac{t_2}{t_1}, \frac{q}{t_5 t_3}, \frac{q}{t_5 t_4}, \frac{pq}{t_5 t_6}; q; p)_N}$$

q-Askey scheme of basic hypergeometric orth. polynomials



Enjoy the rest of the conference!

Motto

On the elliptic level only the most complicated equations remain, but the main ideas are the same.