

q -Stirling numbers of the second kind and q -Bell numbers for graphs

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Abstract

Stirling numbers of the second kind and Bell numbers for graphs were defined by Duncan and Peele in 2009. In a previous paper, one of us, jointly with Nyul, extended the known results for these special numbers by giving new identities, and provided a list of explicit expressions for Stirling numbers of the second kind and Bell numbers for particular graphs. In this work we introduce q -Stirling numbers of the second kind and q -Bell numbers for graphs, and provide a number of explicit examples. Connections are made to q -binomial coefficients and q -Fibonacci numbers.

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1 Introduction

The Stirling numbers of second kind $S(n, k)$ are defined as the following connection coefficients, where $x^{\underline{k}}$ denotes the usual falling factorial of x :

$$x^n = \sum_{k=0}^n S(n, k) x^{\underline{k}}.$$

They count the number of ways to partition a set of n elements into k non-empty subsets. The n -th Bell number B_n counts the number of different ways to partition a set that has exactly n elements, hence

$$B_n = \sum_{k=0}^n S(n, k).$$

Note that a partition of a set S is a collection $\{A_1, A_2, \dots, A_k\}$ of nonempty disjoint subsets of S such that $\bigcup_{i=1}^k A_i = S$. The subsets A_i in a partition are called blocks and without loss of generality we can assume that they are listed in increasing order of their minimal elements, i.e. $\min A_1 < \dots < \min A_k$.

The Stirling numbers of the second kind satisfy the following recursion formula

$$S(n+1, k) = S(n, k-1) + kS(n, k),$$

with $S(n, k) = 0$ for $k < 0$ or $n < k$, and $S(0, 0) := 1$. They also satisfy an explicit formula involving binomial coefficients.

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n.$$

Similarly, we have the following well known identity for Bell numbers:

$$B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k.$$

Various generalizations of Stirling and Bell numbers exist. Particularly attractive are their q -extensions by Carlitz [1]. These involve the q -numbers as explicitly used by Jackson [3]. The classical case is recovered when $q \rightarrow 1$.

Let q be a variable satisfying $0 < |q| < 1$. For complex x , the q -number of x is defined to be

$$[x]_q := \frac{1 - q^x}{1 - q}.$$

Further, for $n, k \in \mathbb{N}_0$, the q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q \cdots [n - k + 1]_q}{[k]_q \cdots [1]_q}.$$

Furthermore, we denote the q -falling factorials by

$$[x]_q^n := [x]_q [x - 1]_q \cdots [x - n + 1]_q \quad \text{if } n = 1, 2, \dots, \quad \text{and } [x]_q^0 := 1.$$

Using these notations, Carlitz' q -Stirling numbers of the second kind are defined as the following connection coefficients:

$$[x]_q^n = \sum_{k=0}^n S_q(n, k) [x]_q^k.$$

They can be shown to satisfy the following recursion:

$$S_q(n + 1, k) = q^{k-1} S_q(n, k - 1) + [k]_q S_q(n, k),$$

which, together with $S_q(n, k) = 0$ for $k < 0$ or $k > n$, and $S_q(0, 0) = 1$, determines them uniquely.

Carlitz [1, Equation (3.3)] gave the following explicit formula,

$$S_q(n, k) = \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [k - j]_q^n.$$

Analogously, the q -Bell numbers are defined by

$$B_{q,n} = \sum_{k=0}^n S_q(n, k).$$

The aforementioned q -analogues of special numbers can be interpreted combinatorially similar to their classical versions where each element of the set has a weight. The weights are given as certain powers of q .

Here we are interested in generalizations for graphs. Stirling numbers of the second kind and Bell numbers for graphs (sometimes called “graphical Stirling numbers”, etc.) were defined by Duncan and Peele [2]. They were further investigated by Kereskényiné Balogh and Nyul [4] who explicitly determined Stirling numbers of the second kind and Bell numbers for several well-known graphs, such as the complete, path, star and cycle graph.

Let G be a simple (finite) graph. A partition of $V(G)$ is called an independent partition if each block is an independent vertex set (i.e. adjacent vertices belong to distinct blocks). Then for a positive integer $k \leq |V(G)|$, the Stirling number of the second kind $S(G, k)$ for the graph G is defined to be the number of independent partitions of $V(G)$ into k subsets (where we set $S(G, 0) = 0$). We further define the Bell number B_G for graph G to be the number of independent partitions of $V(G)$, i.e.

$$B_G = \sum_{k=0}^{|V(G)|} S(G, k).$$

In case of the empty graph E_n with n vertices, there is no restriction on the vertices belonging to a block of a partition and the corresponding graphical Stirling and Bell numbers specialize to the ordinary Stirling and Bell numbers, $S(n, k)$ and B_n , respectively.

Stirling and Bell numbers for special graphs often appear in applications. In particular, consider the “path graph” P_n to be the simple graph having n labeled vertices with its vertices being connected if and only if the difference of their labels is at most one. Independent partitions of path graphs appear implicitly in the literature under several names, e.g., nonconsecutive partitions, Fibonacci, reduced or restricted partitions.

In [4] various new results have been derived for Stirling and Bell numbers for graphs. For instance, we have the following reduction relations:

$$S(G, k) = S(G - e, k) - S(G/e, k) \quad \text{and} \quad B_G = B_{G-e} - B_{G/e},$$

where $e \in E(G)$, $G - e$ and G/e are the simplified graphs obtained by deleting and contracting edge e from G , respectively.

The graphical Stirling numbers of the second kind satisfy the following explicit formula (derivable by using the inclusion-exclusion principle),

$$S(G, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} p_G(k - j)$$

(where p_G denotes the chromatic polynomial of the graph G), while the Bell numbers for graphs satisfy a Dobiński type formula,

$$B_G = \frac{1}{e} \sum_{j=0}^{\infty} \frac{p_G(j)}{j!}.$$

2 q -Stirling numbers of the second kind for graphs

Consider a simple (finite) labeled graph G with n vertices. As before the vertices are partitioned into $k \leq |V(G)|$ independent blocks, $\Pi = (A_1, A_2, \dots, A_k)$ and they are listed in increasing order of their minimal elements. To each such partition we assign the following q -weight

$$w_q(\Pi) = q^{\sum_{i=1}^k (i-1)|A_i|}, \quad (1)$$

where $|A|$ denotes the cardinality of the set A . Then the q -Stirling number of the second kind $S_q(G, k)$ for the graph G is the sum of all weights of independent partitions into k blocks (while we set $S_q(G, 0) = 0$), i.e.

$$S_q(G, k) = \sum_{\text{independent partitions } \Pi \text{ of } V(G)} w_q(\Pi). \quad (2)$$

Further, we define the q -Bell number $B_q(G)$ of G as follows:

$$B_q(G) = \sum_{k=0}^{|V(G)|} S_q(G, k). \quad (3)$$

For $G = E_n$ being the empty graph, the above defined q -Stirling numbers of the second kind and q -Bell numbers for G specialize to the aforementioned q -Stirling and q -Bell numbers of Carlitz, respectively.

Example 2.1 Consider the dual path graph \overline{P}_n , which has labeled vertices being connected if and only if the difference of their labels is at least two.

\overline{P}_4 has the following independent partitions into 3 blocks:

$$\Pi_1 = (\{1, 2\}, \{3\}, \{4\}), \quad \Pi_2 = (\{1\}, \{2, 3\}, \{4\}), \quad \Pi_3 = (\{1\}, \{2\}, \{3, 4\}),$$

with q -weights $w_q(\Pi_1) = q^3$, $w_q(\Pi_2) = q^4$, and $w_q(\Pi_3) = q^5$. Summing up these weights we get

$$S_q(\overline{P}_4, 3) = q^3 + q^4 + q^5 = q^3(1 + q + q^2) = q^3 \frac{(1 - q^3)}{(1 - q)} = q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q.$$

The following result generalizes this specific example to general n and k .

Theorem 2.2 For $n, k \in \mathbb{N}_0$

$$S_q(\overline{P}_n, k) = q^{\binom{k}{2} + \binom{n-k}{2}} \begin{bmatrix} k \\ n - k \end{bmatrix}_q.$$

Proof. We prove the theorem by induction on n . It is true for $n = 0$ as $S_q(\overline{P}_0, 0) = 1$ and $q^{\binom{0}{2} + \binom{0}{2}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$. Let $n \geq 1$ and assume the statement holds for \overline{P}_l where $0 \leq l \leq n$. In case of $l = n + 1$ there are two different cases. The last, $(n + 1)$ -st, element of $V(\overline{P}_{n+1})$ can either form a block alone or can be part of the block where the n -th element is. This gives

$$\begin{aligned} S_q(\overline{P}_{n+1}, k) &= q^{k-1} S_q(\overline{P}_n, k-1) + q^{2\binom{k-1}{2}} S_q(\overline{P}_{n-1}, k-1) \\ &= q^{\binom{k}{2} + \binom{n+1-k}{2}} \left(\begin{bmatrix} k-1 \\ n+1-k \end{bmatrix}_q + q^{2k-n-1} \begin{bmatrix} k-1 \\ n-k \end{bmatrix}_q \right) \\ &= q^{\binom{k}{2} + \binom{n+1-k}{2}} \begin{bmatrix} k \\ n+1-k \end{bmatrix}_q, \end{aligned}$$

by the recursion for the q -binomial coefficients. □

The q -Bell numbers for the dual path graph $G = \overline{P}_n$ turn out to be a variant of the q -Fibonacci numbers:

$$B_q(\overline{P}_n) = \sum_{k=0}^n S_q(\overline{P}_{n+1}, k) = \sum_{k=0}^n q^{\binom{k}{2} + \binom{n-k}{2}} \begin{bmatrix} k \\ n-k \end{bmatrix}_q = F_{q, n+1}. \quad (4)$$

The above example even extends to elliptic weights. In the expanded version of this extended abstract we provide explicit expressions for q -Stirling numbers of the second kind and q -Bell numbers for various particular graphs.

References

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