# A New Multidimensional Matrix Inversion in $A_r$

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*Dedicated to Professor Dick Askey on the occasion of his 65th birthday*

ABSTRACT. We invert a specific infinite $r$-dimensional matrix, thus giving an extension of our previous matrix inversion result. As applications, we derive new summation formulas for series in $A_r$.

## 1. Introduction

A very powerful tool in combinatorics and special functions theory is the application of matrix inversions. In particular, with the help of so-called "inverse relations" (see Section 3), which are immediate consequences of matrix inversions, one can derive and prove identities. This method is especially useful in connection with (basic) hypergeometric series. Though in order to be able to apply this method, *explicit* matrix inversions must be at hand.

Over the last decades, several people discovered and rediscovered useful matrix inversions. A very general matrix inversion was found by Gould and Hsu [**13**] which contained a lot of inverse relations as special cases. The problem, posed by Gould and Hsu, of finding a $q$-analogue of their formula, was solved immediately thereafter by Carlitz [**6**]. However, he did not give any applications. The significance of Carlitz's matrix inversion showed up first when Andrews [**1**] discovered that the Bailey transform [**2**], one of the corner stones in the development of the theory of (basic) hypergeometric series, is just equivalent to a very special case of Carlitz's matrix inversion. Further important contributions to this subject were achieved by Gessel and Stanton [**11**], [**12**], Bressoud [**5**], and Gasper and Rahman [**8**], [**9**]. As all of these authors' matrix inversions have certain termwise similarity, it seemingly was true that there existed some even more general matrix inversion, one which would unify these previous results but still be very explicit.

The desired unification was achieved by Krattenthaler [**19**]. He proved that the matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers) are inverses

1991 *Mathematics Subject Classification.* Primary 33D70; Secondary 15A09, 11B65.

*Key words and phrases.* multidimensional matrix inversion, inverse relations, $A_r$ basic hypergeometric series, $A_r$ $q$-Abel identities, $A_r$ $q$-Rothe identities.

The author was supported by the Austrian Science Foundation FWF, grant P12094-MAT.

of each other, where

$$f_{nk}=\frac{\prod_{j=k}^{n-1}(a_j+b_jc_k)}{\prod_{j=k+1}^{n}(c_j-c_k)},\tag{1.1}$$

and

$$g_{kl}=\frac{(a_l+b_lc_l)}{(a_k+b_kc_k)}\frac{\prod_{j=l+1}^{k}(a_j+b_jc_k)}{\prod_{j=l}^{k-1}(c_j-c_k)}.\tag{1.2}$$

In fact, Krattenthaler's matrix inversion contains all the inversions just mentioned as special cases.

Starting in the late 1970s, Milne and co-authors, in a long series of papers (cf. [**23**], [**24**], [**25**], [**21**], [**26**], [**27**], and the references cited therein), developed a theory of multiple (basic) hypergeometric series associated to root systems. In order to have an equivalent of the (one-dimensional) Bailey transform at hand, to conveniently extend the development of the theory of (one-dimensional) basic hypergeometric series to an analogous theory for multiple series, matrix inversions in this multidimensional setting needed to be found. "Multidimensional" matrix inversions (according to our terminology these are matrix inversions that arise in the theory of multiple series) *associated to root systems* were found by Milne, Lilly and Milne, and by Bhatnagar and Milne. The $A_r$ (or equivalently $U(r+1)$) and $C_r$ inversions (corresponding to the root systems $A_r$ and $C_r$, respectively) of Milne [**25**, Theorem 3.3], and Lilly and Milne [**21**], which are higher-dimensional generalizations of Andrews' Bailey transform matrices, were used to derive $A_r$ and $C_r$ extensions [**25**], [**26**] of many of the classical hypergeometric summation and transformation formulas. Bhatnagar and Milne [**3**, Theorem 3.48] were even able to find an $A_r$ extension of Gasper's bibasic hypergeometric matrix inversion. However, none of these multidimensional matrix inversions contained Krattenthaler's inversion as a special case.

A multidimensional extension of Krattenthaler's matrix inversion (1.1)/(1.2), associated to root systems, was found by the author in [**29**]. Theorems 3.1 and 4.1 of [**29**] cover all the previously discovered multidimensional matrix inversions associated to root systems [**3**], [**21**], [**26**] as special cases. Just recently, another multidimensional extension of Krattenthaler's matrix inverse (1.1)/(1.2) was found [**20**, Theorem 3.1] which covers the inversion of [**7**]. The matrix inverse of [**20**] has applications similar to those in this article although the series considered in [**20**] are of simpler type.

Special cases of [**29**, Theorem 3.1] were used in [**29**] to derive several summation theorems for multidimensional basic hypergeometric series. In particular, a $D_r$ ${}_8\phi_7$ summation theorem, $A_r$ and $D_r$ quadratic, and $D_r$ cubic basic hypergeometric summation theorems were derived. Moreover, the $D_r$ ${}_8\phi_7$ summation theorem of [**29**] lead to new $C_r$ and $D_r$ extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation in [**4**]. In a very recent article [**30**] the author utilized special (non-hypergeometric) cases of the multidimensional matrix inversions in [**29**, Theorems 3.1 and 4.1] to derive some $A_r$ terminating and nonterminating $q$-Abel and $q$-Rothe summations, and also some identities of another type which appear to be new already in the one-dimensional case.

One of the main results of this article is a new multidimensional extension of Krattenthaler's matrix inverse (see Theorem 2.1) which even generalizes [**29**, Theorem 3.1]. Thus, with our new pair of inverse matrices, we are able to derive even more general summation theorems as applications (see Section 3), than it would have been possible by just using [**29**, Theorem 3.1].

Our article is organized as follows. In Section 2 we present our new multidimensional matrix inversion, see Theorem 2.1, its proof using techniques developed in [**18**] and [**29**]. Then, in the following Propositions, we specialize our general matrix inversion result appropriately for the applications in the subsequent Section. In Section 3, we briefly explain the notion and use of inverse relations, together with some standard $q$-series notation. Then, to illustrate the usefulness of our new multidimensional matrix inversion, we derive a number of multiple series identities. In particular, we derive an $A_r$ $q$-Abel-type expansion formula, three $A_r$ $q$-Abel summations, two $A_r$ $q$-Rothe-type expansion formulas, and three $A_r$ $q$-Rothe summations. Our theorems generalize some corresponding results of [**30**].

In Appendix A, we provide a determinant evaluation, Lemma A.1, which generalizes [**29**, Lemma A.1]. It turns out to be crucial for our computations in the proof of Theorem 2.1. Recently, Zagier [**31**] has kindly communicated to us his short and elegant proof of [**29**, Lemma A.1]. Thus, our proof of Lemma A.1 is an extension of Zagier's proof of [**29**, Lemma A.1]. Finally, in Appendix B, we list some background information needed in the proofs of our multiple summation theorems such as some $A_r$ basic hypergeometric summation theorems from Milne [**25**].

## 2. A new multidimensional matrix inversion

Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ (as before, $\mathbb{Z}$ denotes the set of integers) be an infinite lower-triangular $r$-dimensional matrix; i.e. $f_{\mathbf{nk}} = 0$ unless $\mathbf{n} \geq \mathbf{k}$, by which we mean $n_i \geq k_i$ for all $i = 1, \dots, r$. The matrix $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ is said to be the *inverse matrix* of $F$ if and only if

$$\sum_{\mathbf{n}\geq\mathbf{k}\geq\mathbf{l}} f_{\mathbf{nk}} g_{\mathbf{kl}} = \delta_{\mathbf{nl}}$$

for all $\mathbf{n}, \mathbf{l} \in \mathbb{Z}^r$, where $\delta_{\mathbf{nl}}$ is the usual Kronecker delta.

In [**29**] very general multidimensional matrix inversions were derived. In the following Theorem we are able to give a matrix inversion result which extends [**29**, Theorem 3.1] by an additional parameter $m$, thus leading to more general summation theorems in the applications, see Section 3.

For convenience, we introduce the notation $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$. Moreover, we denote by $e_m(\mathbf{c}(\mathbf{k}))$ the elementary symmetric function (see [**22**, p. 19]) of order $m$ in the variables $c_1(k_1), c_2(k_2), \dots, c_r(k_r)$.

THEOREM 2.1. *Let $(a_t)_{t\in\mathbb{Z}}$, $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1, \dots, r$ be arbitrary sequences, $d$ arbitrary, such that none of the denominators in (2.1) or (2.2) vanish. Moreover, let $m$ be a fixed integer such that $0 \leq m \leq r$. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}\left(a_t - \frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\prod_{i=1}^{r}\prod_{t_i=k_i+1}^{n_i}\left(c_i(t_i) - \frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)} \frac{\prod_{i=1}^{r}\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t - c_i(k_i))}{\prod_{i,j=1}^{r}\prod_{t_i=k_i+1}^{n_i}(c_i(t_i) - c_j(k_j))} \tag{2.1}$$

*and*

$$g_{\mathbf{k}\mathbf{l}}=\prod_{1\le i<j\le r}\frac{(c_i(l_i)-c_j(l_j))}{(c_i(k_i)-c_j(k_j))}\\ \times\frac{(d-e_{m+1}(\mathbf{c}(\mathbf{l}))-a_{|\mathbf{l}|}e_m(\mathbf{c}(\mathbf{l})))}{(d-e_{m+1}(\mathbf{c}(\mathbf{k}))-a_{|\mathbf{k}|}e_m(\mathbf{c}(\mathbf{k})))}\prod_{i=1}^r\frac{(a_{|\mathbf{l}|}-c_i(l_i))}{(a_{|\mathbf{k}|}-c_i(k_i))}\\ \times\frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}\left(a_t-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\prod_{i=1}^r\prod_{t_i=l_i}^{k_i-1}\left(c_i(t_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\frac{\prod_{i=1}^r\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t-c_i(k_i))}{\prod_{i,j=1}^r\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}. \tag{2.2}$$

REMARK 2.2. The special case $m=r$ reduces to our previous matrix inversion result, Theorem 3.1 of [**29**].

PROOF OF THEOREM 2.1. The proof is very similar to our proof of [**29**, Theorem 3.1]. We use Krattenthaler's [**18**] operator method and its suitable modification in [**29**, Section 2].

From (2.1) we deduce for $\mathbf{n}\ge\mathbf{k}$ the recursion

$$\left(c_i(n_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(c_i(n_i)-c_s(k_s))f_{\mathbf{n}\mathbf{k}}\\ =\left(a_{|\mathbf{n}|-1}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(a_{|\mathbf{n}|-1}-c_s(k_s))f_{\mathbf{n}-\mathbf{e}_i,\mathbf{k}}, \tag{2.3}$$

for $i=1,\dots,r$, where $\mathbf{e}_i$ denotes the vector of $\mathbb{Z}^r$ where all components are zero except the $i$-th, which is 1. We write

$$f_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{n}\ge\mathbf{k}}\frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}\left(a_t-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\prod_{i=1}^r\prod_{t_i=k_i+1}^{n_i}\left(c_i(t_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\frac{\prod_{i=1}^r\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t-c_i(k_i))}{\prod_{i,j=1}^r\prod_{t_i=k_i+1}^{n_i}(c_i(t_i)-c_j(k_j))}\mathbf{z}^{\mathbf{n}}.$$

Moreover, we define linear operators $\mathcal{A},\mathcal{C}_i$ by $\mathcal{A}\mathbf{z}^{\mathbf{n}}=a_{|\mathbf{n}|}\mathbf{z}^{\mathbf{n}}$ and $\mathcal{C}_i\mathbf{z}^{\mathbf{n}}=c_i(n_i)\mathbf{z}^{\mathbf{n}}$ for all $i=1,\dots,r$. Then we may write (2.3) in the form

$$\left(\mathcal{C}_i-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(\mathcal{C}_i-c_s(k_s))f_{\mathbf{k}}(\mathbf{z})\\ =z_i\left(\mathcal{A}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(\mathcal{A}-c_s(k_s))f_{\mathbf{k}}(\mathbf{z}), \tag{2.4}$$

valid for all $\mathbf{k}\in\mathbb{Z}^r$. We want to write our system of equations in a way such that [**29**, Corollary 2.14] is applicable. In order to achieve this, we expand the products on both sides of (2.4) in terms of the elementary symmetric functions

$$e_j\left(c_1(k_1),c_2(k_2),\dots,c_r(k_r),\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)$$

of order $j$, for which we write $e_j(\bar{\mathbf{c}}(\mathbf{k}))$ for short. Our recurrence system then reads, using $e_{m+1}(\bar{\mathbf{c}}(\mathbf{k})) = d$,

$$\begin{aligned}
(2.5)\quad & \sum_{\substack{1\le j\le r+1\\ j\neq m+1}} e_j(\bar{\mathbf{c}}(\mathbf{k}))[(-\mathcal{C}_i)^{r+1-j} - z_i(-\mathcal{A})^{r+1-j}]f_{\mathbf{k}}(\mathbf{z})\\
&= \sum_{1\le j\le r} e_{j+\chi(j>m)}(\bar{\mathbf{c}}(\mathbf{k}))[(-\mathcal{C}_i)^{r+\chi(j\le m)-j} - z_i(-\mathcal{A})^{r+\chi(j\le m)-j}]f_{\mathbf{k}}(\mathbf{z})\\
&= [z_i(-\mathcal{A})^{r+1} + dz_i(-\mathcal{A})^{r-m} - (-\mathcal{C}_i)^{r+1} - d(-\mathcal{C}_i)^{r-m}]f_{\mathbf{k}}(\mathbf{z}), \quad i=1,\dots,r,
\end{aligned}$$

where $\chi(P)$ equals 1 if $P$ is true and equals 0 otherwise. Now, regarding [**29**, Corollary 2.14], (2.5) is a system of type [**29**, Eq. (2.18)] with $V_{ij} = [(-\mathcal{C}_i)^{r+\chi(j\le m)-j} - z_i(-\mathcal{A})^{r+\chi(j\le m)-j}]$, $W_i = [z_i(-\mathcal{A})^{r+1} + dz_i(-\mathcal{A})^{r-m} - (-\mathcal{C}_i)^{r+1} - d(-\mathcal{C}_i)^{r-m}]$, and $c_j(\mathbf{k}) = e_{j+\chi(j>m)}(\bar{\mathbf{c}}(\mathbf{k}))$. The operators $C_{ij} = (-\mathcal{C}_i)^{r+\chi(j\le m)-j}$, $A_{ij} = -z_i(-\mathcal{A})^{r+\chi(j\le m)-j}$, $W_i^{(c)} = [-(-\mathcal{C}_i)^{r+1} - d(-\mathcal{C}_i)^{r-m}]$, $W_i^{(a)} = [z_i(-\mathcal{A})^{r+1} + dz_i(-\mathcal{A})^{r-m}]$ satisfy [**29**, Eqs. (2.6), (2.7), (2.8), (2.15), (2.16), and (2.17)], the functions $c_j(\mathbf{k})$ satisfy [**29**, Eq. (2.2)]. Hence we may apply [**29**, Corollary 2.14]. The dual system [**29**, Eq. (2.20)] for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ in this case reads

$$\begin{aligned}
& \sum_{\substack{1\le j\le r+1\\ j\neq m+1}} e_j(\bar{\mathbf{c}}(\mathbf{k}))[(-\mathcal{C}_i^*)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j}z_i]h_{\mathbf{k}}(\mathbf{z})\\
&= \sum_{1\le j\le r} e_{j+\chi(j>m)}(\bar{\mathbf{c}}(\mathbf{k}))[(-\mathcal{C}_i^*)^{r+\chi(j\le m)-j} - (-\mathcal{A}^*)^{r+\chi(j\le m)-j}z_i]h_{\mathbf{k}}(\mathbf{z})\\
&= [(-\mathcal{A}^*)^{r+1}z_i + (-\mathcal{A}^*)^{r-m}dz_i - (-\mathcal{C}_i^*)^{r+1} - (-\mathcal{C}_i^*)^{r-m}d]h_{\mathbf{k}}(\mathbf{z}), \quad i=1,\dots,r.
\end{aligned}$$

Equivalently, we have

$$\begin{aligned}
(2.6)\quad & \left(\mathcal{C}_i^* - \frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(\mathcal{C}_i^* - c_s(k_s))h_{\mathbf{k}}(\mathbf{z})\\
&\qquad = \left(\mathcal{A}_i^* - \frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(\mathcal{A}^* - c_s(k_s))z_ih_{\mathbf{k}}(\mathbf{z}),
\end{aligned}$$

for all $i = 1,\dots,r$ and $\mathbf{k}\in\mathbb{Z}^r$. As is easily seen, we have $\mathcal{A}^*\mathbf{z}^{-\mathbf{l}} = a_{|\mathbf{l}|}\mathbf{z}^{-\mathbf{l}}$ and $\mathcal{C}_i^*\mathbf{z}^{-\mathbf{l}} = c_i(l_i)\mathbf{z}^{-\mathbf{l}}$ for $i=1,\dots,r$. Thus, with $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$, by comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (2.6) we obtain

$$\begin{aligned}
& \left(c_i(l_i) - \frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(c_i(l_i) - c_s(k_s))h_{\mathbf{kl}}\\
&\qquad = \left(a_{|\mathbf{l}|} - \frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)\prod_{s=1}^r(a_{|\mathbf{l}|} - c_s(k_s))h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i}.
\end{aligned}$$

If we set $h_{\mathbf{kk}}=1$, we get

$$h_{\mathbf{kl}}=\frac{\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1}\left(a_t-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\prod_{i=1}^{r}\prod_{t_i=l_i}^{k_i-1}\left(c_i(t_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\,\frac{\prod_{i=1}^{r}\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1}(a_t-c_i(k_i))}{\prod_{i,j=1}^{r}\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}.$$

Taking into account [**29**, Eq. (2.19)], we have to compute the action of

$$\overrightarrow{\det_{1\le i,j\le r}}(V_{ij}^*)=\overrightarrow{\det_{1\le i,j\le r}}[(-\mathcal{C}_i^*)^{r+\chi(j\le m)-j}-(-\mathcal{A}^*)^{r+\chi(j\le m)-j}z_i] \tag{2.7}$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{l}\le\mathbf{k}}\frac{\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1}\left(a_t-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\prod_{i=1}^{r}\prod_{t_i=l_i}^{k_i-1}\left(c_i(t_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\,\frac{\prod_{i=1}^{r}\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1}(a_t-c_i(k_i))}{\prod_{i,j=1}^{r}\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}\mathbf{z}^{-\mathbf{l}}.$$

Since

$$z_ih_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{l}\le\mathbf{k}}\frac{\left(c_i(l_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\left(a_{|\mathbf{l}|}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\prod_{j=1}^{r}\frac{(c_i(l_i)-c_j(k_j))}{(a_{|\mathbf{l}|}-c_j(k_j))}h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}},$$

we conclude that

$$\overrightarrow{\det_{1\le i,j\le r}}V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{l}\le\mathbf{k}}\det_{1\le i,j\le r}(v_{ij})h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}, \tag{2.8}$$

where

$$\begin{aligned}v_{ij}&=(-c_i(l_i))^{r+\chi(j\le m)-j}\\&\qquad-(-a_{|\mathbf{l}|})^{r+\chi(j\le m)-j}\frac{\left(c_i(l_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\left(a_{|\mathbf{l}|}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\prod_{s=1}^{r}\frac{(c_i(l_i)-c_s(k_s))}{(a_{|\mathbf{l}|}-c_s(k_s))}.\end{aligned}$$

(This follows in a similar way as in the proof of Theorem 3.1 of [**29**].) For the computation of $\det_{1\le i,j\le r}(v_{ij})$ we utilize Lemma A.1 with $x_i=-c_i(l_i)$, $y_i=-c_i(k_i)$, $i=1,\dots,r$, $y_0=\frac{e_{m+1}(\mathbf{c}(\mathbf{k}))-d}{e_m(\mathbf{c}(\mathbf{k}))}$, and $a=-a_{|\mathbf{l}|}$, obtaining

$$\begin{aligned}\det_{1\le i,j\le r}(v_{ij})&=(-1)^m\,e_m(\mathbf{c}(\mathbf{l}))\frac{\left(a_{|\mathbf{l}|}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{l}))}{e_m(\mathbf{c}(\mathbf{l}))}\right)}{\left(a_{|\mathbf{l}|}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\prod_{i=1}^{r}\frac{(a_{|\mathbf{l}|}-c_i(l_i))}{(a_{|\mathbf{l}|}-c_i(k_i))}\\&\qquad\times\prod_{1\le i<j\le r}(c_j(l_j)-c_i(l_i)).\end{aligned}$$

Plugging this determinant evaluation into (2.8) leads to

$$\underset{1\le i,j\le r}{\overrightarrow{\det}}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}}\Bigg(\prod_{1\le i<j\le r}(c_j(l_j)-c_i(l_i))$$
$$\times(-1)^m e_m(\mathbf{c}(\mathbf{l}))\frac{\left(a_{|\mathbf{l}|}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{l}))}{e_m(\mathbf{c}(\mathbf{l}))}\right)}{\left(a_{|\mathbf{k}|}-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\prod_{i=1}^r\frac{(a_{|\mathbf{l}|}-c_i(l_i))}{(a_{|\mathbf{k}|}-c_i(k_i))}$$
$$\times\frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}\left(a_t-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}{\prod_{i=1}^r\prod_{t_i=l_i}^{k_i-1}\left(c_i(t_i)-\frac{d-e_{m+1}(\mathbf{c}(\mathbf{k}))}{e_m(\mathbf{c}(\mathbf{k}))}\right)}\frac{\prod_{i=1}^r\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t-c_i(k_i))}{\prod_{i,j=1}^r\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}\mathbf{z}^{-\mathbf{l}}\Bigg). \tag{2.9}$$

Note that since $f_{\mathbf{k}\mathbf{k}}=1$, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \overrightarrow{\det}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in (2.9). Thus, [**29**, Eq. (2.19)] reads

$$g_{\mathbf{k}}(\mathbf{z}) = (-1)^m e_m(\mathbf{c}(\mathbf{k}))^{-1}\prod_{1\le i<j\le r}(c_j(k_j)-c_i(k_i))^{-1}\underset{1\le i,j\le r}{\overrightarrow{\det}}(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}), \tag{2.10}$$

where $g_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{l}\le\mathbf{k}}g_{\mathbf{k}\mathbf{l}}\mathbf{z}^{-\mathbf{l}}$. So, extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in (2.10) we obtain exactly (2.2). □

**2.1. Specializations.** In the following specializations of Theorem 2.1, we employ the (short) standard notation for the $q$-rising factorial in (3.6),(3.8).

First, we deduce a multidimensional matrix inversion which is used in Section 3 for deriving a nonterminating $A_r$ $q$-Abel expansion formula.

PROPOSITION 2.3. *Let $a$, $b$, and $x_1,\dots,x_r$ be indeterminate, and suppose that none of the denominators in* (2.11) *or* (2.12) *vanish. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then $(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{n}\mathbf{k}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|}q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}}\left(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})\right)^{|\mathbf{n}|-|\mathbf{k}|}$$
$$\times\prod_{i,j=1}^r\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}^{-1} \tag{2.11}$$

*and*

$$g_{\mathbf{k}\mathbf{l}} = \left(a+be_m(x_1q^{l_1},\dots,x_rq^{l_r})\right)\left(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})\right)^{|\mathbf{k}|-|\mathbf{l}|-1}$$
$$\times\prod_{i,j=1}^r\left(\frac{x_i}{x_j}q^{1+l_i-l_j}\right)_{k_i-l_i}^{-1}. \tag{2.12}$$

PROOF. In Theorem 2.1 we set $a_t\mapsto -d/a$ and $c_i(t_i)\mapsto b^{1/m}x_iq^{t_i}$ for $i=1,\dots,r$. After some elementary manipulations, which include Lemma B.1, we let $d\to\infty$ and obtain the inverse pair (2.11)/(2.12). □

In Proposition 2.3, if we interchange $a$ and $b$ and transfer some factors from one matrix to the other we obtain the equivalent Proposition 2.4. In this form the inverse matrices are used in Section 3 for deriving some terminating $A_r$ $q$-Abel summations.

Proposition 2.4. *Let $a$, $b$, and $x_1,\dots,x_r$ be indeterminate, and suppose that none of the denominators in* (2.13) *or* (2.14) *vanish. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}}=(ae_m(x_1q^{n_1},\dots,x_rq^{n_r})+b)\left(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b\right)^{|\mathbf{n}|-1}\\ \times(-1)^{|\mathbf{k}|}q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}}\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right] \tag{2.13}$$

*and*

$$g_{\mathbf{kl}}=(-1)^{|\mathbf{l}|}\left(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b\right)^{-|\mathbf{l}|}\\ \times\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{k_i}}{\left(\frac{x_i}{x_j}q\right)_{l_i}\left(\frac{x_i}{x_j}q^{1+l_i-l_j}\right)_{k_i-l_i}}\right]. \tag{2.14}$$

By the following specialization of Theorem 2.1, we deduce a multidimensional matrix inversion which is used in Section 3 for deriving some nonterminating $A_r$ $q$-Rothe expansions.

Proposition 2.5. *Let $a$, $b$, and $x_1,\dots,x_r$ be indeterminate, and suppose that none of the denominators in* (2.15) *or* (2.16) *vanish. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}}=(-1)^{|\mathbf{n}|-|\mathbf{k}|}q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}}\left(\frac{q^{|\mathbf{k}|}}{(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r}))}\right)_{|\mathbf{n}|-|\mathbf{k}|}\\ \times\left(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})\right)^{|\mathbf{n}|-|\mathbf{k}|}\prod_{i,j=1}^r\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)^{-1}_{n_i-k_i} \tag{2.15}$$

*and*

$$g_{\mathbf{kl}}=(-1)^{|\mathbf{k}|-|\mathbf{l}|}q^{\binom{|\mathbf{k}|+1}{2}-\binom{|\mathbf{l}|+1}{2}}\frac{(a+be_m(x_1q^{l_1},\dots,x_rq^{l_r})-q^{|\mathbf{l}|})}{(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})-q^{|\mathbf{k}|})}\\ \times\left(aq^{-|\mathbf{k}|}+bq^{-|\mathbf{k}|}e_m(x_1q^{k_1},\dots,x_rq^{k_r})\right)_{|\mathbf{k}|-|\mathbf{l}|}\prod_{i,j=1}^r\left(\frac{x_i}{x_j}q^{1+l_i-l_j}\right)^{-1}_{k_i-l_i}. \tag{2.16}$$

Proof. In Theorem 2.1 we set $a_t\mapsto d/(q^t-a)$ and $c_i(t_i)\mapsto b^{1/m}x_iq^{t_i}$ for $i=1,\dots,r$. After some elementary manipulations, which include Lemma B.1, we let $d\to\infty$ and obtain the inverse pair (2.15)/(2.16). ☐

In Proposition 2.5, if we interchange $a$ and $b$ and transfer some factors from one matrix to the other we obtain the equivalent Proposition 2.6. In this form the inverse matrices are used in Section 3 for deriving some terminating $A_r$ $q$-Rothe summations.

Proposition 2.6. *Let $a$, $b$, and $x_1,\dots,x_r$ be indeterminate, and suppose that none of the denominators in* (2.17) *or* (2.18) *vanish. Moreover, let $m$ be a fixed* *integer such that* $0 \le m \le r$. *Then* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}} \frac{(b + ae_m(x_1q^{n_1},\dots,x_rq^{n_r}) - q^{|\mathbf{n}|})}{(b + ae_m(x_1q^{k_1},\dots,x_rq^{k_r}) - q^{|\mathbf{n}|})} \times \left(\frac{q}{(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{n}|} \big(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b\big)^{|\mathbf{n}|} \times \prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right] \tag{2.17}$$

*and*

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{l}|} \frac{\big(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b\big)^{-|\mathbf{l}|}}{\left(\frac{q}{(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{l}|}} \times \prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q\right)_{k_i}}{\left(\frac{x_i}{x_j}q\right)_{l_i}\left(\frac{x_i}{x_j}q^{1+l_i-l_j}\right)_{k_i-l_i}}\right]. \tag{2.18}$$

## 3. Applications to $A_r$ series

**3.1. Preliminaries.** Here we introduce the basic concept of "inverse relations" and introduce some standard $q$-series notation.

Probably, the most important application of matrix inversion is the derivation of (hypergeometric) series identities. There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. **[1]**, **[11]**, **[28]**). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are lower triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}}a_{\mathbf{k}} = b_{\mathbf{n}} \tag{3.1}$$

*if and only if*

$$\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{3.2}$$

If either (3.1) or (3.2) is known, then the other produces another summation formula. We will also use another version, the so-called "rotated inversion", which can be used to derive nonterminating summations. It reads

$$\sum_{\mathbf{n}\ge\mathbf{k}} f_{\mathbf{nk}}a_{\mathbf{n}} = b_{\mathbf{k}} \tag{3.3}$$

*if and only if*

$$\sum_{\mathbf{k}\ge\mathbf{l}} g_{\mathbf{kl}}b_{\mathbf{k}} = a_{\mathbf{l}}, \tag{3.4}$$

subject to suitable convergence conditions. Again, if one of (3.3) or (3.4) is known, the other produces a possibly new identity.

Subsequently, we use special cases of our Theorem 2.1 to derive a couple of higher dimensional summation formulas for $q$-series.

Before we start to develop the applications of our multidimensional matrix inversion, we recall some standard notation for $q$-series. Let $q$ be a complex number such that $0<|q|<1$. Define

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j), \tag{3.5}$$

and the *$q$-rising factorial*,

$$(a;q)_0 := 1, \qquad (a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}), \tag{3.6}$$

where $k$ is a nonnegative integer. If $a$ is not a negative integer power of $q$, (3.6) may also be written as

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{3.7}$$

As usual, we define the *$q$-binomial coefficient* as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}},$$

for nonnegative integers $n,k$ (cf. [**10**, Eq. (I.39)]).

As there is no confusion with any other notation, we use the short form

$$(a)_k \equiv (a;q)_k \tag{3.8}$$

for all $q$-rising factorials throughout this article.

For a thorough exposition on basic hypergeometric series including lists of selected summation and transformation formulas, we refer the reader to [**10**]. Here, however, we do not make use of the compact ${}_s\phi_t$ notation for basic hypergeometric series (cf. [**10**, Eq. (1.2.22)]), since for the series occurring in this article the latter notation cannot be applied.

Concerning the nonterminating multiple series given in this article, we have stated their regions of convergence explicitly. The absolute convergence of these series can be checked by application of the multiple power series ratio test [**14**], [**17**]. However, in our proofs we have omitted such calculations. Proofs of absolute convergence of series which are very similar to those in this paper are given in [**30**, Appendix C].

**3.2. Multiple $q$-Abel- and $q$-Rothe-type identities.** For illustration of the usefulness of our new multidimensional matrix inversion, we give new multiple series extensions of the $q$-Abel-type expansion

$$1=\sum_{k=0}^{\infty}\frac{(a+b)(a+bq^k)^{k-1}}{(q)_k}\left(z(a+bq^k)\right)_\infty z^k, \tag{3.9}$$

being valid for $|az|<1$ (see [**20**, Eq. (7.3)]), the $q$-Abel summation

$$1=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a+b)\left(a+bq^k\right)^{k-1} c^k \left(c(a+bq^k)\right)_{n-k} \tag{3.10}$$

(see [**20**, Eq. (8.1)]), the $q$-Rothe-type expansion

$$(z)_\infty=\sum_{k=0}^{\infty}\frac{1-(a+b)}{1-(aq^{-k}+b)}\frac{\left(aq^{-k}+b\right)_k}{(q)_k}(-1)^k q^{\binom{k}{2}}\left(z(a+bq^k)\right)_\infty z^k, \tag{3.11}$$

being valid for $|az| < 1$ (see [**20**, Eq. (7.4)]), and the $q$-Rothe summation

$$(c)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-(a+b)}{1-(aq^{-k}+b)} \left(aq^{-k}+b\right)_k \left(c\left(a+bq^k\right)\right)_{n-k} (-1)^k q^{\binom{k}{2}} c^k \tag{3.12}$$

(see [**20**, Eq. (8.5)]), respectively. Other terminating $q$-Abel and $q$-Rothe summations can be found in [**15**], [**16**], and [**30**]. Multiple series extensions of the above formulas have also been considered in [**20**, Sections 7 and 8] and [**30**, Sections 3–6] (which were also derived by multidimensional inverse relations.) There the (non-trivial) specializations which lead to the well-known Abel- and Rothe-type formulas for ordinary series (i.e. the "$q = 1$ case") are given explicitly.

First, we give a multiple series extension of (3.9). Theorem 3.1 generalizes the $A_r$ $q$-Abel-type expansion in [**30**, Theorem 3.2], to which it reduces for $m = r$ and $b \mapsto b/\prod_{i=1}^r x_i$.

THEOREM 3.1 (An $A_r$ $q$-Abel-type expansion). *Let $a$, $b$, $z$, and $x_1, \dots, x_r$ be indeterminate, and let $m$ be a fixed integer such that $0 \le m \le r$. Then there holds*

$$\begin{aligned} 1 = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( & \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) \prod_{i,j=1}^{r} \left(\frac{x_i}{x_j}q\right)_{k_i}^{-1} \\ & \times \left(a+be_m(x_1,\dots,x_r)\right) \left(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})\right)^{|\mathbf{k}|-1} \\ & \times (-1)^{(r-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \\ & \times z^{|\mathbf{k}|} \left(z(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r}))\right)_\infty \Bigg), \end{aligned} \tag{3.13}$$

*provided $|az| < \left|q^{\frac{r-1}{2}} x_j^{-r} \prod_{i=1}^r x_i\right|$ for $j = 1, \dots, r$.*

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.11)/(2.12). Then (3.3) holds for

$$\begin{aligned} a_{\mathbf{n}} = (-1)^{(r-1)|\mathbf{n}|} q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^r\binom{n_i}{2}+\sum_{i=1}^r(i-1)n_i} z^{|\mathbf{n}|} \prod_{i=1}^{r} x_i^{rn_i-|\mathbf{n}|} \\ \times \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right) \end{aligned}$$

and

$$\begin{aligned} b_{\mathbf{k}} = (-1)^{(r-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i} z^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \\ \times \left(z(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r}))\right)_\infty \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right) \end{aligned}$$

by the $A_r$ ${}_0\phi_0$-summation (B.2) in Theorem B.4. This implies the inverse relation (3.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \dots, r$, and the substitutions $x_i \mapsto x_i q^{-l_i}$, $i = 1, \dots, r$, we get rid of the $l_i$'s and eventually obtain (3.13). $\square$

Next, we give three $A_r$ $q$-Abel summations. These are multiple series extensions of (3.10). Theorems 3.2, 3.3, and 3.4 generalize the $A_r$ $q$-Abel summations in [**30**, Theorems 4.6, 4.7, and 4.8], respectively, to which they reduce for $m=r$ and $a \mapsto a\prod_{i=1}^r x_i$.

THEOREM 3.2 (An $A_r$ $q$-Abel summation). *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then there holds*

$$
\begin{aligned}
(3.14)\quad 1 = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^r & \left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right] \\
\times\, (ae_m(1/x_1,\dots,1/x_r)+b) & \left(aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+bq^{|\mathbf{k}|}\right)^{|\mathbf{k}|-1} \\
& \times c^{|\mathbf{k}|}\left(c(aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+bq^{|\mathbf{k}|})\right)_{|\mathbf{n}|-|\mathbf{k}|}\Bigg).
\end{aligned}
$$

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.13)/(2.14). Then (3.2) holds for

$$a_{\mathbf{k}} = \left(\frac{q}{c(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{k}|}$$

and

$$b_{\mathbf{l}} = q^{\binom{|\mathbf{l}|+1}{2}}c^{-|\mathbf{l}|}$$

by the $A_r$ terminating $q$-binomial theorem (B.3) in Theorem B.6. This implies the inverse relation (3.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i\mapsto n_i-k_i$, $i=1,\dots,r$. After performing the substitutions $c\mapsto cq^{|\mathbf{n}|}$, $x_i\mapsto q^{-n_i}/x_i$, $i=1,\dots,r$, we eventually obtain (3.14). ☐

THEOREM 3.3 (An $A_r$ $q$-Abel summation). *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then there holds*

$$
\begin{aligned}
(3.15)\quad 1 = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^r & \left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right] c^{|\mathbf{k}|} \\
\times\, (ae_m(1/x_1,\dots,1/x_r)+b) & \left(aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+bq^{|\mathbf{k}|}\right)^{|\mathbf{k}|-1} \\
& \times q^{-e_2(\mathbf{k})}\prod_{i=1}^r x_i^{k_i}\left(cx_iq^{k_i}(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b)\right)_{n_i-k_i}\Bigg),
\end{aligned}
$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.*

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.13)/(2.14). Then (3.2) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^r\left(\frac{qx_i}{c(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{|\mathbf{l}|+\sum_{i=1}^r \binom{l_i}{2}} c^{-|\mathbf{l}|} \prod_{i=1}^r x_i^{l_i}$$

by the $A_r$ terminating $q$-binomial theorem (B.4) in Theorem B.6. This implies the inverse relation (3.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \dots, r$. After performing the substitutions $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \dots, r$, we eventually obtain (3.15). ☐

THEOREM 3.4 (An $A_r$ $q$-Abel summation). *Let $a$, $b$, $c$, and $x_1, \dots, x_r$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Moreover, let $m$ be a fixed integer such that $0 \le m \le r$. Then there holds*

$$\begin{aligned}(3.16)\quad 1 = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^r \left[ \frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}} \right] q^{e_2(\mathbf{k})} \\ \times \left(ae_m(1/x_1, \dots, 1/x_r) + b\right) \left(aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1, \dots, q^{-k_r}/x_r) + bq^{|\mathbf{k}|}\right)^{|\mathbf{k}|-1} \\ \times c^{|\mathbf{k}|} \prod_{i=1}^r x_i^{-k_i} \left( \frac{c}{x_i} q^{|\mathbf{n}|-n_i} (aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1, \dots, q^{-k_r}/x_r) + bq^{|\mathbf{k}|}) \right)_{n_i-k_i} \Bigg),\end{aligned}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \dots, k_r\}$.*

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.13)/(2.14). Then (3.2) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^r \left( \frac{q^{1+|\mathbf{k}|-k_i}}{cx_i(ae_m(x_1q^{k_1}, \dots, x_rq^{k_r}) + b)} \right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{e_2(\mathbf{l})+\binom{|\mathbf{l}|+1}{2}} c^{-|\mathbf{l}|} \prod_{i=1}^r x_i^{-l_i}$$

by the $A_r$ terminating $q$-binomial theorem (B.5) in Theorem B.6. This implies the inverse relation (3.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \dots, r$. After performing the substitutions $c \mapsto cq^{2|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \dots, r$, we eventually obtain (3.16). ☐

In the following, we give two $A_r$ $q$-Rothe-type expansions. These are multiple series extensions of (3.11). Theorems 3.5 and 3.6 generalize the $A_r$ $q$-Rothe-type expansions in [**30**, Theorems 5.1 and 5.2], respectively, to which they reduce for $m = r$ and $b \mapsto b/\prod_{i=1}^r x_i$.

THEOREM 3.5 (An $A_r$ $q$-Rothe-type expansion). *Let $a$, $b$, $z$, and $x_1,\dots,x_r$ be indeterminate, and let $m$ be a fixed integer such that $0 \le m \le r$. Then there holds*

$$
\begin{aligned}
(3.17)\quad (z)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( & \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j}q \right)_{k_i}^{-1} \\
& \times (-1)^{r|\mathbf{k}|} \frac{(1-(a+be_m(x_1,\dots,x_r)))}{(1-(aq^{-|\mathbf{k}|}+bq^{-|\mathbf{k}|}e_m(x_1q^{k_1},\dots,x_rq^{k_r})))} \\
& \times \left( aq^{-|\mathbf{k}|}+bq^{-|\mathbf{k}|}e_m(x_1q^{k_1},\dots,x_rq^{k_r}) \right)_{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \\
& \times q^{r\sum_{i=1}^{r}\binom{k_i}{2}+\sum_{i=1}^{r}(i-1)k_i} z^{|\mathbf{k}|} \left( z(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})) \right)_\infty \Bigg),
\end{aligned}
$$

*provided* $|az| < \left| q^{\frac{r-1}{2}} x_j^{-r} \prod_{i=1}^{r} x_i \right|$ *for* $j=1,\dots,r$.

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.15)/(2.16). Then (3.3) holds for

$$
\begin{aligned}
a_{\mathbf{n}} = (-1)^{(r-1)|\mathbf{n}|} & q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^{r}\binom{n_i}{2}+\sum_{i=1}^{r}(i-1)n_i} \prod_{i=1}^{r} x_i^{rn_i-|\mathbf{n}|} \\
& \times z^{|\mathbf{n}|} (z)_{|\mathbf{n}|}^{-1} \prod_{1\le i<j\le r} \left( 1-\frac{x_i}{x_j}q^{n_i-n_j} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
b_{\mathbf{k}} = (-1)^{(r-1)|\mathbf{k}|} & q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^{r}\binom{k_i}{2}+\sum_{i=1}^{r}(i-1)k_i} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \\
& \times z^{|\mathbf{k}|} \frac{\left( z(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})) \right)_\infty}{(z)_\infty} \prod_{1\le i<j\le r} \left( 1-\frac{x_i}{x_j}q^{k_i-k_j} \right)
\end{aligned}
$$

by the $A_r$ ${}_1\phi_1$-summation (B.6) in Theorem B.8. This implies the inverse relation (3.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i=1,\dots,r$, and the substitutions $a \mapsto aq^{|\mathbf{l}|}$, $b \mapsto bq^{|\mathbf{l}|}$, $z \mapsto zq^{-|\mathbf{l}|}$, $x_i \mapsto x_i q^{-l_i}$, $i=1,\dots,r$, we get rid of the $l_i$'s and eventually obtain (3.17). $\square$

THEOREM 3.6 (An $A_r$ $q$-Rothe-type expansion). *Let $a$, $b$, $z$, and $x_1,\dots,x_r$ be indeterminate, and let $m$ be a fixed integer such that $0 \le m \le r$. Then there holds*

$$
\begin{aligned}
(3.18)\quad (z)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( & \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j}q \right)_{k_i}^{-1} \\
& \times \frac{(1-(a+be_m(x_1,\dots,x_r)))}{(1-(aq^{-|\mathbf{k}|}+bq^{-|\mathbf{k}|}e_m(x_1q^{k_1},\dots,x_rq^{k_r})))} \\
& \times \left( aq^{-|\mathbf{k}|}+bq^{-|\mathbf{k}|}e_m(x_1q^{k_1},\dots,x_rq^{k_r}) \right)_{|\mathbf{k}|} (-1)^{|\mathbf{k}|} \\
& \times q^{\binom{|\mathbf{k}|}{2}+\sum_{i=1}^{r}(i-1)k_i} z^{|\mathbf{k}|} \left( z(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r})) \right)_\infty \Bigg),
\end{aligned}
$$

*provided* $|az| < 1$.

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.15)/(2.16). Then (3.3) holds for

$$a_{\mathbf{n}} = z^{|\mathbf{n}|}\,(z)_{|\mathbf{n}|}^{-1}\,q^{\sum_{i=1}^r (i-1)n_i} \prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)$$

and

$$b_{\mathbf{k}} = z^{|\mathbf{k}|}\frac{\big(z(a+be_m(x_1q^{k_1},\dots,x_rq^{k_r}))\big)_\infty}{(z)_\infty}\,q^{\sum_{i=1}^r (i-1)k_i} \prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)$$

by the $A_r$ ${}_1\phi_1$-summation (B.7) in Theorem B.8. This implies the inverse relation (3.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1,\dots,r$, and the substitutions $a \mapsto aq^{|\mathbf{l}|}$, $b \mapsto bq^{|\mathbf{l}|}$, $z \mapsto zq^{-|\mathbf{l}|}$, $x_i \mapsto x_iq^{-l_i}$, $i = 1,\dots,r$, we get rid of the $l_i$'s and eventually obtain (3.18). □

Finally, we give three $A_r$ $q$-Rothe summations. These are multiple series extensions of (3.12). Theorems 3.7, 3.8, and 3.9 generalize the $A_r$ $q$-Rothe summations in [**30**, Theorems 6.3, 6.4, and 6.5], respectively, to which they reduce for $m = r$ and $a \mapsto a\prod_{i=1}^r x_i$.

THEOREM 3.7 (An $A_r$ $q$-Rothe summation). *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Moreover, let $m$ be a fixed integer such that $0 \le m \le r$. Then there holds*

$$\prod_{i=1}^r (cx_i)_{n_i} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]q^{\sum_{i=1}^r\binom{k_i}{2}}\right.$$
$$\times\frac{(1-(ae_m(1/x_1,\dots,1/x_r)+b))}{(1-(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b))}\,\big(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b\big)_{|\mathbf{k}|}$$
$$\left.\times(-1)^{|\mathbf{k}|}c^{|\mathbf{k}|}\prod_{i=1}^r x_i^{k_i}\big(cx_iq^{k_i}(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b)\big)_{n_i-k_i}\right). \tag{3.19}$$

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.17)/(2.18). Then (3.2) holds for

$$a_{\mathbf{k}} = \left(\frac{q}{(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{k}|}^{-1}\prod_{i=1}^r\left(\frac{qx_i}{c(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{|\mathbf{l}|+\sum_{i=1}^r\binom{l_i}{2}}\,c^{-|\mathbf{l}|}\prod_{i=1}^r x_i^{l_i}\left(\frac{c}{x_i}q^{|\mathbf{l}|-l_i}\right)_{l_i}$$

by the $A_r$ $q$-Chu–Vandermonde summation (B.8) in Theorem B.10. This implies the inverse relation (3.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1,\dots,r$. After performing the substitutions $a \mapsto aq^{|\mathbf{n}|}$, $b \mapsto bq^{|\mathbf{n}|}$, $c \mapsto cq^{-|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1,\dots,r$, we eventually obtain (3.19). □

THEOREM 3.8 (An $A_r$ $q$-Rothe summation). *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then there holds*

$$\prod_{i=1}^r\left(\frac{c}{x_i}q^{|\mathbf{n}|-n_i}\right)_{n_i}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\right.$$
$$\times q^{e_2(\mathbf{k})+\binom{|\mathbf{k}|}{2}}\frac{(1-(ae_m(1/x_1,\dots,1/x_r)+b))}{(1-(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b))}$$
$$\times\left(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b\right)_{|\mathbf{k}|}(-1)^{|\mathbf{k}|}c^{|\mathbf{k}|}$$
$$\left.\times\prod_{i=1}^r x_i^{-k_i}\left(\frac{c}{x_i}q^{|\mathbf{n}|-n_i}(aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+bq^{|\mathbf{k}|})\right)_{n_i-k_i}\right),\tag{3.20}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.*

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.17)/(2.18). Then (3.2) holds for

$$a_{\mathbf{k}}=\left(\frac{q}{(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{k}|}^{-1}\prod_{i=1}^r\left(\frac{q^{1+|\mathbf{k}|-k_i}}{cx_i(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{k_i}$$

and

$$b_{\mathbf{l}}=q^{e_2(\mathbf{l})+\binom{|\mathbf{l}|+1}{2}}c^{-|\mathbf{l}|}\prod_{i=1}^r x_i^{-l_i}(cx_i)_{l_i}$$

by the $A_r$ $q$-Chu–Vandermonde summation (B.9) in Theorem B.10. This implies the inverse relation (3.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i\mapsto n_i-k_i$, $i=1,\dots,r$. After performing the substitutions $a\mapsto aq^{|\mathbf{n}|}$, $b\mapsto bq^{|\mathbf{n}|}$, $c\mapsto cq^{|\mathbf{n}|}$, $x_i\mapsto q^{-n_i}/x_i$, $i=1,\dots,r$, we eventually obtain (3.20). □

THEOREM 3.9 (An $A_r$ $q$-Rothe summation). *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Moreover, let $m$ be a fixed integer such that $0\le m\le r$. Then there holds*

$$(c)_{|\mathbf{n}|}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\right.$$
$$\times\frac{(1-(ae_m(1/x_1,\dots,1/x_r)+b))}{(1-(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b))}\left(ae_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+b\right)_{|\mathbf{k}|}$$
$$\left.\times(-1)^{|\mathbf{k}|}q^{\binom{|\mathbf{k}|}{2}}c^{|\mathbf{k}|}\left(c(aq^{|\mathbf{k}|}e_m(q^{-k_1}/x_1,\dots,q^{-k_r}/x_r)+bq^{|\mathbf{k}|})\right)_{|\mathbf{n}|-|\mathbf{k}|}\right).\tag{3.21}$$

PROOF. Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (2.17)/(2.18). Then (3.2) holds for

$$a_{\mathbf{k}}=\left(\frac{q}{(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{k}|}^{-1}\left(\frac{q}{c(ae_m(x_1q^{k_1},\dots,x_rq^{k_r})+b)}\right)_{|\mathbf{k}|}$$

and

$$b_{\mathbf{l}} = q^{\binom{|\mathbf{l}|+1}{2}} c^{-|\mathbf{l}|} (c)_{|\mathbf{l}|}$$

by the $A_r$ $q$-Chu–Vandermonde summation (B.10) in Theorem B.10. This implies the inverse relation (3.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \dots, r$. After performing the substitutions $a \mapsto aq^{|\mathbf{n}|}$, $b \mapsto bq^{|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \dots, r$, we eventually obtain (3.21). □

CONCLUDING REMARK 3.10. In this Section, we derived multidimensional $q$-Abel- and $q$-Rothe-type identities by inverse relations. If we specialized our inverse pair of matrices as in the Propositions of Section 2 but with $c_i(t_i) \mapsto b^{1/m}(x_i + t_i)$ instead of $c_i(t_i) \mapsto b^{1/m} x_i q^{t_i}$ (and similarly adjusting the specialization of $a_t$), in combination with certain $A_r$ hypergeometric series, we could have derived some Abel and Rothe summations for "ordinary" ($q = 1$) $A_r$ series (see [**3**] and [**30**] for other $A_r$ Abel and Rothe summations). However, we did not aim to give a complete treatment of possible applications of our matrix inversion in Theorem 2.1, but rather wanted to provide a few examples for evidence.

## Appendix A. A determinant evaluation

We need the following Lemma in the proof of Theorem 2.1.

LEMMA A.1. *Let $x_1, \dots, x_r$, $y_0, y_1, \dots, y_r$, and $a$ be indeterminate. Let $P(x) = (x - y_0) \dots (x - y_r)$, and let $m$ be a fixed integer with $0 \le m \le r$. Then*

$$\begin{aligned} P(a) \cdot \det_{1 \le i,j \le r} \left( x_i^{r+\chi(j\le m)-j} - a^{r+\chi(j\le m)-j} \frac{P(x_i)}{P(a)} \right) \\ = \left( a e_m(x_1, \dots, x_r) + e_{m+1}(x_1, \dots, x_r) - e_{m+1}(y_0, \dots, y_r) \right) \\ \times \prod_{i=1}^{r} (a - x_i) \prod_{1 \le i < j \le r} (x_i - x_j). \end{aligned} \tag{A.1}$$

REMARK A.2. This generalizes [**29**, Lemma A.1], which is the special case $m = r$. For $m = r$ the following proof reduces to Don Zagier's [**31**] shorter and more elegant proof of [**29**, Lemma A.1] which he kindly communicated to us.

PROOF OF LEMMA A.1. From the standard identity $\det\begin{pmatrix} M & \eta \\ \xi & c \end{pmatrix} = c \det(M - c^{-1}\xi\eta)$ applied to $M = (x_i^{r+\chi(j\le m)-j})$, $\xi = (a^{r+\chi(j\le m)-j})$, $\eta = (P(x_i))$ and $c = P(a)$, we see that the left-hand side of (A.1) is just the determinant $D$ of the $(r+1)\times(r+1)$ matrix with rows $(x_i^{r+\chi(1\le m)-1}, x_i^{r+\chi(2\le m)-2}, \dots, x_i^{\chi(r\le m)}, P(x_i))$, where $x_{r+1} = a$. Substracting from the last column a linear combination of the other columns to replace $P(x)$ by $x^{r+1} + (-1)^{m+1} e_{m+1}(y_0, \dots, y_r) x^{r-m}$ we can simplify $D$ to

$$\begin{aligned} (-1)^r \det_{1 \le i,j \le r+1} \left( x_i^{r+1+\chi(j \le m+1)-j} \right) + (-1)^{r+1} e_{m+1}(y_0, \dots, y_r) \det_{1 \le i,j \le r+1} \left( x_i^{r+1-j} \right) \\ = \left( e_{m+1}(x_1, \dots, x_r, a) - e_{m+1}(y_0, \dots, y_r) \right) \prod_{i=1}^{r} (a - x_i) \prod_{1 \le i < j \le r} (x_i - x_j), \end{aligned}$$

where we made use of [**22**, (3.1) and (3.9)]. □

## Appendix B. Background information – $A_r$ basic hypergeometric summation theorems

Here we state a simplification lemma, and some $A_r$ basic hypergeometric summations from Milne [**25**] we need in the proofs of our theorems of Section 3.

LEMMA B.1.

$$\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}q^{l_i-l_j}}\right)\prod_{i,j=1}^r\frac{\left(\frac{x_i}{x_j}q^{l_i-k_j}\right)_{k_i-l_i}}{\left(\frac{x_i}{x_j}q^{1+l_i-l_j}\right)_{k_i-l_i}}=(-1)^{|\mathbf{k}|-|\mathbf{l}|}q^{-\binom{|\mathbf{k}|-|\mathbf{l}|}{2}-\sum_{i=1}^r i(k_i-l_i)}.$$

REMARK B.2. Lemma B.1 is equivalent to Lemma 4.3 of [**25**], which is proved by some elementary manipulations.

REMARK B.3. When reversing order of the summations in the proofs of Section 3, we permanently make use of the fact that the "$A_r$ $q$-binomial coefficient"

$$\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\tag{B.1}$$

remains unchanged after performing the substitutions $k_i\mapsto n_i-k_i$ and $x_i\mapsto q^{-n_i}/x_i$, for $i=1,\dots,r$. This can be seen using Lemma B.1.

THEOREM B.4 (An $A_r$ ${}_0\phi_0$-summation). *Let $a$ and $x_1,\dots,x_r$ be indeterminate. Then there holds*

$$(a)_\infty=\sum_{k_1,\dots,k_r=0}^\infty\left(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)\prod_{i,j=1}^r\left(\frac{x_i}{x_j}q\right)_{k_i}^{-1}\right.$$
$$\left.\times(-1)^{r|\mathbf{k}|}a^{|\mathbf{k}|}q^{r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i}\prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}\right).\tag{B.2}$$

REMARK B.5. Theorem B.4 can be obtained from Theorem 1.47 of [**25**] by substituting $z\mapsto z/\prod_{i=1}^r a_i$, then letting $a_1\to\infty,\dots,a_r\to\infty$, and relabelling $z\mapsto a$.

THEOREM B.6 ($A_r$ terminating $q$-binomial theorems). *Let $z$ and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$(z)_{|\mathbf{n}|}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right](-1)^{|\mathbf{k}|}q^{\binom{|\mathbf{k}|}{2}}z^{|\mathbf{k}|}\right),\tag{B.3}$$

$$\prod_{i=1}^r(zx_i)_{n_i}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^r\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\right.$$
$$\left.\times(-1)^{|\mathbf{k}|}q^{\sum_{i=1}^r\binom{k_i}{2}}z^{|\mathbf{k}|}\prod_{i=1}^r x_i^{k_i}\right),\tag{B.4}$$

$$\prod_{i=1}^{r}\left(\frac{z}{x_i}q^{|\mathbf{n}|-n_i}\right)_{n_i}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\right.$$
$$\left.\times(-1)^{|\mathbf{k}|}q^{e_2(\mathbf{k})+\binom{|\mathbf{k}|}{2}}z^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{-k_i}\right), \tag{B.5}$$

*where* $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\{k_1,\dots,k_r\}$.

REMARK B.7. The summations (B.3), (B.4), and (B.5) are Theorems 5.44, 5.46, and 5.48 of [**25**], respectively (slightly rewritten using Lemma B.1).

THEOREM B.8 ($A_r$ ${}_1\phi_1$-summations). *Let* $a$, $c$, *and* $x_1,\dots,x_r$ *be indeterminate. Then there holds*

$$\frac{(c/a)_\infty}{(c)_\infty}=\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)\prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q\right)_{k_i}^{-1}\right.$$
$$\left.\times\frac{(a)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}}(-1)^{r|\mathbf{k}|}q^{r\sum_{i=1}^{r}\binom{k_i}{2}+\sum_{i=1}^{r}(i-1)k_i}\left(\frac{c}{a}\right)^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{rk_i-|\mathbf{k}|}\right), \tag{B.6}$$

$$\frac{(c/a)_\infty}{(c)_\infty}=\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)\prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q\right)_{k_i}^{-1}\right.$$
$$\left.\times\frac{(a)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}}(-1)^{|\mathbf{k}|}q^{\binom{|\mathbf{k}|}{2}+\sum_{i=1}^{r}(i-1)k_i}\left(\frac{c}{a}\right)^{|\mathbf{k}|}\right). \tag{B.7}$$

REMARK B.9. The summation (B.6) can be obtained from Theorem 7.6 of [**25**] by letting $a_1\to\infty,\dots,a_r\to\infty$, and relabelling $b\mapsto a$. The summation (B.7) can be obtained from Theorem 7.9 of [**25**] by letting $b\to\infty$.

THEOREM B.10 ($A_r$ $q$-Chu–Vandermonde summations). *Let* $a$, $c$, *and* $x_1,\dots,x_r$ *be indeterminate, and let* $n_1,\dots,n_r$ *be nonnegative integers. Then there holds*

$$\frac{\prod_{i=1}^{r}(cx_i/a)_{n_i}}{(c)_{|\mathbf{n}|}}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\right.$$
$$\left.\times\frac{\prod_{i=1}^{r}\left(\frac{a}{x_i}q^{|\mathbf{k}|-k_i}\right)_{k_i}}{(c)_{|\mathbf{k}|}}(-1)^{|\mathbf{k}|}q^{\sum_{i=1}^{r}\binom{k_i}{2}}\left(\frac{c}{a}\right)^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{k_i}\right), \tag{B.8}$$

$$\frac{\prod_{i=1}^{r}\left(\frac{c}{ax_i}q^{|\mathbf{n}|-n_i}\right)_{n_i}}{(c)_{|\mathbf{n}|}}=\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\left(\prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}}\right]\right.$$
$$\left.\times\frac{\prod_{i=1}^{r}(ax_i)_{k_i}}{(c)_{|\mathbf{k}|}}(-1)^{|\mathbf{k}|}q^{e_2(\mathbf{k})+\binom{|\mathbf{k}|}{2}}\left(\frac{c}{a}\right)^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{-k_i}\right), \tag{B.9}$$

*where* $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\{k_1,\dots,k_r\}$,

$$\frac{(c/a)_{|\mathbf{n}|}}{(c)_{|\mathbf{n}|}} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q\right)_{n_i}}{\left(\frac{x_i}{x_j}q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j}\right)_{n_i-k_i}} \right] \times \frac{(a)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}} \left(\frac{c}{a}\right)^{|\mathbf{k}|} \right). \tag{B.10}$$

REMARK B.11. The summations (B.8) and (B.9) are Theorems 5.28 and 5.32 of [**25**], respectively (slightly rewritten using Lemma B.1). The summation (B.10) can be obtained from Theorem 7.6 of [**25**] by letting $a_i \to q^{-n_i}$, $i=1,\dots,r$, and relabelling $b \mapsto a$.

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