ELLIPTIC EXTENSIONS OF THE ALPHA-PARAMETER MODEL AND THE ROOK MODEL FOR MATCHINGS

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Abstract. We construct elliptic extensions of the alpha-parameter rook model introduced by Goldman and Haglund and of the rook model for matchings of Haglund and Remmel. In particular, we extend the product formulas of these models to the elliptic setting. By specializing the parameter α in our elliptic extension of the alpha-parameter model and the shape of the Ferrers board in different ways, we obtain elliptic analogues of the Stirling numbers of the first kind and of the Abel polynomials, and an a,q-analogue of the matching numbers. We further generalize the rook theory model for matchings by introducing $\mathbf{l}$-lazy graphs which correspond to $\mathbf{l}$-shifted boards, where $\mathbf{l}$ is a finite vector of positive integers. The corresponding elliptic product formula generalizes Haglund and Remmel's product formula for matchings already in the non-elliptic basic case.

1. Introduction

Since the introduction of rook theory by Kaplansky and Riordan [12], the theory has thrived and developed further by revealing connections to, for instance, orthogonal polynomials [7, 9], hypergeometric series [10], q-analogues and permutation statistics [2, 5], algebraic geometry [3, 4], and many more. Within rook theory itself, various models have been introduced, including a p,q-analogue of rook numbers [1, 14, 20], the j-attacking model [14], the matching model [11], the augmented rook model [13] which includes all other models as special cases, etc. In previous work [17, 18], the authors have constructed elliptic extensions of the aforementioned rook theory models and obtained corresponding product formulas.

In this work, we construct elliptic extensions of two rook theory models: the alpha parameter rook model introduced by Goldman and Haglund [8] and the matching model of Haglund and Remmel [11]. The alpha-parameter model, as explained in [8], is a slight generalization of the i-creation model. It connects to several other combinatorial models, including polynomial sequences of binomial type, permutations of multisets, Abel polynomials, Bessel polynomials and matchings, and so on. Our elliptic extension lays the foundations for raising those connections to the elliptic level.

In our construction of an elliptic analogue of the matching model, we actually consider a model that generalizes the original model of Haglund and Remmel already in the non-elliptic, basic case. In particular, we consider matchings on specific graphs which we call "$\mathbf{l}$-lazy graphs" with respect to an N-dimensional vector of positive integers, $\mathbf{l} = (l_1, l_2, \dots, l_N)$. The original matching model can be realized from the generalized model by setting $N = 2n-1$ and $\mathbf{l} = (1, 1, \dots, 1)$. For the new model, we are able to prove a product formula involving elliptic

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rook numbers for matchings on $\mathbf{l}$-lazy graphs, a result which generalizes the corresponding product formula of Haglund and Remmel [11].

In Section 2 we define elliptic weights and review some of the elementary identities useful for dealing with them. Section 3 is devoted to the elliptic extension of the alpha-parameter model, together with some applications. Finally, Section 4 features an elliptic extension of the rook theory of matchings.

2. Elliptic weights

In this section, we define the elliptic weights which we utilize to weight cells in Ferrers boards. (For the definition of Ferrers boards, see Section 3). We start by explaining what elliptic functions are.

A complex function is called elliptic, if it is a doubly-periodic, meromorphic function on $\mathbb{C}$. It is well-known that such functions can be expressed in terms of ratios of theta functions (cf. [21]). We will use the following (multiplicative) notation for theta functions. First, we define the *modified Jacobi theta function* with argument x and nome p by

$$\theta(x;p)=\prod_{j\geq 0}((1-p^jx)(1-p^{j+1}/x)), \qquad \theta(x_1,\dots,x_m;p)=\prod_{k=1}^m\theta(x_k;p),$$

where $x,x_1,\dots,x_m\neq 0$, $|p|<1$. Further, we define the *theta shifted factorial* (or q,p*-shifted factorial*) by

$$(a;q,p)_n=\begin{cases}\prod_{k=0}^{n-1}\theta(aq^k;p), & n=1,2,\dots\,,\\ 1, & n=0,\\ 1/\prod_{k=0}^{-n-1}\theta(aq^{n+k};p), & n=-1,-2,\dots.\end{cases}$$

We frequently write

$$(a_1,a_2,\dots,a_m;q,p)_n=\prod_{k=1}^m(a_k;q,p)_n,$$

for brevity. Notice that for $p=0$ we have $\theta(x;0)=1-x$ and, hence, $(a;q,0)_n=(a;q)_n$ is just the usual *q-shifted factorial* in base q (cf. [6]). The parameters q and p in $(a;q,p)_n$ are called the *base* and *nome*, respectively. In analogy to the theories of ordinary and basic hypergeometric series (cf. [6]) there exists also a (rather young) theory of hypergeometric series involving theta shifted factorials, namely of theta, modular, and elliptic hypergeometric series, see [6, Chapter 11].

The modified Jacobi theta functions satisfy the following basic properties which are essential in the theory of elliptic hypergeometric series:

$$\begin{aligned}\theta(x;p)&=-x\,\theta(1/x;p),\\ \theta(px;p)&=-\frac{1}{x}\,\theta(x;p),\end{aligned}\tag{2.1}$$

and the *addition formula*

$$\theta(xy,x/y,uv,u/v;p)-\theta(xv,x/v,uy,u/y;p)=\frac{u}{y}\,\theta(yv,y/v,xu,x/u;p)\tag{2.2}$$

(cf. [22, p. 451, Example 5]).

The three-term relation in (2.2), containing four variables and four factors of theta functions in each term, is the "smallest" addition formula connecting products of theta functions with general arguments. Note that in the theta function $\theta(x;p)$ one cannot let $x\to 0$ (unless one first lets $p \to 0$) since x is an essential singularity. (For this reason elliptic analogues of q-series identities usually contain many parameters.)

The elliptic identities we shall consider all involve terms which are elliptic (with the same periods) in all of their parameters. Spiridonov [19] refers to such multivariate functions as *totally elliptic*; they are by nature *well-poised* and *balanced* (see also [6, Chapter 11]).

Following the setup used in our earlier paper on elliptic rook numbers [17], we define the *elliptic weights* $w_{a,b;q,p}(k)$ and $W_{a,b;q,p}(k)$, depending on two independent parameters a and b, base q, nome p, and integer parameter k by

$$w_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, bq^k, aq^{k-2}/b; p)}{\theta(aq^{2k-1}, bq^{k+2}, aq^k/b; p)} q, \tag{2.3a}$$

and

$$W_{a,b;q,p}(k) = \frac{\theta(aq^{1+2k}, bq, bq^2, aq^{-1}/b, a/b; p)}{\theta(aq, bq^{k+1}, bq^{k+2}, aq^{k-1}/b, aq^k/b; p)} q^k, \tag{2.3b}$$

respectively. It is clear that if k is a positive integer, Equations (2.3a) and (2.3b) imply that

$$W_{a,b;q,p}(k) = \prod_{j=1}^{k} w_{a,b;q,p}(j). \tag{2.4}$$

We refer to the $w_{a,b;q,p}(k)$ as *small* weights (these will correspond to the weights of single squares in the Ferrers boards) and to the $W_{a,b;q,p}(k)$ as *big* weights (these will correspond to partial columns of a Ferrers board). Note that the weights $w_{a,b;q,p}(k)$ and $W_{a,b;q,p}(k)$ also can be defined for arbitrary (complex) value k which is clear from the definition.

We will make frequent use of the following two properties:

$$w_{a,b;q,p}(k+n) = w_{aq^{2k},bq^k;q,p}(n), \tag{2.5a}$$
$$W_{a,b;q,p}(k+n) = W_{a,b;q,p}(k)\, W_{aq^{2k},bq^k;q,p}(n). \tag{2.5b}$$

Remark 2.1. (1) The small weight $w_{a,b;q,p}(k)$ (and so the big one) is indeed elliptic in its parameters (i.e., totally elliptic). If we write $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, $a = q^\alpha$ and $b = q^\beta$ with complex σ, τ, α, β and k, then the small weight $w_{a,b;q,p}(k)$ is clearly periodic in α with period σ^{-1}. A simple computation involving (2.1) further shows that $w_{a,b;q,p}(k)$ is also periodic in α with period $\tau\sigma^{-1}$. The same applies to $w_{a,b;q,p}(k)$ as a function in β (or k) with the same two periods σ^{-1} and $\tau\sigma^{-1}$.

(2) For $p \to 0$, the small and big weights reduce to

$$w_{a,b;q}(k) = \frac{(1-aq^{2k+1})(1-bq^k)(1-aq^{k-2}/b)}{(1-aq^{2k-1})(1-bq^{k+2})(1-aq^k/b)} q, \quad \text{and} \tag{2.6a}$$

$$W_{a,b;q}(k) = \frac{(1-aq^{1+2k})(1-bq)(1-bq^2)(1-aq^{-1}/b)(1-a/b)}{(1-aq)(1-bq^{k+1})(1-bq^{k+2})(1-aq^{k-1}/b)(1-aq^k/b)} q^k, \tag{2.6b}$$

respectively. In the $a,b;q$-weights in (2.6), we may let $b \to 0$ (or $b \to \infty$) to obtain "$a,0;q$-weights", or in short, "$a;q$-weights":

$$w_{a;q}(k) = \frac{(1-aq^{2k+1})}{(1-aq^{2k-1})} q^{-1}, \qquad \text{and} \qquad W_{a;q}(k) = \frac{(1-aq^{1+2k})}{(1-aq)} q^{-k}.$$

Note that by writing $q = e^{ix}$ and $a = e^{i(2c+1)x}$, $c \in \mathbb{N}$, the $a;q$-weights can be written as quotients of Chebyshev polynomials of the second kind.

Also, in (2.6), we may let $a \to 0$ (or $a \to \infty$) to obtain "$b;q$-weights". Importantly, if in (2.6) we first let $b \to 0$ and then $a \to \infty$ (or, equivalently, first let $a \to 0$ and then $b \to 0$), we obtain the familiar q-weights

$$w_q(k) = q \qquad \text{and} \qquad W_q(k) = q^k,$$

respectively.

We also define an elliptic analogue of the q-number $[z]_q = \frac{1-q^z}{1-q}$ by

$$[z]_{a,b;q,p} = \frac{\theta(q^z, aq^z, bq^2, a/b; p)}{\theta(q, aq, bq^{z+1}, aq^{z-1}/b; p)}.$$

Using the addition formula for theta functions (2.2), it is straightforward to verify that the thus defined elliptic numbers satisfy

$$[z]_{a,b;q,p} = [z-1]_{a,b;q,p} + W_{a,b;q,p}(z-1). \tag{2.7}$$

In case $z = n$ is a nonnegative integer, (2.7) constitutes a recursion which, together with $W_{a,b;q,p}(0) = 1$, uniquely defines any elliptic number $[n]_{a,b;q,p}$, namely,

$$[n]_{a,b;q,p} = 1 + W_{a,b;q,p}(1) + \cdots + W_{a,b;q,p}(n-1).$$

More generally, by (2.2) we have the following useful identity

$$[z]_{a,b;q,p} = [y]_{a,b;q,p} + W_{a,b;q,p}(y)[z-y]_{aq^{2y},bq^y;q,p} \tag{2.8}$$

which reduces to (2.7) for $y = z-1$.

Remark 2.2. In [16], the first author, in analogy to the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^{1+k};q)_{n-k}}{(q;q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[0]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q!$, defined the elliptic binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q,p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q,p)_{n-k}}. \tag{2.9}$$

These satisfy a nice recursion

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{aq^{k-1},bq^{2k-2};q,p}(n+1-k) \qquad \text{for } n,k \in \mathbb{N}_0,$$

with the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{a,b;q,p} = 1, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = 0 \qquad \text{for } n \in \mathbb{N}_0, \text{ and } k \in -\mathbb{N} \text{ or } k > n, \tag{2.10}$$

where $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and of nonnegative integers, respectively.

Combinatorially, the elliptic binomial coefficient in (2.9) can be interpreted in terms of weighted lattice paths in $\mathbb{Z}^2$ see [15]. More precisely, (2.9) is the weighted-area generating function for paths starting in $(0,0)$ and ending in $(k, n-k)$ composed of unit steps going north or east only, when the weight of each cell (with (s,t) as the north-east corner) "covered" by the path is defined to be $w_{aq^{s-1},bq^{2s-2};q,p}(t)$. Then it can be shown that the sum of weighted areas below the paths satisfies the same recursion (2.10) by distinguishing the last step of the path whether it is vertical or horizontal. Now, the elliptic number $[n]_{a,b;q,p}$ is just a short-hand notation for

$$[n]_{a,b;q,p} = \begin{bmatrix} n \\ 1 \end{bmatrix}_{a,b;q,p},$$

the weighted enumeration of all paths starting in $(0,0)$ and ending in $(1, n-1)$.

If we take the limit $p \to 0$, $a \to 0$, and $b \to 0$ (in this order), then we recover the usual q-binomial coefficients, i.e.,

$$\lim_{b\to 0}(\lim_{a\to 0}(\lim_{p\to 0}[z]_{a,b;q,p})) = [z]_q.$$

3. Elliptic extension of the alpha-parameter model

In [8, Section 6], Goldman and Haglund introduced the *alpha-parameter model* which generalizes the original rook theory [12] as well as the i-creation model that they introduced in the same paper [8]. In [8, Section 7], they also derived q-analogues of their results. The purpose of this section is to extend the alpha-parameter model to the elliptic setting. Before we give an explicit description in the elliptic setting, we review the classical case.

We consider a *board* to be a finite subset of the $\mathbb{N} \times \mathbb{N}$ grid, and label the columns from left to right with $1, 2, 3, \dots$, and the rows from bottom to top with $1, 2, 3, \dots$. We use the notation (i, j) to denote the cell in the i-th column from the left and the j-th row from the bottom. For technical reasons, in our proofs, we sometimes find it convenient to extend the $\mathbb{N}\times\mathbb{N}$ grid to $\mathbb{N}\times\mathbb{Z}$ where the row below the row 1 is labeled by 0 and the row labels decrease by 1 as they go down.

Let $B(b_1, \dots, b_n)$ denote the set of cells

$$B = B(b_1, \dots, b_n) = \{(i, j) \mid 1 \le i \le n,\ 1 \le j \le b_i\}.$$

Note that b_i's are allowed to be zero. If a board B can be represented by the set $B(b_1, \dots, b_n)$ for some nonnegative integer b_i's, then the board B is called a *skyline board*. If in addition those b_i's are nondecreasing, that is, $0 \le b_1 \le \cdots \le b_n$, then the board $B = B(b_1, \dots, b_n)$ is called a *Ferrers board*.

Classical rook theory studies the number of ways of choosing k cells in the given board B, denoted by $r_k(B)$, so that no two have a common coordinate, that is, no two cells lie in the same row or in the same column. For this, we say that we place k-nonattacking rooks on the board.

The *i-creation rook placement* can be obtained by the following process: first choose the columns to place rooks. Then as nonattacking rooks are placed in columns, from left to right, to the right of each rook, i new rows are created until the end of the row, strictly above where the rook is placed. Given a Ferrers board B, the *i-rook number*, $r_k^{(i)}(B)$, counts the number of i-creation rook placements of k rooks on B, with $r_0^{(i)}(B) = 1$. Figure 1 shows an $(i = 1)$-creation rook placement in which rooks are denoted by X's.

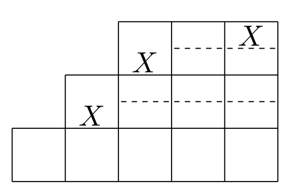

Figure 1. An i-creation rook placement for $i = 1$.

The i-rook number $r_k^{(i)}(B)$ can be generalized by introducing weights to each row of the board. Now we allow to place more than one rook in the respective rows, keeping the condition that each column contains at most one rook. Such placements are called *file placements* and we use $\mathcal{F}_k(B)$ to denote the set of all k-rook file placements in B. Rooks do not create i rows to the right, but we instead weight the rows: if there are u rooks in a given row, then the weight of the row is

$$\begin{cases} 1 & \text{if } \ 0 \le u \le 1, \\ \alpha(2\alpha-1)(3\alpha-2)\cdots((u-1)\alpha-(u-2)), & \text{if } \ u \ge 2. \end{cases}$$

The weight of a placement is defined to be the product of the weights of all the rows and $r_k^{(\alpha)}(B)$ is defined by the sum of the weights of all placements $P \in \mathcal{F}_k(B)$. If $\alpha = 0$, then $r_k^{(0)}(B)$ reduces to the original rook number, and if α is a positive integer i, then $r_k^{(i)}(B)$ is the i-rook number of the i-creation model. The numbers $r_k^{(\alpha)}(B)$ satisfy the following product formula, or α-factorization theorem (see [8]):

$$\prod_{j=1}^{n}(z+b_j+(j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B)\prod_{i=1}^{k}(z+(i-1)(\alpha-1)).$$

We establish an elliptic analogue of the alpha-parameter model by assigning elliptic weights to the cells in the board B. This elliptic analogue contains the q-analogue of $r_k^{(\alpha)}(B)$ that Goldman and Haglund constructed in [8, Section 7] as a limit case.

Let B be a Ferrers board and $P \in \mathcal{F}_k(B)$ a file placement in B. For each cell $c \in B$ in (i,j), we define the weight of c to be

$$wt_\alpha(c) = \begin{cases} 1, & \text{if there is a rook above and in the same column as } c, \\ [(\alpha-1)v(c)+1]_{aq^{2(-j+(\alpha-1)(1-i+r_c(P)))},bq^{-j+(\alpha-1)(1-i+r_c(P))};q,p}, & \text{if } c \text{ contains a rook,} \\ W_{aq^{2(-j+(\alpha-1)(1-i+r_c(P)))},bq^{-j+(\alpha-1)(1-i+r_c(P))};q,p}((\alpha-1)v(c)+1), & \text{otherwise,} \end{cases} \tag{3.1}$$

where $v(c)$ is the number of rooks strictly to the left of, and in the same row as c, and $r_c(P)$ is the number of rooks in the north-west region of c. The weight of the rook placement P is defined to be the product of the weights of all cells:

$$wt_\alpha(P) = \prod_{c\in B} wt_\alpha(c).$$

We define an elliptic analogue of $r_k^{(\alpha)}(B)$ by setting

$$r_k^{(\alpha)}(a,b;q,p;B) = \sum_{P\in\mathcal{F}_k(B)} wt_\alpha(P).$$

This $r_k^{(\alpha)}(a,b;q,p;B)$ satisfies the following recursion which can be proved by considering the cases whether there is a rook or not in the last column of the board.

Proposition 3.1. *Let B be a Ferrers board with l columns of height at most m, and $B\cup m$ denote the board obtained by adding the $(l+1)$-st column of height m to B. Then, for any integer k, we have*

$$r_{k+1}^{(\alpha)}(a,b;q,p;B\cup m) = [m+(\alpha-1)k]_{aq^{-2(m+(\alpha-1)l)},bq^{-m-(\alpha-1)l};q,p}\, r_k^{(\alpha)}(a,b;q,p;B)$$

$$+\,W_{aq^{-2(m+(\alpha-1)l)},bq^{-m-(\alpha-1)l};q,p}(m+(\alpha-1)(k+1))\,r_{k+1}^{(\alpha)}(a,b;q,p;B), \quad (3.2)$$

assuming that

$$r_k^{(\alpha)}(a,b;q,p;B)=0 \qquad \text{for } k<0 \text{ or } k>l, \text{ and}$$
$$r_0^{(\alpha)}(a,b;q,p;B)=1 \qquad \text{for } l=0, \text{ i.e. for } B \text{ being the empty board.}$$

Proof. Let us compute the coefficient of $r_{k+1}^{(\alpha)}(a,b;q,p;B)$ which corresponds to the case when there is no rook in the last column. To see how the computation goes, we first assume that there is only one row in B containing all $k+1$ rooks in the row and let the row coordinate be j. Then the product of the weight of the cells in the last column would be, from the top,

$$\begin{aligned}
&W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(1)\cdot W_{aq^{-2(m-1+(\alpha-1)l)},bq^{-(m-1+(\alpha-1)l)};q,p}(1)\\
&\quad\cdots W_{aq^{-2(j+1+(\alpha-1)l)},bq^{-(j+1+(\alpha-1)l)};q,p}(1)\cdot W_{aq^{-2(j+(\alpha-1)l)},bq^{-(j+(\alpha-1)l)};q,p}((\alpha-1)(k+1)+1)\\
&\cdot W_{aq^{-2(j-1+(\alpha-1)(l-k-1))},bq^{-(j-1+(\alpha-1)(l-k-1))};q,p}(1)\cdot W_{aq^{-2(j-2+(\alpha-1)(l-k-1))},bq^{-(j-2+(\alpha-1)(l-k-1))};q,p}(1)\\
&\qquad\qquad\cdots W_{aq^{-2(1+(\alpha-1)(l-k-1))},bq^{-(1+(\alpha-1)(l-k-1))};q,p}(1).
\end{aligned}$$

By applying the identities (2.4) and (2.5), it is not very hard to see that the above product eventually reduces to

$$W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m+(\alpha-1)(k+1)).$$

Even if there are several rows containing rooks, say the j_ith row contains v_{j_i} rooks, for $1\le i\le k+1$ and $\sum_{i=1}^{k} v_{j_i}=k+1$, by the way we defined the weights of the cells, the product of the weight of the cells in the last column eventually reduces to

$$W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m+(\alpha-1)(k+1)),$$

which is the coefficient of $r_{k+1}^{(\alpha)}(a,b;q,p;B)$ in (3.2).

Now we consider the case when there is a rook in the last column and sum up the weights coming from all the possible rook placements. Let us assume that the rows $j_1,j_2,\ldots,j_k$, from the top, contain v_{j_i} rooks respectively, for $i=1,\ldots,k$, and $\sum_{i=1}^{k} v_{j_i}=k$. If we place the rook in the top cell of the last column, then the top cell containing the rook has the weight $[1]_{aq^{2(-m+(\alpha-1)(-l))},bq^{-m+(\alpha-1)(-l)};q,p}$, which is just 1, and all the cells below, being below a rook, have weight 1. If we place the rook in the second top cell, then the top cell itself has the weight $W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(1)$ and the other cells have weight 1. If we place the rook in the third top row, then the weight of the placement would be

$$\begin{aligned}
&W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(1)\cdot W_{aq^{-2(m-1+(\alpha-1)l)},bq^{-(m-1+(\alpha-1)l)};q,p}(1)\\
&=W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(2),
\end{aligned}$$

by application of (2.5b). Continuing in this way, we can see that the weights grow as we place rooks in lower rows of the last column. If we place the rook in the row j_1, containing v_{j_1} rooks to the left, the weight of the placement is

$$W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m-j_1)[(\alpha-1)v_{j_1}+1]_{aq^{-2(j_1+(\alpha-1)l)},bq^{-(j_1+(\alpha-1)l)};q,p}.$$

This is the result after simplifying the product of the weights coming from the top $m-j_1$ cells using (2.5b). The sum of the rook placements so far is

$$\begin{aligned}
&[m-j_1]_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}+\\
&W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m-j_1)[(\alpha-1)v_{j_1}+1]_{aq^{-2(j_1+(\alpha-1)l)},bq^{-(j_1+(\alpha-1)l)};q,p}\\
&=[m-j_1+(\alpha-1)v_{j_1}+1]_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p},
\end{aligned}$$

by applying (2.8). The placement of the last rook in row $j_1 - 1$ has the weight

$$\underbrace{W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m-j_1)}_{\text{product of weights of top } m-j_1 \text{ cells}} \cdot \underbrace{W_{aq^{-2(j_1+(\alpha-1)l)},bq^{-(j_1+(\alpha-1)l)};q,p}((\alpha-1)v_{j_1}+1)}_{\text{weight of the cell in row } j_1}$$
$$= W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m-j_1+(\alpha-1)v_{j_1}+1),$$

by (2.5), and hence, the sum of the weights of the rook placements so far is

$$[m-j_1+(\alpha-1)v_{j_1}+2]_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}.$$

We can see that placing a rook to the right of v_{j_i} rooks contribute $(\alpha-1)v_{j_i}$ discrepancy and the elliptic number continues to increase as we place the rook in lower rows. We continue in this way. If we place the rook in the bottom most cell in the last column, the weight of the placement is

$$W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m-2) \cdot W_{aq^{-2(2+(\alpha-1)(l-k))},bq^{-(2+(\alpha-1)(l-k))};q,p}(1)$$
$$= W_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p}(m+k(\alpha-1)-1).$$

Adding this weight to the weighted sum of the rook placements so far, which is

$$[m+k(\alpha-1)-1]_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p},$$

gives

$$[m+k(\alpha-1)]_{aq^{-2(m+(\alpha-1)l)},bq^{-(m+(\alpha-1)l)};q,p},$$

which is the coefficient of $r_k^{(\alpha)}(a,b;q,p;B)$. □

We can now prove an elliptic analogue of the α-factorization theorem.

Theorem 3.2. *For any Ferrers board $B = B(b_1, b_2, \ldots, b_n)$, we have*

$$\prod_{j=1}^{n}[z+b_j+(j-1)(\alpha-1)]_{aq^{-2(b_j+(j-1)(\alpha-1))},bq^{-(b_j+(j-1)(\alpha-1))};q,p}$$
$$= \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(a,b;q,p;B) \prod_{i=1}^{k}[z+(i-1)(\alpha-1)]_{aq^{-2(i-1)(\alpha-1)},bq^{-(i-1)(\alpha-1)};q,p}. \tag{3.3}$$

Proof. Let us extend the board by attaching z rows of width n below the board B, denoted by B_z, and compute

$$\sum_{P\in\mathcal{F}_n(B_z)} wt_\alpha(P)$$

in two different ways. The left-hand side of (3.3) is the result of computing the above weight sum columnwise, and the right-hand side can be obtained by computing the weight of the cells in B and the cells in the extended part separately.

If we place rooks columnwise starting from the first column, then by using a similar argument as in Proposition 3.1, it is not hard to see that all the possible rook placements in the j-th column contribute to the weight sum $[z+b_j+(j-1)(\alpha-1)]_{aq^{-2(b_j+(j-1)(\alpha-1))},bq^{-(b_j+(j-1)(\alpha-1))};q,p}$ and the entire weight sum is the product of all those factors.

On the other hand, consider the way that we choose a placement $P \in \mathcal{F}_{n-k}(B)$ and extend it to an n-rook placement in $F_n(B_z)$ by placing k rooks in $B_z - B$. Then the weighted sum of all such rook placements is

$$\sum_{\substack{P'\in\mathcal{F}_n(B_z)\\ P'\cap B=P}} wt_\alpha(P') = \sum_{\substack{P'\in\mathcal{F}_n(B_z)\\ P'\cap B=P}} wt_\alpha(P)\cdot wt_\alpha(P'\cap(B_z-B))$$

$$= wt_\alpha(P)\prod_{i=1}^{k}[z+(i-1)(\alpha-1)]_{aq^{-2(i-1)(\alpha-1)},bq^{-(i-1)(\alpha-1)};q,p},$$

where the last line comes from the weight computation of placing k rooks columnwise. We sum the above weight over all the file placements in $F_{n-k}(B)$:

$$\begin{aligned}&\sum_{P\in\mathcal{F}_{n-k}(B)}\Big(wt_\alpha(P)[z+(i-1)(\alpha-1)]_{aq^{-2(i-1)(\alpha-1)},bq^{-(i-1)(\alpha-1)};q,p}\Big)\\&=\left(\sum_{P\in\mathcal{F}_{n-k}(B)}wt_\alpha(P)\right)[z+(i-1)(\alpha-1)]_{aq^{-2(i-1)(\alpha-1)},bq^{-(i-1)(\alpha-1)};q,p}\\&=r_{n-k}^{(\alpha)}(a,b;q,p;B)[z+(i-1)(\alpha-1)]_{aq^{-2(i-1)(\alpha-1)},bq^{-(i-1)(\alpha-1)};q,p}.\end{aligned}$$

The right-hand side of (3.3) is the result of summing the above weights over all k, for $0\le k\le n$. This completes the proof. □

Remark 3.3. (1) If we take the limit $p\to 0$, $a\to 0$, and then $b\to 0$ (or, $p\to 0$, $b\to 0$, then $a\to\infty$), then $r_k^{(\alpha)}(0,0;q,0;B)$ becomes $R_k^{(\alpha)}(B)$ which is a q-analogue of $r_k^{(\alpha)}(B)$ defined by Goldman and Haglund [8, Section 7]. Thus, Theorem 3.2 gives a q-analogue of the α-factorization theorem:

$$\begin{aligned}&\prod_{j=1}^{n}[z+b_j+(j-1)(\alpha-1)]_q\\&=\sum_{k=0}^{n}R_k^{(\alpha)}(B)[z]_q[z+(\alpha-1)]_q\cdots[z+(n-k-1)(\alpha-1)]_q.\end{aligned}$$

(2) In [18], we obtained the elliptic analogue of the α-factorization theorem (3.3) by a different approach. There we introduced an elliptic analogue of a generalized rook model of Miceli and Remmel [13] utilizing augmented rook boards. The alpha-parameter model can be obtained from it by specializing some parameters. For full details, see [18].

3.1. **The $\alpha=1$ case.** Note that the number of file placements $f_k(B):=|\mathcal{F}_k(B)|$ is called a *file number*. An elliptic analogue of the file number has been defined in [17] by assigning weights $w_{a,b;q,p}(1-j)$ to the cell in (i,j) which are neither below nor contain any rook in $P\in\mathcal{F}_k(B)$. In the definition of $wt_\alpha(c)$ in (3.1), if we set $\alpha=1$, then

$$wt_{\alpha=1}(c)=\begin{cases}1, & \text{if there is a rook above or in } c,\\ W_{aq^{-2j},bq^{-j};q,p}(1), & \text{otherwise.}\end{cases}$$

Since $W_{aq^{-2j},bq^{-j};q,p}(1)=w_{aq^{-2j},bq^{-j};q,p}(1)=w_{a,b;q,p}(1-j)$, $r_k^{(1)}(a,b;q,p;B)$ coincides with the elliptic analogue of the file number $f_k(a,b;q,p;B)$, defined in [17, Section 5]. In fact, $f_k(a,b;q,p;B)$ is defined for any skyline board.

Example 3.4 (Elliptic Stirling number of the first kind)*.* Consider the staircase shape board $St_n=B(0,1,2,\ldots,n-1)$. The file placement of $n-k$ rooks in St_n counts the number of permutations of $\{1,2,\ldots,n\}$ with k cycles, or the signless Stirling numbers of the first kind, denoted by $c(n,k)$. Hence, $r_{n-k}^{(1)}(a,b;q,p;St_n)$ can be defined as an elliptic analogue of

$c(n,k)$. Let us use the notation $c_{a,b;q,p}(n,k) := r^{(1)}_{n-k}(a,b;q,p;St_n)$. The recurrence relation in Proposition 3.1 gives a recurrence relation for $c_{a,b;q,p}(n,k)$:

$$c_{a,b;q,p}(n+1,k) = [n]_{aq^{-2n},bq^{-n}} c_{a,b;q,p}(n,k) + W_{aq^{-2n},bq^{-n}}(n) c_{a,b;q,p}(n,k-1), \tag{3.4}$$

with the initial conditions $c_{a,b;q,p}(0,0) = 1$ and $c_{a,b;q,p}(n,k) = 0$ for $k < 0$ or $k > n$.

Furthermore, if we consider the truncated staircase board $St_n^{(r)} = B(b_1,\dots,b_n)$ with $b_i = 0$ for $i = 1,\dots,r$ and $b_i = i-1$ for $i = r+1,\dots,n$, then $f_{n-k}(St_n^{(r)})$ equals the number of permutations with k cycles such that the first r numbers $1,2,\dots,r$ are in distinct cycles; these are called the *r-restricted Stirling number of the first kind.* One can now define $c^{(r)}_{a,b;q,p}(n,k) := r^{(1)}_{n-k}(a,b;q,p;St_n^{(r)})$ as an elliptic analogue of the r-restricted Stirling numbers of the first kind. They satisfy the same recursion (3.4) but with the initial conditions $c^{(r)}_{a,b;q,p}(r-1,r-1) = 1$ and $c^{(r)}_{a,b;q,p}(n,k) = 0$ for $k < r-1$ or $k > n$. For details of the correspondence between file placements and permutations with certain number of cycles, see [17, Subsections 5.1 and 5.2].

Example 3.5 (Elliptic analogue of Abel polynomials). The polynomials $z(z+\mathfrak{a}n)^{n-1}$ are called *Abel polynomials* and the file numbers $f_{n-k}(A_n)$ for the board $A_n := B(0,\mathfrak{a}n,\dots,\mathfrak{a}n)$ are its coefficients at the monomials z^k, that is,

$$z(z+\mathfrak{a}n)^{n-1} = \sum_{k=0}^{n} f_{n-k}(A_n) z^k.$$

The coefficient $f_{n-k}(A_n)$ counts the number of forests on n labeled vertices composed of k rooted trees where each of the vertices can be colored by one of $\mathfrak{a}$ colors and all the k roots must have the first color. If we apply Theorem 3.2 for $B = A_n$, then the left-hand side of (3.3) becomes an elliptic extension of the Abel polynomial. More precisely, we get

$$[z]_{a,b;q,p}([z+\mathfrak{a}n]_{aq^{-2\mathfrak{a}n},bq^{-2\mathfrak{a}n};q,p})^{n-1} = \sum_{k=0}^{n} r^{(1)}_{n-k}(a,b;q,p;A_n)([z]_{a,b;q,p})^k.$$

In particular, $r^{(1)}_{n-k}(a,b;q,p;A_n)$ has a nice closed form expression

$$r^{(1)}_{n-k}(a,b;q,p;A_n) = \binom{n-1}{k-1} (W_{aq^{-2\mathfrak{a}n},bq^{-\mathfrak{a}n};q,p}(\mathfrak{a}n))^{k-1} ([\mathfrak{a}n]_{aq^{-2\mathfrak{a}n},bq^{-\mathfrak{a}n};q,p})^{n-k}$$

which can easily be proved by the recurrence relation (3.2). For a detailed description of the combinatorial interpretation for $f_{n-k}(A_n)$ and more general cases, see [17, Section 5.3].

3.2. **The $\alpha = 2$ case.** In [8], Goldman and Haglund observe that

$$R^{(2)}_k(St_n) = q^{\binom{n-k}{2}} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q \prod_{j=1}^{k} [2j-1]_q.$$

Unfortunately, the elliptic analogue $r^{(2)}_k(a,b;q,p;St_n)$ does not factor nicely. However, if we take the limit $p \to 0$ and $b \to 0$, then the corresponding a,q-analogue $r^{(2)}_k(a,q;St_n) := r^{(2)}_k(a,0;q,0;St_n)$ has a closed form expression

$$r^{(2)}_k(a,q;St_n) = q^{-\binom{n+k}{2}+k(k+2)} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q \prod_{j=1}^{k} [2j-1]_q \frac{(aq;q^{-2})_{n-k}(aq^{1-2n};q^2)_k}{(aq;q^{-4})_n}.$$

Substituting this expression into the α-factorization theorem gives the identity

$$\prod_{j=1}^{n}[z+2(j-1)]_{aq^{-4(j-1)};q}=\sum_{k=0}^{n} r_k^{(2)}(a,q;St_n)\prod_{i=1}^{n-k}[z+i-1]_{aq^{-2(i-1)};q},$$

where we used the simplified notation $[z]_{a;q} := [z]_{a,0;q,0}$. If we express this identity in basic hypergeometric notation, then it reduces to the following terminating ${}_4\phi_3$ sum (which is equivalent to the terminating q-analogue of Whipple's ${}_3F_2$ sum listed in [6, (II.19)]):

$$\frac{(q^{z+2},aq^{z-2n};q^2)_n}{(q^{z+1},aq^{z-n};q)_n}=\sum_{k=0}^{n}\frac{(q^{-n},q^{n+1},a^{1/2}q^{-n-1/2},-a^{1/2}q^{-n-1/2};q)_k}{(q,-q,q^{-z-n},aq^{z-n};q)_k}q^k.$$

Remark 3.6. In [8], Goldman and Haglund give a bijective proof of

$$r_k^{(2)}(St_n)=m_k(K_{n+k-1}),$$

where $m_k(K_n)$ is the number of k-edge matchings in the complete graph K_n of n vertices. Since $m_k(K_n)=\binom{n}{2k}\frac{(2k)!}{k!2^k}$, the bijection gives

$$r_k^{(2)}(St_n)=\binom{n+k-1}{2k}\frac{(2k)!}{k!2^k}.$$

4. Rook theory for matchings

Haglund and Remmel [11] studied the rook theory for matchings for which they replace permutations by perfect matchings. Rather than $[n]\times[n]$ (the relevant board for considering permutations of n numbers), they consider the following shifted board B_{2n} (not to be confused with B_z, considered earlier) pictured in Figure 2.

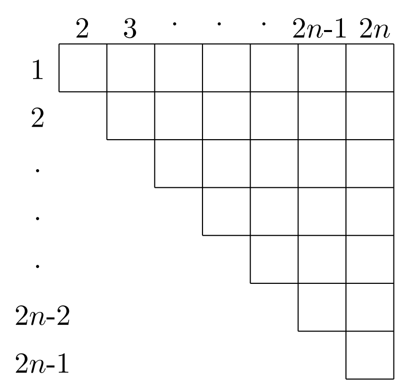

Figure 2. B_{2n}.

Note that any rook placement P in $[n]\times[n]$ is a partial permutation which can be extended to a placement P_σ of n rooks corresponding to some permutation $\sigma\in S_n$, where S_n is the set of permutations of n numbers, $1,2,\dots,n$. For the board B_{2n}, we replace permutations by perfect matchings of the complete graph K_{2n} on vertices $1,2,\dots,2n$. That is, for each perfect matching M of K_{2n} consisting of n pairwise vertex disjoint edges in K_{2n}, we let

$$P_M=\{(i,j)\mid i<j \text{ and } \{i,j\}\in M\}$$

where (i,j) denotes the square in row i and column j of B_{2n} according to the labeling of rows and columns pictured in Figure 2. We now define a rook placement to be a subset of some

P_M for a perfect matching M of K_{2n}. Given a board $B \subseteq B_{2n}$, we let $\mathcal{M}_k(B)$ denote the set of k element rook placements in B. The analogue of a skyline board in this setting is a board $B(a_1, a_2, \ldots, a_{2n-1}) = \{(i, i+j) \mid 1 \le i \le 2n-1, 1 \le j \le a_i\}$. It is called a *shifted Ferrers board* if $2n-1 \ge a_1 \ge a_2 \ge \cdots \ge a_{2n-1} \ge 0$ and the nonzero entries of a_i's are strictly decreasing. A rook in (i,j) with $i < j$ in a rook placement *cancels* all cells (i,s) in B_{2n} with $i < s < j$ and all cells (t,j) and (t,i) with $t < i$. See Figure 3 for a specific example of a Ferrers board and the cells being cancelled by a rook on the shifted board B_8.

Given a shifted Ferrers board $B = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$ and a rook placement $P \in$

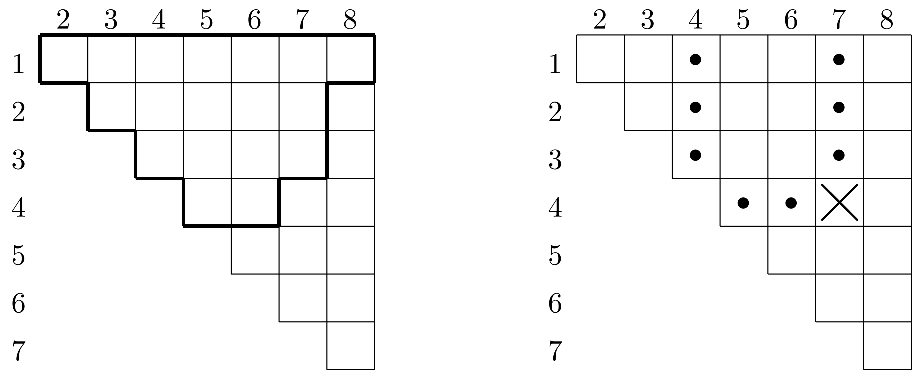

FIGURE 3. The shifted Ferrers board $B = (7, 5, 4, 2, 0, 0, 0) \subseteq B_8$, and the cells cancelled by a rook in $(4, 7)$ on B_8.

$\mathcal{M}_k(B)$, we let $u_B(P)$ denote the number of cells in B which are neither in P nor rook-cancelled by a rook in P. Then Haglund and Remmel proved the following product formula.

Theorem 4.1. [11] *For a shifted Ferrers board $B = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$, define*

$$m_k(q; B) = \sum_{P \in \mathcal{M}_k(B)} q^{u_B(P)}.$$

Then we have

$$\prod_{i=1}^{2n-1} [z + a_{2n-i} - 2i + 2]_q = \sum_{k=0}^{n} m_k(q; B)[z]\downarrow\downarrow_{2n-1-k}$$

where $[z]_q \downarrow\downarrow_k = [z]_q [z-2]_q \cdots [z-2k+2]_q$.

Here, we shall consider a more general case. Let $\mathbf{l} = (l_1, \ldots, l_N)$ be a fixed N-dimensional vector of positive integers. For convenience, define $L_0 = 0$ and $L_j = \sum_{s=1}^{j} l_s$, so that $l_j = L_j - L_{j-1}$, for $1 \le j \le N$. Now we extend B_{2n} to an $\mathbf{l}$-*shifted board* with $L_N = l_1 + \cdots + l_N$ columns and N rows as in Figure 4, denoted by $B_N^{\mathbf{l}}$. Notice that the row labels successively increase by the increments $l_N, l_{N-1}, \ldots, l_1$. For $N = 2n-1$ and $\mathbf{l} = (1, \ldots, 1)$, the $\mathbf{l}$-shifted board $B_N^{\mathbf{l}}$ reduces to the shifted board B_{2n} considered by Haglund and Remmel. A rook placed in $B_N^{\mathbf{l}}$, say $\mathbf{r} \in (i,j)$, $i < j$, attacks the cells in the same row, the same column, and the cells in the i-th column. We can interpret a rook placement in $B_N^{\mathbf{l}}$ in the following way. We call a labeled graph of at most $L_N + 1$ vertices from the set $\{1, 2, \ldots, L_N + 1\}$ *lazy*[1] with

[1]We have chosen the name "lazy" since in contrast to the complete graph we are (lazily) leaving out many edges. Also, compared to the staircase with a unit increment in column height after each step, the lazy graph features a staircase where the unit increments in column height may be delayed.

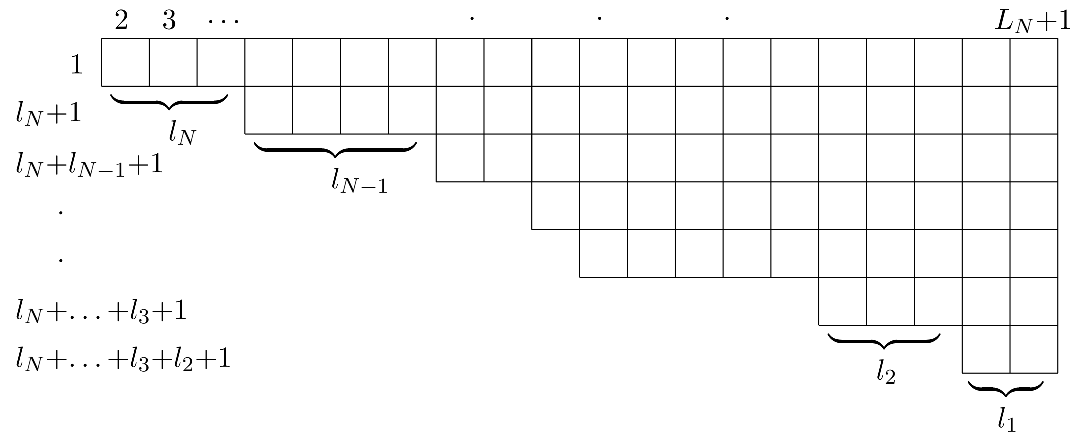

FIGURE 4. $B_N^{\mathbf{l}}$.

respect to $\mathbf{l} = (l_1, \ldots, l_N)$ (or, an $\mathbf{l}$-*lazy graph*, in short) if it only contains edges (i, j) with $i < j$ when i is of the form $l_N + \cdots + l_{N-s+1} + 1 = L_N - L_{N-s} + 1$ for $s \in \{0, 1, \ldots, N-1\}$. Then a k-rook placement on $B_N^{\mathbf{l}}$ is a k-matching of $K_{L_N+1}^{\mathbf{l}}$, the complete $\mathbf{l}$-lazy graph on $L_N + 1$ vertices.

Given a board $B \subseteq B_N^{\mathbf{l}}$, we let $\mathcal{M}_k^{\mathbf{l}}(B)$ denote the set of placements of k nonattacking rooks in B. An $\mathbf{l}$-*shifted skyline board* is the set of cells

$$B(a_1, a_2, \ldots, a_N) = \{(L_N - L_{N-i+1} + 1, L_N - L_{N-i+1} + 1 + j) \mid 1 \le i \le N, 1 \le j \le a_i\}.$$

It is called an $\mathbf{l}$-*shifted Ferrers board* if $L_N \ge a_1 \ge a_2 \ge \cdots \ge a_N \ge 0$ and the nonzero entries of a_i satisfy $a_i - a_{i+1} \ge l_{N+1-i}$ for $1 \le i \le N-1$. As in the ordinary shifted case, a rook in (i, j) with $i < j$ in a rook placement cancels all cells (i, s) in $B_N^{\mathbf{l}}$ with $i < s < j$ and all cells (t, j) and (t, i) in $B_N^{\mathbf{l}}$ with $t < i$. Given an $\mathbf{l}$-shifted Ferrers board $B = B(a_1, \ldots, a_N) \subseteq B_N^{\mathbf{l}}$ and a rook placement $P \in \mathcal{M}_k^{\mathbf{l}}(B)$, let $u_B^{(\mathbf{l})}(P)$ denote the number of cells in B which are neither in P nor cancelled by any rook in P. Define

$$m_k^{(\mathbf{l})}(q; B) = \sum_{P \in \mathcal{M}_k^{\mathbf{l}}(B)} q^{u_B^{(\mathbf{l})}(P)}.$$

Then we can prove the following product formula.

Theorem 4.2. *For any $\mathbf{l}$-shifted Ferrers board $B = B(a_1, \ldots, a_N) \subseteq B_N^{\mathbf{l}}$, we have*

$$\prod_{i=1}^{N} [z + a_{N-i+1} - 2i + 2]_q = \sum_{k=0}^{N} m_k^{(\mathbf{l})}(q; B)[z]\downarrow\downarrow_{N-k} \tag{4.1}$$

where $[z]_q\downarrow\downarrow_k = [z]_q[z-2]_q \cdots [z-2k+2]_q$.

Proof. It suffices to prove the theorem for nonnegative integer values of z. Then the result follows by analytic continuation.

We extend the board $B_N^{\mathbf{l}}$ by attaching z many columns of height N to the right of $B_N^{\mathbf{l}}$, as pictured in Figure 5, and denote it by $B_{N,z}^{\mathbf{l}}$. Now we define the set of cells that a rook in (i, j) *attacks* in the extended board $B_{N,z}^{\mathbf{l}}$. If $(i, j) \in B_N^{\mathbf{l}}$, then it attacks, as we explained above, the cells in the same row, in the same column, and the cells in the i-th column. If $(i, j) \in B_{N,z}^{\mathbf{l}} - B_N^{\mathbf{l}}$, then the cells that a rook in (i, j) attacks in a rook placement P depend on other rooks in $P \cap (B_{N,z}^{\mathbf{l}} - B_N^{\mathbf{l}})$. That is, if the rook in (i, j), say $\mathbf{r}_1$, is the lowest rook in

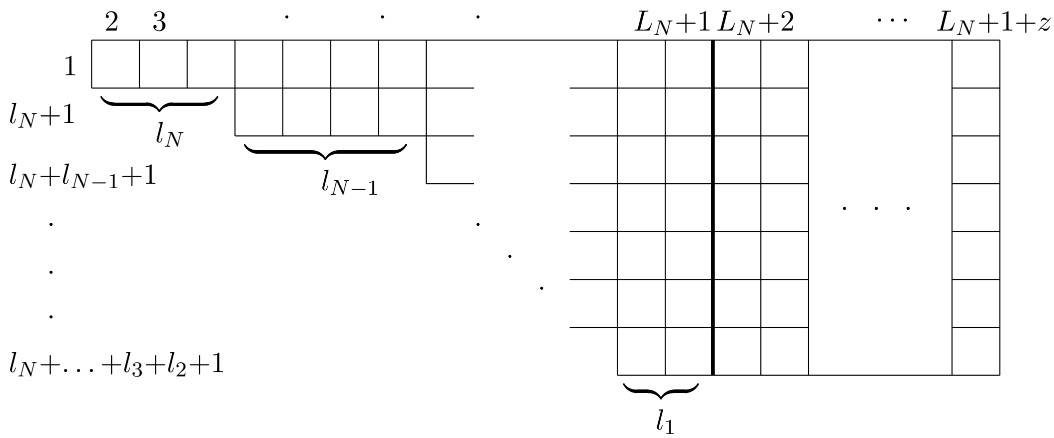

FIGURE 5. $B^{\mathbf{l}}_{N,z}$.

$P\cap(B^{\mathbf{l}}_{N,z}-B^{\mathbf{l}}_N)$, then $\mathbf{r}_1$ attacks all cells in row i and column j other than (i,j) and all cells in column $j-1$ if $L_N+2<j$. If $j=L_N+2$, then $\mathbf{r}_1$ attacks all cells in row i and column j other than (i,j) plus all cells in column L_N+1+z. In general, if the rook in (i,j) is the k-th lowest rook, say $\mathbf{r}_k$, then $\mathbf{r}_k$ attacks all cells in row i and column j other than (i,j) and all cells in the first column of the following list of columns, $j-1, j-2, \ldots, L_N+2, L_N+1+z, L_N+z, \ldots, j+1$, that contain a square which is not attacked by any of the $k-1$ lower rooks in $B^{\mathbf{l}}_{N,z}-B^{\mathbf{l}}_N$. In other words, if $\mathbf{r}$ is in (i,j), then $\mathbf{r}$ attacks all cells in column j and the cells in the first column s in the extended part to the left of column j which has a cell that is not attacked by any lower rooks in $P\cap(B^{\mathbf{l}}_{N,z}-B^{\mathbf{l}}_N)$. If there is no such column to the left of column j, then start scanning from column L_N+1+z and look for the right-most column containing a square which is not attacked by any lower rooks in $P\cap(B^{\mathbf{l}}_{N,z}-B^{\mathbf{l}}_N)$. Note that the existence of such a column s is guaranteed if $z\geq 2N$.

Now let B be an $\mathbf{l}$-shifted Ferrers board contained in $B^{\mathbf{l}}_N$ and assume that $z\geq 2N$. Let $\mathcal{N}_N(B^{\mathbf{l}}_{N,z})$ denote the set of all placements P of N rooks in $B^{\mathbf{l}}_{N,z}$ such that no cell which contains a rook in P is attacked by another rook in P and any rook in $B^{\mathbf{l}}_N\cap P$ is contained in B, namely, rooks are not placed outside of the $\mathbf{l}$-shifted Ferrers board B in $B^{\mathbf{l}}_N$. To prove (4.1), we define a rook cancellation for a rook placement P in $\mathcal{N}_N(B^{\mathbf{l}}_{N,z})$. If a rook $\mathbf{r}$ is in $(i,j)\in B$, then we say $\mathbf{r}$ $\mathcal{N}$-cancels all cells in

$$\{(r,j): r<i\}\cup\{(i,s): i+1\leq s<j\}\cup\{(t,i): t<i\}\\ \cup\{(i,u): u>j \text{ and } (i,u)\notin B\}.$$

Then note that the cells from the first three sets in this union agree with the cells that are cancelled by $\mathbf{r}$ in B relative to the $u^{(\mathbf{l})}_B(P\cap B)$ statistic and the last set in the union contains the cells to the right of $\mathbf{r}$ in $B^{\mathbf{l}}_{N,z}$ which are not in B. If $\mathbf{r}$ is in $(i,j)\in B^{\mathbf{l}}_{N,z}-B^{\mathbf{l}}_N$, then let $\mathcal{A}^N_{(i,j)}$ denote the set of cells attacked by $\mathbf{r}$. The rook $\mathbf{r}$ then $\mathcal{N}$-cancels all cells in $\mathcal{A}^N_{(i,j)}$ that lie in rows s with $s<i$ plus all cells in row i that are either in $B^{\mathbf{l}}_N-B$ or to the right of (i,j). Now we let $u^{(\mathbf{l})}_{\mathcal{N}}(P)$ denote the number of squares in $B^{\mathbf{l}}_{N,z}-P$ which are not $\mathcal{N}$-cancelled by

any rooks in P. Then (4.1) is the result of computing the sum

$$\sum_{P \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})} q^{u^{(1)}_{\mathcal{N}}(P)}$$

in two different ways. First, we could place N rooks row by row. Then starting from the right-most cell in the bottom row of B, we move the rook to the left, and then again start from column $L_N + 2$ (which is the left-most cell of the extended part) and move the rook to the right. For a graphical explanation, see Figure 6. In the figure, the board B is outlined by thick lines and the boundary of the extended board is denoted by double lines. All the

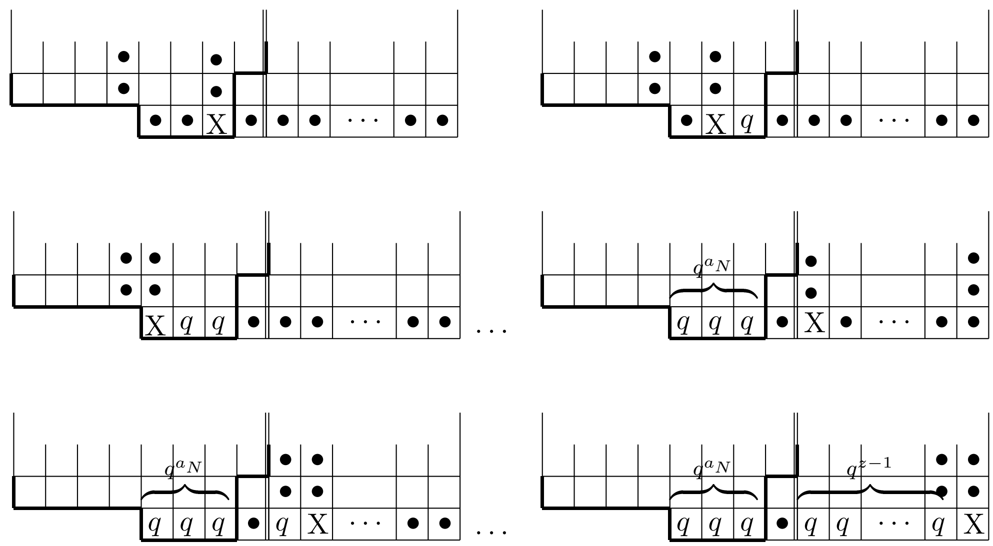

FIGURE 6. Possible rook placements in the bottom first row.

possible rook placements in the bottom row contribute $1 + q + \cdots + q^{z+a_N-1} = [z + a_N]_q$. Since this rook $\mathcal{N}$-cancels exactly two cells in the above row, the possible rook placements in the second row from the bottom contribute $[z + a_{N-1} - 2]_q$. Continuing this way, we get

$$\sum_{P \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})} q^{u^{(1)}_{\mathcal{N}}(P)} = \prod_{i=1}^{N} [z + a_{N-i+1} - 2i + 2]_q.$$

On the other hand, we could fix a placement $P \in \mathcal{M}^{\mathbf{l}}_k(B)$ and consider the sum

$$\sum_{\substack{P' \in \mathcal{N}_N(B^{l}_{N,z}) \\ P' \cap B = P}} q^{u^{(1)}_{\mathcal{N}}(P')}.$$

The way how the $\mathcal{N}$-cancellation is defined ensures that for any $P' \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})$ such that $P' \cap B = P$, the number of squares of $B^{\mathbf{l}}_N - P$ which are not $\mathcal{N}$-cancelled by some rook in P' is consistent with $u^{(1)}_B(P)$. The same type of argument that we used in the first case applies for the possible placements of $N - k$ rooks in $B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N$ which gives $[z]_q[z-2]_q \cdots [z - 2(N -$

$k)+2]_q = [z]_q \downarrow\downarrow_{N-k}$. Thus we have

$$\begin{aligned}\sum_{P\in\mathcal{N}_N(B^l_{N,z})} q^{u^{(\mathbf{l})}_N(P)} &= \sum_{k=0}^{N} \sum_{P\in\mathcal{M}^{\mathbf{l}}_k(B)} q^{u^{(\mathbf{l})}_B(P)}[z]_q \downarrow\downarrow_{N-k} \\ &= \sum_{k=0}^{N} m^{(\mathbf{l})}_k(q;B)[z]_q \downarrow\downarrow_{N-k},\end{aligned}$$

as desired. □

Remark 4.3. Note that when $\mathbf{l} = (1,\dots,1)$ and $N = 2n-1$, we recover Theorem 4.1 of Haglund and Remmel. It is clear from the definition of attacking rooks that in this case there can be at most n rooks on the board, so on the right-hand side of (4.1) the index k can only vary between 0 and n. Hence, the placements of k nonattacking rooks on B_{2n} correspond to k-matchings of the complete graph K_{2n} and n-matchings are perfect matchings. In the general case, the placements of k nonattacking rooks on $B^{\mathbf{l}}_N$ correspond to k-matchings of $K^{\mathbf{l}}_{L_N+1}$, while for $L_N+1 \geq 2N$ any such placements of N nonattacking rooks correspond to *maximal matchings.*

The $z=0$ case of Theorem 4.2 immediately gives the following product formula for $m^{(\mathbf{l})}_k(q;B)$ when $k=N$.

Corollary 4.4. *Given an* $\mathbf{l}$*-shifted Ferrers board* $B = B(a_1,\dots,a_N) \subseteq B^{\mathbf{l}}_N$*, we have*

$$m^{(\mathbf{l})}_N(q;B) = \prod_{i=1}^{N}[a_{N-i+1} - 2i + 2]_q.$$

In particular, if $B = B^{\mathbf{l}}_N$ *(the full* $\mathbf{l}$*-shifted board), then*

$$m^{(\mathbf{l})}_N(q;B^{\mathbf{l}}_N) = \prod_{i=1}^{N}[L_i - 2i + 2]_q.$$

Now we work out an elliptic analogue of Theorem 4.2. We essentially assume the same rook cancellation. However, for the purpose of conveniently computing the elliptic weights of cells, we label the rows and columns of $B^{\mathbf{l}}_N$ as in Figure 7, namely, we label the columns from 1 to L_N, from right to left, and label the rows from 1 to N from the bottom. When we use the labeling in Figure 7, then we denote the board by ${}^wB^{\mathbf{l}}_N$ and use $(i,j)^w$ to denote a cell in row i and column j with respect to this labeling.

Given an $\mathbf{l}$-shifted Ferrers board $B = B(a_1,\dots,a_N) \subseteq B^{\mathbf{l}}_N$ and a rook placement $P \in \mathcal{M}^{(\mathbf{l})}_k(B)$, let $U^{\mathbf{l}}_B(P)$ denote the set of cells in $B-P$ which are not cancelled by any rook of P. Define

$$wt_m(P) = \prod_{(i,j)^w\in U^{\mathbf{l}}_B(P)} w_{a,b;q,p}\left(i+j-1-l_i-2r_{(i,j)}(P)-s_{(i,j)}(P)\right), \tag{4.2}$$

where the elliptic weight $w_{a,b;q,p}(l)$ of an integer l is defined in (2.3a), $r_{(i,j)}(P)$ is the number of rooks in P positioned south-east of $(i,j)^w$ such that the two columns cancelled by those rooks are to the right of the column j, and $s_{(i,j)}(P)$ is the number of rooks in P which are in the south-east region of $(i,j)^w$ such that only one cancelled column (the column containing

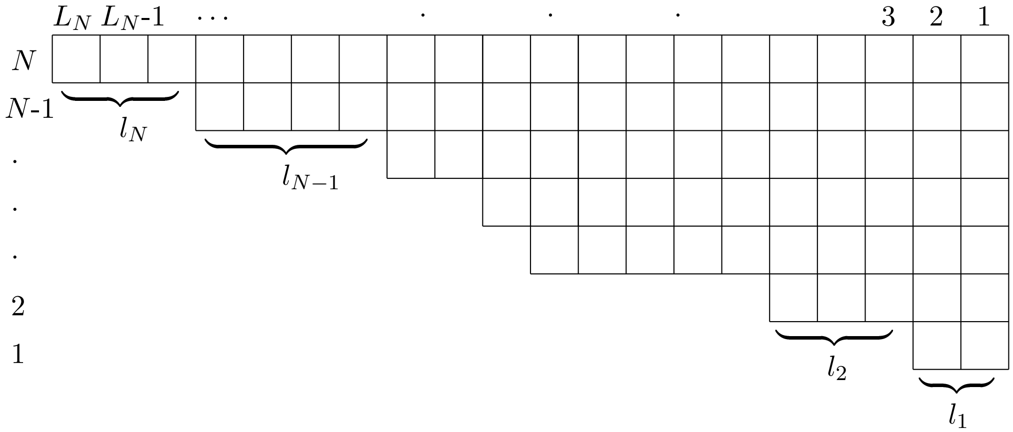

FIGURE 7. Labeling for elliptic weights in B^{l}_{N}.

the rook) is to the right of column j. Then we define

$$m_k^{(\mathbf{1})}(a,b;q,p;B)=\sum_{P\in\mathcal{M}_k^{\mathbf{l}}(P)} wt_m(P).$$

Theorem 4.5. *For any* $\mathbf{l}$*-shifted Ferrers board* $B=B(a_1,\dots,a_N)\subseteq B^{\mathbf{l}}_N$*, we have*

$$\begin{aligned}\prod_{i=1}^{N}&[z+a_{N-i+1}-2i+2]_{aq^{2(L_{i-1}+i-1-a_{N-i+1})},bq^{L_{i-1}+i-1-a_{N-i+1}};q,p}\\&=\sum_{k=0}^{N} m_k^{(\mathbf{1})}(a,b;q,p;B)\prod_{j=1}^{N-k}[z-2j+2]_{aq^{2(L_{j-1}+j-1)},bq^{L_{j-1}+j-1};q,p}.\end{aligned}\tag{4.3}$$

Proof. It suffices to prove the theorem for nonnegative integer values of z. The full result follows then by analytic continuation.

The proof is similar to the proof of Theorem 4.2. Here we also consider the extended board $B^{\mathbf{l}}_{N,z}$. However, we use a different labeling for the sake of elliptic weight computation. For the cells in $B^{\mathbf{l}}_N$, we use the labeling described in Figure 7, and for the extended part, we use the labeling described in Figure 8. We consider the rook placements in $\mathcal{N}_N(B^{\mathbf{l}}_{N,z})$.

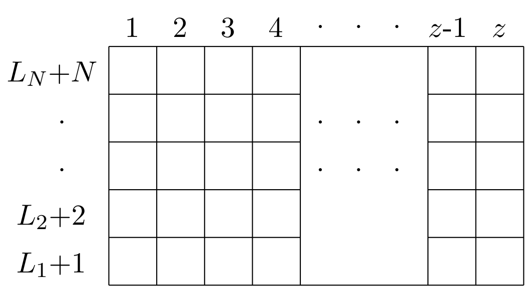

FIGURE 8. Labeling of the extended part in $B^{\mathbf{l}}_{N,z}$ for the elliptic weights.

We assume the same $\mathcal{N}$-cancellation as defined in the proof of Theorem 4.2. For a rook placement $P \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})$, let $U^{\mathbf{l}}_{\mathcal{N}}(P)$ denote the set of cells in $B^{\mathbf{l}}_{N,z} - P$ which are not $\mathcal{N}$-cancelled by any rooks in P. Consider the cells in $U^{\mathbf{l}}_{\mathcal{N}}(P)$. Due to the inconsistency of the column labeling in $B^{\mathbf{l}}_N$ and the extended part of $B^{\mathbf{l}}_{N,z}$, we give slightly different weights to the cells in $B \cap U^{\mathbf{l}}_{\mathcal{N}}(P)$ and $(B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N) \cap U^{\mathbf{l}}_{\mathcal{N}}(P)$. That is, to the cells in $B \cap U^{\mathbf{l}}_{\mathcal{N}}(P)$, we assign the elliptic weight as defined in (4.2). To the cells in $(B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N) \cap U^{\mathbf{l}}_{\mathcal{N}}(P)$, say to $(i,j)^w \in (B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N) \cap U^{\mathbf{l}}_{\mathcal{N}}(P)$ in terms of the weight labeling described in Figure 8, we assign the weight $w_{a,b;q,p}(i+j-1-l_i-2\tilde{r}_{(i,j)}(P)-\tilde{s}_{(i,j)}(P))$, where $\tilde{r}_{(i,j)}(P)$ is the number of rooks in $(B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N) \cap P$ which are in the south-west region of $(i,j)^w$ and both of the columns cancelled by those rooks are to the left of column j, and $\tilde{s}_{(i,j)}(P)$ is the number of rooks in $(B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N) \cap P$ which are in the south-west region of $(i,j)^w$ with only one column cancelled by the respective rooks being to the left of column j (and so the other cancelled column is to the right of column j). Then the product formula (4.3) is the result of computing

$$\sum_{P \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})} \widetilde{wt}_m(P),$$

where

$$\begin{aligned}\widetilde{wt}_m(P) = &\prod_{(i,j)^w \in B \cap U^{\mathbf{l}}_{\mathcal{N}}(P)} w_{a,b;q,p}(i+j-1-l_i-2r_{(i,j)}(P)-s_{(i,j)}(P))\\ &\times \prod_{(i,j)^w \in (B^{\mathbf{l}}_{N,z}-B^{\mathbf{l}}_N)\cap U^{\mathbf{l}}_{\mathcal{N}}(P)} w_{a,b;q,p}(i+j-1-l_i-2\tilde{r}_{(i,j)}(P)-\tilde{s}_{(i,j)}(P))\end{aligned}$$

in two different ways. We first place N rooks row by row, starting from the bottom row. Starting from the right-most cell in the bottom row of B we move a rook to the left, and then start from the left-most cell of the extended part and move to the right. Again we refer to Figure 6 for a graphical example. From these possible rook placements, we get the sum of weights

$$\begin{aligned}&1 + w_{a,b;q,p}(1-a_N) + w_{a,b;q,p}(1-a_N)w_{a,b;q,p}(2-a_N) + \cdots + \prod_{i=1+l_1-a_N}^{l_1+z-1} w_{a,b;q,p}(i-l_1)\\ &= [z+a_N]_{aq^{-2a_N},bq^{-a_N};q,p}.\end{aligned}$$

We remark that the labeling of the extended part in $B^{\mathbf{l}}_{N,z}$ was defined so that the above sum continues to add up to an elliptic number. Note that if the first rook $\mathbf{r}_1$ is placed in a cell in $B^{\mathbf{l}}_N$, then it $\mathcal{N}$-cancels exactly two cells in $B^{\mathbf{l}}_N$ in every row above the bottom row. If $\mathbf{r}_1$ is placed in $B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N$, then it $\mathcal{N}$-cancels exactly two cells in every row in $B^{\mathbf{l}}_{N,z} - B^{\mathbf{l}}_N$ above the bottom row. Hence, after placing a rook in the bottom row, the weight sum over all possible rook placements in the second row from the bottom is of size $z + a_{N-1} - 2$. However, as it happens in the bottom case, we have to shift a and b up to the coordinate of the first uncancelled cell which depends on the difference between the length of the row i and the size of a_{N-i+1}, in general. Also, moving one row up shifts a by q^2 and b by q. Thus, the weight sum coming from all possible rook placements in the i-th row from the bottom becomes $[z+a_{N-i+1}-2i+2]_{aq^{2(L_{i-1}+i-1-a_{N-i+1})},bq^{L_{i-1}+i-1-a_{N-i+1}};q,p}$, and so finally we obtain

$$\sum_{P \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})} \widetilde{wt}_m(P) = \prod_{i=1}^{N} [z+a_{N-i+1}-2i+2]_{aq^{2(L_{i-1}+i-1-a_{N-i+1})},bq^{L_{i-1}+i-1-a_{N-i+1}};q,p}.$$

On the other hand, we can fix a placement $P \in \mathcal{M}_k^{\mathbf{l}}(B)$ and consider the sum

$$\sum_{\substack{P' \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z}) \\ P' \cap B = P}} \widetilde{wt}_m(P').$$

Then by the same reasoning used in the proof of Theorem 4.2, we obtain

$$\begin{aligned}\sum_{P \in \mathcal{N}_N(B^{\mathbf{l}}_{N,z})} \widetilde{wt}_m(P) &= \sum_{k=0}^{N} \sum_{P \in \mathcal{M}_k^{\mathbf{l}}(B)} wt(P) \prod_{j=1}^{N-k} [z-2j+2]_{aq^{2(L_{j-1}+j-1)},bq^{L_{j-1}+j-1};q,p} \\ &= \sum_{k=0}^{n} m_k^{(\mathbf{l})}(a,b;q,p;B) \prod_{j=1}^{N-k} [z-2j+2]_{aq^{2(L_{j-1}+j-1)},bq^{L_{j-1}+j-1};q,p},\end{aligned}$$

as desired. □

The first corollary is a consequence of specializing the value $z=0$ in Theorem 4.5.

Corollary 4.6. *Given an* $\mathbf{l}$*-shifted Ferrers board* $B = B(a_1,\ldots,a_N) \subseteq B^{\mathbf{l}}_N$*, we have*

$$m_N^{(\mathbf{l})}(a,b;q,p;B) = \prod_{i=1}^{N} [a_{N-i+1}-2i+2]_{aq^{2(L_{i-1}+i-1-a_{N-i+1})},bq^{L_{i-1}+i-1-a_{N-i+1}};q,p}.$$

In particular, if $B = B^{\mathbf{l}}_N$*, then we have*

$$m_N^{(\mathbf{l})}(a,b;q,p;B^{\mathbf{l}}_N) = \prod_{i=1}^{N} [L_i-2i+2]_{aq^{2(i-1-l_i)},bq^{i-1-l_i};q,p}.$$

The next corollary concerns the case $\mathbf{l} = (1,\ldots,1)$ and $N = 2n-1$ of Theorem 4.5 which gives an elliptic analogue of Theorem 4.1. For the case $\mathbf{l} = (1,\ldots,1)$, we use $m_k(a,b;q,p;B)$ to denote $m_k^{(1,1,\ldots,1)}(a,b;q,p;B)$.

Corollary 4.7. *Given a shifted Ferrers board* $B = B(a_1,\ldots,a_{2n-1}) \subseteq B_{2n}$*, we have*

$$\begin{aligned}&\prod_{i=1}^{2n-1} [z+a_{2n-i}-2i+2]_{aq^{2(2i-2-a_{2n-i})},bq^{2i-2-a_{2n-i}};q,p} \\ &= \sum_{k=0}^{n} m_k(a,b;q,p;B) \prod_{j=1}^{2n-1-k} [z-2j+2]_{aq^{4j-4},bq^{2j-2};q,p}.\end{aligned} \tag{4.4}$$

The following result concerns the elliptic enumeration of (perfect) matchings on K_{2n}.

Corollary 4.8. *Given a shifted Ferrers board* $B = B(a_1,\ldots,a_{2n-1}) \subseteq B_{2n}$*, we have*

$$m_n(a,b;q,p;B) = \frac{\prod_{i=1}^{2n-1} [a_{2n-i}+2n-2i]_{aq^{2(2i-2-a_{2n-i})},bq^{2i-2-a_{2n-i}};q,p}}{\prod_{i=1}^{n-1} [2n-2i]_{aq^{4i-4},bq^{2i-2};q,p}}.$$

In particular, for the full shifted Ferrers board $B_{2n} = B(2n-1,2n-2,\ldots,1)$ *we have*

$$\begin{aligned}m_n(a,b;q,p;B_{2n}) &= \frac{\prod_{i=1}^{2n-1} [2n-i]_{aq^{2i-4},bq^{i-2};q,p}}{\prod_{i=1}^{n-1} [2n-2i]_{aq^{4i-4},bq^{2i-2};q,p}} \\ &= [2n-1]_{aq^{-2},bq^{-1};q,p}[2n-3]_{aq^2,bq;q,p}\ldots[1]_{aq^{4n-6},bq^{2n-3};q,p}.\end{aligned}$$

Proof. In Corollary 4.7 we let $z \to 2n-2$. Since

$$\prod_{j=1}^{2n-1-k} [2n-2j]_{aq^{4j-4},bq^{2j-2};q,p} = 0,$$

for $1 \le k < n$, the right-hand side of (4.4) reduces to a single term only, corresponding to $k = n$. Simplification then yields the result. $\square$

Similarly, the case when $\mathbf{l} = (1, \dots, 1)$ and $N = 2n$ of Theorem 4.5 gives the following result.

Corollary 4.9. *Given a shifted Ferrers board $B = B(a_1, \dots, a_{2n}) \subseteq B_{2n+1} = B(2n, 2n-1, \dots, 1)$, we have*

$$\begin{aligned}&\prod_{i=1}^{2n}[z + a_{N-i+1} - 2i + 2]_{aq^{2(2i-2-a_{N-i+1})},bq^{2i-2-a_{N-i+1}};q,p}\\ &= \sum_{k=0}^{2n} m_k(a,b;q,p;B) \prod_{j=1}^{2n-k} [z-2j+2]_{aq^{4j-4},bq^{2j-2};q,p}.\end{aligned} \tag{4.5}$$

As a special case, we obtain an explicit expression for the elliptic enumeration of (maximal) matchings on K_{2n+1}.

Corollary 4.10. *Given a shifted Ferrers board $B = B(a_1, \dots, a_{2n}) \subseteq B_{2n+1}$, we have*

$$m_n(a,b;q,p;B) = \frac{\prod_{i=1}^{2n}[a_{2n-i+1} + 2n - 2i + 2]_{aq^{2(2i-2-a_{2n-i+1})},bq^{2i-2-a_{2n-i+1}};q,p}}{\prod_{i=1}^{n}[2n-2i+2]_{aq^{4i-4},bq^{2i-2};q,p}}.$$

In particular, for the full shifted Ferrers board $B = B_{2n+1} = B(2n, 2n-1, \dots, 1)$ we have

$$\begin{aligned}m_n(a,b;q,p;B_{2n+1}) &= \frac{\prod_{i=1}^{2n}[2n-i+2]_{aq^{2i-4},bq^{i-2};q,p}}{\prod_{i=1}^{n}[2n-2i+2]_{aq^{4i-4},bq^{2i-2};q,p}}\\ &= [2n+1]_{aq^{-2},bq^{-1};q,p}[2n-1]_{aq^{2},bq;q,p}\dots[3]_{aq^{4n-6},bq^{2n-3};q,p}.\end{aligned}$$

Proof. In Corollary 4.9 we let $z \to 2n$. As in the proof of Corollary 4.8 the right-hand side of (4.5) then reduces to a single term only, corresponding to $k = n$. Simplification then yields the result. $\square$

When the board is the full $\mathbf{l}$-shifted Ferrers board $B = B_N^{\mathbf{l}}$, the elliptic matchings number $m_k^{(\mathbf{l})}(a,b;q,p;B_N^{\mathbf{l}})$ satisfies the following recursion which can be proved by considering whether there is a rook or not in the top row.

Proposition 4.11.

$$\begin{aligned}m_k^{(\mathbf{l})}(a,b;q,p;B_N^{\mathbf{l}}) &= [L_N - 2k + 2]_{aq^{2(N-1-l_N)},bq^{N-1-l_N};q,p} m_{k-1}^{(\mathbf{l})}(a,b;q,p;B_{N-1}^{\mathbf{l}'})\\ &\quad + W_{aq^{2(N-1-l_N)},bq^{N-1-l_N};q,p}(L_N - 2k) m_k^{(\mathbf{l})}(a,b;q,p;B_{N-1}^{\mathbf{l}'}),\end{aligned}$$

where $\mathbf{l}' = (l_1, \dots, l_{N-1})$ and $B_{N-1}^{\mathbf{l}'}$ is the board obtained by removing the top row from $B_N^{\mathbf{l}}$.

Remark 4.12. In the limit case $p \to 0$ and $b \to 0$, for the shifted Ferrers board $B_{2n} = B(2n-1, 2n-2, \dots, 2, 1)$, $m_k(a,q;B_{2n}) := m_k(a,0;q,0;B_{2n})$ has a closed form

$$m_k(a,q;B_{2n}) = q^{k^2-\binom{2n}{2}} \begin{bmatrix}2n\\2k\end{bmatrix}_q \prod_{j=1}^{k}[2j-1]_q \frac{(aq^{4n-2k-3};q^2)_{2n-k-1}}{(aq^{-1};q^4)_{2n-k-1}}$$

which can be proved by the recursion in Proposition 4.11.

If we let $a \to \infty$ in $m_k(a, q; B_{2n})$, then we obtain

$$m_k(q; B_{2n}) = q^{\binom{2n-2k}{2}} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \prod_{j=1}^{k} [2j-1]_q,$$

which is a q-analogue of the k-matching number $\binom{2n}{2k}(2k-1)!!$ of the complete graph K_{2n}.

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