MULTILATERAL INVERSION OF A_r, C_r AND D_r BASIC HYPERGEOMETRIC SERIES

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Abstract. In [Electron. J. Combin. **10** (2003), #R10], the author presented a new basic hypergeometric matrix inverse with applications to bilateral basic hypergeometric series. This matrix inversion result was directly extracted from an instance of Bailey's very-well-poised ${}_6\psi_6$ summation theorem, and involves two infinite matrices which are not lower-triangular. The present paper features three different multivariable generalizations of the above result. These are extracted from Gustafson's A_r and C_r extensions and from the author's recent A_r extension of Bailey's ${}_6\psi_6$ summation formula. By combining these new multidimensional matrix inverses with A_r and D_r extensions of Jackson's ${}_8\phi_7$ summation theorem three balanced very-well-poised ${}_8\psi_8$ summation theorems associated to the root systems A_r and C_r are derived.

1. Introduction

In [23, Th. 3.1], we presented the following matrix inverse:

Let $|q| < 1$, and a, b and c be indeterminates. The infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are *inverses* of each other where

$$f_{nk} = \frac{(aq/b, bq/a, aq/c, cq/a, bq, q/b, cq, q/c)_\infty}{(q, q, aq, q/a, aq/bc, bcq/a, cq/b, bq/c)_\infty} \times \frac{(1-bcq^{2n}/a)}{(1-bc/a)} \frac{(b)_{n+k}\,(a/c)_{k-n}}{(cq)_{n+k}\,(aq/b)_{k-n}} \tag{1.1a}$$

and

$$g_{kl} = \frac{(1-aq^{2k})}{(1-a)} \frac{(c)_{k+l}\,(a/b)_{k-l}}{(bq)_{k+l}\,(aq/c)_{k-l}}\, q^{k-l}. \tag{1.1b}$$

(The notation is explained in Section 2.)

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This result was directly extracted from an instance of Bailey's [3, Eq. (4.7)] very-well-poised ${}_6\psi_6$ summation formula,

$$ {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{a^2q}{bcde}\right] = \frac{(q, aq, q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2q/bcde)_\infty}, \quad (1.2) $$

where $|a^2q/bcde| < 1$ (cf. [10, Eq. (5.3.1)]).

If we let $c \to a$ in (1.1), we obtain a matrix inverse found by Bressoud [7] which he directly extracted from the terminating very-well-poised ${}_6\phi_5$ summation (a special case of (1.2)). If, after letting $c \to a$, we additionally let $a \to 0$, we obtain Andrews' [1, Lemma 3] "Bailey transform matrices", a matrix inversion underlying the powerful Bailey lemma. While Bressoud's matrix inverse underlies Andrews' WP-Bailey lemma [2] (WP stands for "well-poised") which generalizes the classical Bailey lemma, the "bilateral" matrix inverse (1.1) underlies the BWP-Bailey lemma, a bilateral generalization of the WP-Bailey lemma, see [26].

In [23], several applications of (1.1) to bilateral basic hypergeometric series were given. One of them included a new very-well-poised ${}_8\psi_8$ summation formula, see Proposition 2.1 in this paper.

Here we apply part of the analysis of [23] to multiple sums. In fact, by appropriately specializing Gustafson's A_r and C_r ${}_6\psi_6$ summations [11, 12], and an A_r ${}_6\psi_6$ summation by the author [25], we derive three multidimensional extensions of the bilateral matrix inverse (1.1) and deduce three multidimensional ${}_8\psi_8$ summations, associated with the respective root systems of types A_r and C_r, as applications. These are obtained via *multidimensional inverse relations*, applied to A_r and D_r extensions of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summations, taken from [17, 22, 25].

Our paper is organized as follows. In Section 2, we review some preliminaries on basic hypergeometric series. In the same section, we also explain some facts we need on multidimensional basic hypergeometric series associated with root systems. We list several multi-sum identities explicitly there for easy reference. Section 3 is devoted to multidimensional matrix inversions. In particular, we give three new explicit multilateral matrix inverses, which are directly extracted from corresponding multivariate ${}_6\psi_6$ summation formulae. Our applications, see Section 4, include three balanced very-well-poised ${}_8\psi_8$ summation formulae, two of them associated with the root system A_r, the third with the root system C_r. These new multivariate ${}_8\psi_8$ summations comprise, via specialization and analytic continuation, corresponding multivariate ${}_8\phi_7$ and ${}_6\psi_6$ summation formulae. Finally, we show in the Appendix how an incorrect application of multidimensional inverse relations leads to a false result, namely a divergent D_r very-well-poised

${}_6\psi_6$ summation (which however remains true for $r = 1$, or whenever the series terminates).

2. Preliminaries

2.1. Basic hypergeometric series. Here we recall some standard notation for basic hypergeometric series (cf. [10]).

Let q be a complex number such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a)_\infty := \prod_{j=0}^{\infty}(1 - aq^j) \qquad \text{and} \qquad (a)_k := \frac{(a)_\infty}{(aq^k)_\infty}.$$

For brevity, we employ the condensed notation

$$(a_1, \ldots, a_m)_k := (a_1)_k \ldots (a_m)_k$$

where k is an integer or infinity. Further, we utilize

$${}_s\phi_{s-1}\left[\begin{matrix}a_1, a_2, \ldots, a_s\\ b_1, b_2, \ldots, b_{s-1}\end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_s)_k}{(q, b_1, \ldots, b_{s-1})_k} z^k, \tag{2.1}$$

and

$${}_s\psi_s\left[\begin{matrix}a_1, a_2, \ldots, a_s\\ b_1, b_2, \ldots, b_s\end{matrix}; q, z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_s)_k}{(b_1, b_2, \ldots, b_s)_k} z^k, \tag{2.2}$$

to denote the *basic hypergeometric ${}_s\phi_{s-1}$ series*, and the *bilateral basic hypergeometric ${}_s\psi_s$ series*, respectively. In (2.1) or (2.2), $a_1, \ldots, a_s$ are called the *upper parameters*, $b_1, \ldots, b_s$ the *lower parameters*, z is the *argument*, and q the *base* of the series. See [10, p. 5 and p. 137] for the criteria of when these series terminate, or, if not, when they converge.

The classical theory of basic hypergeometric series contains numerous summation and transformation formulae involving ${}_s\phi_{s-1}$ or ${}_s\psi_s$ series. Many of these summation theorems require that the parameters satisfy the condition of being either balanced and/or very-well-poised. An ${}_s\phi_{s-1}$ basic hypergeometric series is called *balanced* if $b_1 \cdots b_{s-1} = a_1 \cdots a_s q$ and $z = q$. An ${}_s\phi_{s-1}$ series is *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_s b_{s-1}$. An ${}_s\phi_{s-1}$ basic hypergeometric series is called *very-well-poised* if it is well-poised and if $a_2 = -a_3 = q\sqrt{a_1}$. Note that the factor

$$\frac{1 - a_1 q^{2k}}{1 - a_1} \tag{2.3}$$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series. Similarly, a bilateral ${}_s\psi_s$ basic hypergeometric series is well-poised if $a_1 b_1 = a_2 b_2 \cdots = a_s b_s$ and very-well-poised if, in addition, $a_1 = -a_2 = q b_1 = -q b_2$. Further, we call a bilateral ${}_s\psi_s$ basic hypergeometric series balanced if $b_1 \cdots b_s = a_1 \cdots a_s q^2$ and $z = q$.

A standard reference for basic hypergeometric series is Gasper and Rahman's text [10]. In our computations in the subsequent sections we frequently use some elementary identities of q-shifted factorials, listed in [10, Appendix I].

One of the most important theorems in the theory of basic hypergeometric series is F. H. Jackson's [14] terminating balanced very-well-poised ${}_8\phi_7$ summation (cf. [10, Eq. (2.6.2)]):

$$
{}_8\phi_7\left[\begin{matrix} a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix}; q, q\right]
= \frac{(aq, aq/bc, aq/bd, aq/cd)_n}{(aq/b, aq/c, aq/d, aq/bcd)_n}. \qquad (2.4)
$$

A combinatorial proof of the *elliptic* extension of (2.4), namely of Frenkel and Turaev's [9] ${}_{10}V_9$ summation, which degenerates to a combinatorial proof of (2.4) in the "trigonometric" special case, has recently been given in [24].

In [23, Thm. 4.1], Jackson's summation (2.4) was utilized, together with the bilateral matrix inverse (1.1), to derive the following balanced very-well-poised ${}_8\psi_8$ summation formula:

Proposition 2.1. *Let a, b, c and d be indeterminates, let k be an arbitrary integer and M a nonnegative integer. Then*

$$
{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, dq^k, aq^{-k}/c, aq^{1+M}/b, aq^{-M}/d \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1-k}/d, cq^{1+k}, bq^{-M}, dq^{1+M} \end{matrix}; q, q\right]
$$
$$
= \frac{(aq/bc, cq/b, dq, dq/a)_M}{(cdq/a, dq/c, q/b, aq/b)_M} \frac{(cd/a, bd/a, cq, cq/a, dq^{1+M}/b, q^{-M})_k}{(q, cq/b, d/a, d, bcq^{-M}/a, cdq^{1+M}/a)_k}
$$
$$
\times \frac{(q, q, aq, q/a, cdq/a, aq/cd, cq/d, dq/c)_\infty}{(cq, q/c, dq, q/d, cq/a, aq/c, dq/a, aq/d)_\infty}. \qquad (2.5)
$$

Note that two of the upper parameters of the ${}_8\psi_8$ series in (2.5) (namely b and aq^{1+M}/b) differ multiplicatively from corresponding lower parameters by q^M, (namely bq^{-M} and aq/b, respectively) a nonnegative integral power of q.

One can also derive (or verify) (2.5) by adequately specializing M. Jackson's [15, Eq. (2.2)] transformation formula for a very-well-poised ${}_8\psi_8$ series into a sum of two (multiples of) ${}_8\phi_7$ series (cf. [10, Eq. (5.6.2)]):

$$
{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, g \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g \end{matrix}; q, \frac{a^3q^2}{bcdefg}\right]
$$
$$
= \frac{(q, aq, q/a, d, d/a, bq/c, bq/e, bq/f, bq/g, aq/bc, aq/be, aq/bf, aq/bg)_\infty}{(q/b, q/c, q/e, q/f, q/g, aq/b, aq/c, aq/e, aq/f, aq/g, d/b, bd/a, b^2q/a)_\infty}
$$
$$
\times\, {}_8\phi_7\left[\begin{matrix} b^2/a,\, qb/\sqrt{a}, -qb/\sqrt{a}, bc/a, bd/a, be/a, bf/a, bg/a \\ b/\sqrt{a}, -b/\sqrt{a}, bq/c, bq/d, bq/e, bq/f, bq/g \end{matrix}; q, \frac{a^3q^2}{bcdefg}\right]
$$

$$+\frac{(q,aq,q/a,b,b/a,dq/c,dq/e,dq/f,dq/g,aq/cd,aq/de,aq/df,aq/dg)_\infty}{(q/c,q/d,q/e,q/f,q/g,aq/c,aq/d,aq/e,aq/f,aq/g,b/d,bd/a,d^2q/a)_\infty}$$
$$\times\, {}_8\phi_7\left[\begin{matrix}d^2/a,\ qd/\sqrt{a},-qd/\sqrt{a},bd/a,cd/a,de/a,df/a,dg/a\\ d/\sqrt{a},-d/\sqrt{a},dq/b,dq/c,dq/e,dq/f,dq/g\end{matrix};q,\frac{a^3q^2}{bcdefg}\right], \quad (2.6)$$

where $|a^3q^2/bcdefg|<1$, for convergence. In particular, substituting $d\mapsto dq^k$, and then letting $e\to aq^{-k}/b$, $f\to aq^{1+M}/b$, $g\to aq^{-M}/d$, the coefficient of the first ${}_8\phi_7$ series on the right-hand side becomes zero (as it contains $(q^{-M})_\infty$), while the second ${}_8\phi_7$ series can be summed by an application of Jackson's terminating ${}_8\phi_7$ summation in (2.4). This way of establishing (2.5) works so far only in the classical one-dimensional case, as no multiple series extension of (2.6) is yet known. Our multivariate extensions of Proposition 2.1 in Section 4 of this paper, see Theorems 4.1, 4.3 and 4.5 (obtained by suitable extensions of the analysis applied in [23]), which we find attractive by themselves, can be understood as a first step in the quest of finding multivariate extensions of the very-well-poised ${}_8\psi_8$ transformation formula (2.6), or of even more general transformations.

Two special cases of Proposition 2.1 are worth pointing out:

(1) If $c\to a$ (or $c\to q^{-k}$), then the bilateral series in (2.5) gets truncated from below and from above so that the sum is finite. By a polynomial argument, q^M can then be replaced by any complex number. If we replace it by bc/aq, perform the substitution $d\mapsto dq^{-k}$, and finally replace k by n, we obtain exactly Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation in (2.4).

(2) If, in (2.5), we let $M\to\infty$, perform the substitution $d\mapsto dq^{-k}$, and rewrite the products of the form $(x)_k$ as $(x)_\infty/(xq^k)_\infty$, we can apply analytic continuation to replace q^k by a/ce (in order to relax the integrality condition of k) where e is a new complex parameter. We then obtain exactly Bailey's very-well-poised ${}_6\psi_6$ summation in (1.2).

2.2. **Multidimensional basic hypergeometric series associated with root systems.** A_r (or, equivalently, $U(r+1)$) hypergeometric series were motivated by the work of Biedenharn, Holman, and Louck [13] in theoretical physics. The theory of A_r basic hypergeometric series (or "multiple basic hypergeometric series associated with the root system A_r", or "associated with the unitary group $U(r+1)$"), analogous to the classical theory of one-dimensional series, has been developed originally by R. A. Gustafson, S. C. Milne, and their co-workers, and later others (see [11, 12, 17, 18, 20, 21] for a very small selection of papers in this area). Notably, several higher-dimensional extensions have been derived (in each case) for the q-binomial theorem, the q-Chu–Vandermonde summation, the q-Pfaff–Saalschütz summation, Jackson's ${}_8\phi_7$ summation, Bailey's ${}_{10}\phi_9$ transformation, and other important summation and transformation theorems. See [19] for a survey on some of the main results and techniques from the theory of A_r

basic hypergeometric series. Multiple basic hypergeometric series associated with other roots systems besides A_r have been first defined by Gustafson [12] who succeeded in giving several multivariable extensions of Bailey's ${}_6\psi_6$ summation. In particular, some important results for C_r and D_r basic hypergeometric series have been derived in [4, 6, 8, 12, 16, 20, 22] (– again, this is a very imcomplete listing).

We note the conventions for naming our series as A_r, C_r or D_r basic hypergeometric series. We consider multiple series of the form $\sum_{k_1,k_2,\dots,k_r} S(\mathbf{k})$, where $\mathbf{k} = (k_1, \dots, k_r)$, which reduce to classical basic hypergeometric series when $r = 1$. We call such a multiple basic hypergeometric series *balanced* if it reduces to a balanced series when $r = 1$. Well-poised and very-well-poised series are defined similarly.

Further, such a multiple series is called a C_r basic hypergeometric series if the summand $S(\mathbf{k})$ contains the factor

$$\prod_{1\le i<j\le r} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{1\le i\le j\le r} \frac{1 - x_i x_j q^{k_i+k_j}}{1 - x_i x_j} \tag{2.7}$$

Note that when $r = 1$, (2.7) reduces to

$$\frac{1 - x_1^2 q^{2k_1}}{1 - x_1^2},$$

which is (2.3) with x_1^2 acting like the special parameter of a very-well poised series. In our statements of C_r theorems, we set $x_i \mapsto \sqrt{a}x_i$ for $i = 1, \dots, r$, and make similar changes to other parameters in $S(\mathbf{k})$. This is done in order to follow the classical notation in [10] as closely as possible. A typical example of a C_r basic hypergeometric series is the left-hand side of (2.12).

D_r multiple basic series are closely related to C_r series. Instead of (2.7), $S(\mathbf{k})$ only has the following factors:

$$\prod_{1\le i<j\le r} \frac{(x_i q^{k_i} - x_j q^{k_j})(1 - x_i x_j q^{k_i+k_j})}{(x_i - x_j)(1 - x_i x_j)}. \tag{2.8}$$

A typical example is the left-hand side of (2.13) (with $x_i \mapsto \sqrt{cd}x_i$ for $i = 1, \dots, r$).

A_r basic hypergeometric series only have

$$\prod_{1\le i<j\le r} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \tag{2.9}$$

as a factor of $S(\mathbf{k})$. Typical examples are the left-hand sides of (2.10) and (2.11). A reason for naming these series as A_r, C_r or D_r series is that (2.9), (2.8), and (2.7) are closely associated with the product side of the Weyl denominator formulae for the respective root systems, see [4, 12].

For compact notation, we usually write

$$|\mathbf{k}| := k_1 + \cdots + k_r, \quad \text{where} \quad \mathbf{k} = (k_1, \dots, k_r),$$

and

$$C := c_1 \cdots c_r, \quad E := e_1 \cdots e_r, \quad \text{etc.}$$

We now list several multivariable extensions of Jackson's ${}_8\phi_7$ summation (2.4). The first identity is taken from [17, Thm. 6.17].

Proposition 2.2 ((Milne) An A_r terminating balanced very-well-poised ${}_8\phi_7$ summation formula)**.** *Let a, b, $c_1, \dots, c_r$, d and $x_1, \dots, x_r$ be indeterminate, and let M be a nonnegative integer. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le M}} \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i=1}^r \frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \prod_{i,j=1}^r \frac{(c_jx_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(ax_i)_{|\mathbf{k}|}\,(dx_i, a^2x_iq^{1+M}/bCd)_{k_i}}{(ax_iq/c_i)_{|\mathbf{k}|}\,(ax_iq/b, ax_iq^{1+M})_{k_i}} \cdot \frac{(b,q^{-M})_{|\mathbf{k}|}}{(aq/d, bCdq^{-M}/a)_{|\mathbf{k}|}}\, q^{|\mathbf{k}|}$$
$$= \frac{(aq/bd, aq/Cd)_M}{(aq/d, aq/bCd)_M} \prod_{i=1}^r \frac{(ax_iq, ax_iq/bc_i)_M}{(ax_iq/b, ax_iq/c_i)_M}. \tag{2.10}$$

The following identity was recently obtained in [25, Eq. (4.3)].

Proposition 2.3 ((S.) An A_r terminating balanced very-well-poised ${}_8\phi_7$ summation formula)**.** *Let a, b, $c_1, \dots, c_r$, d and $x_1, \dots, x_r$ be indeterminate, and let M be a nonnegative integer. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le M}} \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i,j=1}^r \frac{(c_jx_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(aq/Cx_id)_{|\mathbf{k}|-k_i}\,(b/x_i)_{|\mathbf{k}|}\,(dx_i)_{k_i}}{(b/x_i)_{|\mathbf{k}|-k_i}\,(ac_iq/Cx_id)_{|\mathbf{k}|}\,(ax_iq/b)_{k_i}} \cdot \frac{(a, a^2q^{1+M}/bCd, q^{-M})_{|\mathbf{k}|}}{(aq/C, bCdq^{-M}/a, aq^{1+M})_{|\mathbf{k}|}}\, q^{|\mathbf{k}|}$$
$$= \frac{(aq, aq/bd)_M}{(aq/C, aq/bCd)_M} \prod_{i=1}^r \frac{(aq/Cx_id, ax_iq/bc_i)_M}{(ax_iq/b, ac_iq/Cx_id)_M}. \tag{2.11}$$

The following identity was derived in [8, Thm. 4.1], and, independently, in [20, Thm. 6.13].

Proposition 2.4 ((Denis–Gustafson; Milne–Lilly) A C_r terminating balanced very-well-poised ${}_8\phi_7$ summation formula)**.** *Let a, b, c, d and $x_1, \dots, x_r$ be indeterminate and let $m_1, \dots, m_r$ be nonnegative integers. Then*

$$\sum_{\substack{0\le k_i\le m_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{1\le i\le j\le r} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \prod_{i,j=1}^r \frac{(q^{-m_j}x_i/x_j, ax_ix_j)_{k_i}}{(ax_ix_jq^{1+m_j}, qx_i/x_j)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(bx_i, cx_i, dx_i, a^2x_iq^{1+|\mathbf{m}|}/bcd)_{k_i}}{(ax_iq/b, ax_iq/c, ax_iq/d, bcdx_iq^{-|\mathbf{m}|}/a)_{k_i}} \cdot q^{|\mathbf{k}|}$$
$$= \prod_{1\le i<j\le r} (ax_ix_jq)^{-1}_{m_i+m_j} \prod_{i,j=1}^{r} (ax_ix_jq)_{m_i}$$
$$\times \frac{(aq/bc, aq/bd, aq/cd)_{|\mathbf{m}|}}{\prod_{i=1}^{r}(ax_iq/b, ax_iq/c, ax_iq/d, aq^{1+|\mathbf{m}|-m_i}/bcdx_i)_{m_i}}. \quad (2.12)$$

The last extension of (2.4) we need was derived in [22, Thm. 5.17].

Proposition 2.5 ((S.) A D_r terminating balanced very-well-poised ${}_8\phi_7$ summation formula)**.** *Let a, b, c, $c_1, \dots, c_r$, d, and $x_1, \dots, x_r$ be indeterminate and let M be a nonnegative integer. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le M}} \prod_{1\le i<j\le r} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i=1}^{r} \frac{1 - ax_iq^{k_i+|\mathbf{k}|}}{1 - ax_i} \prod_{i=1}^{r} \frac{(ax_i)_{|\mathbf{k}|}\,(aq/cdx_i)_{|\mathbf{k}|-k_i}}{(ax_iq/c_i, ac_iq/cdx_i)_{|\mathbf{k}|}}$$
$$\times \prod_{1\le i<j\le r} (cdx_ix_j)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j, cdx_ix_j/c_j)_{k_i}}{(qx_i/x_j)_{k_i}}$$
$$\times \frac{(b, a^2q^{1+M}/bcd, q^{-M})_{|\mathbf{k}|}}{\prod_{i=1}^{r}(ax_iq/b, bcdx_iq^{-M}/a, ax_iq^{1+M})_{k_i}} q^{|\mathbf{k}|}$$
$$= \prod_{i=1}^{r} \frac{(ax_iq, ax_iq/bc_i, ac_iq/bcdx_i, aq/cdx_i)_M}{(aq/bcdx_i, ac_iq/cdx_i, ax_iq/c_i, ax_iq/b)_M}. \quad (2.13)$$

A closely related D_r terminating balanced very-well-poised ${}_8\phi_7$ summation, equivalent to Proposition 2.5 by reversing summations of the "rectangular form" of Proposition 2.5, given in [22, Thm. 5.6], was derived by Bhatnagar, see [4, Thm. 7].

Next, we list several multivariable extensions of Bailey's ${}_6\psi_6$ summation in (1.2). The first of these was derived in [11, Thm. 1.15].

Proposition 2.6 ((Gustafson) An A_r very-well-poised ${}_6\psi_6$ summation formula)**.** *Let a, b, $c_1, \dots, c_r$, d, $e_1, \dots, e_r$ and $x_1, \dots, x_r$ be indeterminate. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i=1}^{r} \frac{1 - ax_iq^{k_i+|\mathbf{k}|}}{1 - ax_i} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j)_{k_i}}{(ax_iq/e_jx_j)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(e_ix_i)_{|\mathbf{k}|}\,(dx_i)_{k_i}}{(ax_iq/c_i)_{|\mathbf{k}|}\,(ax_iq/b)_{k_i}} \cdot \frac{(b)_{|\mathbf{k}|}}{(aq/d)_{|\mathbf{k}|}} \left(\frac{a^{r+1}q}{bCdE}\right)^{|\mathbf{k}|}$$

$$= \frac{(aq/bd, a^rq/bE, aq/Cd)_\infty}{(a^{r+1}q/bCdE, aq/d, q/b)_\infty} \prod_{i,j=1}^{r} \frac{(ax_iq/c_ie_jx_j, qx_i/x_j)_\infty}{(qx_i/c_ix_j, ax_iq/e_jx_j)_\infty} \times \prod_{i=1}^{r} \frac{(ax_iq/bc_i, aq/de_ix_i, ax_iq, q/ax_i)_\infty}{(ax_iq/b, ax_iq/c_i, q/dx_i, q/e_ix_i)_\infty}, \quad (2.14)$$

provided $|a^{r+1}q/bCdE| < 1$.

The following identity was recently obtained in [25, Thm. 6.1].

Proposition 2.7 ((S.) An A_r very-well-poised ${}_6\psi_6$ summation formula)**.** *Let* a, b, $c_1,\dots,c_r$, d, $e_1,\dots,e_r$ *and* $x_1,\dots,x_r$ *be indeterminate. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j)_{k_i}}{(ax_iq/e_jx_j)_{k_i}} \times \prod_{i=1}^{r} \frac{(aq/Cdx_i)_{|\mathbf{k}|-k_i}\,(bE/a^{r-1}e_ix_i)_{|\mathbf{k}|}\,(dx_i)_{k_i}}{(bE/a^rx_i)_{|\mathbf{k}|-k_i}\,(ac_iq/Cdx_i)_{|\mathbf{k}|}\,(ax_iq/b)_{k_i}} \cdot \frac{(E/a^{r-1})_{|\mathbf{k}|}}{(aq/C)_{|\mathbf{k}|}} \left(\frac{a^{r+1}q}{bCdE}\right)^{|\mathbf{k}|}$$
$$= \frac{(aq, q/a, aq/bd)_\infty}{(aq/C, a^{r+1}q/bCdE, a^{r-1}q/E)_\infty} \prod_{i,j=1}^{r} \frac{(qx_i/x_j, ax_iq/c_ie_jx_j)_\infty}{(qx_i/c_ix_j, ax_iq/e_jx_j)_\infty} \times \prod_{i=1}^{r} \frac{(a^rx_iq/bE, aq/e_idx_i, aq/Cdx_i, ax_iq/bc_i)_\infty}{(a^{r-1}e_ix_iq/bE, q/dx_i, ax_iq/b, ac_iq/Cdx_i)_\infty}, \quad (2.15)$$

provided $|aq^{r+1}/bCdE| < 1$.

The third extension of (1.2) we need is taken from [12, Thm. 5.1].

Proposition 2.8 ((Gustafson) A C_r very-well-poised ${}_6\psi_6$ summation formula)**.** *Let* a, b, $c_1,\dots,c_r$, d, $e_1,\dots,e_r$ *and* $x_1,\dots,x_r$ *be indeterminate. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{1\le i\le j\le r} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \times \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j, e_jx_ix_j)_{k_i}}{(ax_ix_jq/c_j, ax_iq/e_jx_j)_{k_i}} \prod_{i=1}^{r} \frac{(bx_i, dx_i)_{k_i}}{(ax_iq/b, ax_iq/d)_{k_i}} \cdot \left(\frac{a^{r+1}q}{bCdE}\right)^{|\mathbf{k}|}$$
$$= \prod_{1\le i<j\le r} (ax_ix_jq/c_ic_j, aq/e_ie_jx_ix_j)_\infty \prod_{1\le i\le j\le r} (ax_ix_jq, q/ax_ix_j)_\infty \times \frac{(aq/bd)_\infty}{(a^{r+1}q/bCdE)_\infty} \prod_{i,j=1}^{r} \frac{(ax_iq/c_ie_jx_j, qx_i/x_j)_\infty}{(ax_iq/e_jx_j, q/e_jx_ix_j, ax_ix_jq/c_i, qx_i/c_ix_j)_\infty} \times \prod_{i=1}^{r} \frac{(ax_iq/bc_i, aq/be_ix_i, ax_iq/c_id, aq/de_ix_i)_\infty}{(ax_iq/b, q/bx_i, ax_iq/d, q/dx_i)_\infty}, \quad (2.16)$$

provided $|a^{r+1}q/bCdE|<1$.

Having listed several of the most fundamental summation formulae of the theory of multidimensional basic hypergeometric series associated with root systems, we are now ready to turn to the derivation of new results.

3. Multidimensional matrix inversions

Let $\mathbb{Z}$ denote the sets of integers. In the following, we consider infinite r-dimensional matrices $F=(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $G=(g_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$, and infinite sequences $(a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^r}$ and $(b_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^r}$, of complex numbers.

Clearly, F is the *left-inverse* of G, if and only if the following orthogonality relation holds:

$$\sum_{\mathbf{k}\in\mathbb{Z}^r} f_{\mathbf{nk}}g_{\mathbf{kl}}=\delta_{\mathbf{nl}} \qquad \text{for all} \quad \mathbf{n},\mathbf{l}\in\mathbb{Z}^r. \tag{3.1}$$

Further, F is the *right-inverse* of G, if and only if the following orthogonality relation holds:

$$\sum_{\mathbf{k}\in\mathbb{Z}^r} g_{\mathbf{nk}}f_{\mathbf{kl}}=\delta_{\mathbf{nl}} \qquad \text{for all} \quad \mathbf{n},\mathbf{l}\in\mathbb{Z}^r. \tag{3.2}$$

If F is the left-inverse *and* the right-inverse of G we simply say that F and G are *mutually inverse* or *inverses of each other*.

Note that in (3.1) and (3.2) we are *not* requiring that the infinite r-dimensional matrices are lower-triangular (which would mean that $f_{\mathbf{nk}}=g_{\mathbf{nk}}=0$ unless $\mathbf{n}\ge\mathbf{k}$, where by the latter we mean $n_i\ge k_i$ for all $i=1,\dots,r$). If they were lower-triangular, the multiple series on the left-hand sides of (3.1) and (3.2) would be in fact finite sums (and both relations must then hold at the same time). In the general case, the sums are infinite and we will require the series to be absolutely convergent. Note that convergence of one of the sums does not necessarily imply convergence of the other.

Now consider the following two equations (a.k.a. "inverse relations"):

$$\sum_{\mathbf{k}\in\mathbb{Z}^r} f_{\mathbf{nk}}a_{\mathbf{k}}=b_{\mathbf{n}} \qquad \text{for all } \mathbf{n}, \tag{3.3a}$$

and

$$\sum_{\mathbf{l}\in\mathbb{Z}^r} g_{\mathbf{kl}}b_{\mathbf{l}}=a_{\mathbf{k}} \qquad \text{for all } \mathbf{k}. \tag{3.3b}$$

It is immediate from the orthogonality relations (3.1) and (3.2) that if F is the left-inverse of G, the relation (3.3b) implies (3.3a), while if F is the right-inverse of G, the relation (3.3a) implies (3.3b), subject to convergence.

Similarly, one may consider another pair of equations, where one sums over the *first* (instead of the second) multi-index of the matrix:

$$\sum_{\mathbf{n}\in\mathbb{Z}^r} f_{\mathbf{nk}}a_{\mathbf{n}}=b_{\mathbf{k}} \qquad \text{for all } \mathbf{k}, \tag{3.4a}$$

and

$$\sum_{\mathbf{k}\in\mathbb{Z}^r} g_{\mathbf{kl}} b_{\mathbf{k}} = a_{\mathbf{l}} \qquad \text{for all } \mathbf{l}. \tag{3.4b}$$

Again, it is immediate from the orthogonality relations (3.1) and (3.2) that if F is the left-inverse of G, the relation (3.4a) implies (3.4b), while if F is the right-inverse of G, the relation (3.4b) implies (3.4a), again subject to convergence.

We are ready to state and prove three multidimensional matrix inverses, all of them as consequences of corresponding multivariate ${}_6\psi_6$ summations which have been stated in Section 2.

Theorem 3.1 (An A_r multilateral matrix inverse). *Let a, b, $c_1,\dots,c_r$, and $x_1,\dots,x_r$ be indeterminate. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are* inverses *of each other where*

$$\begin{aligned} f_{\mathbf{nk}} = {} & \frac{(bq,q/b)_\infty}{(bq/C,Cq/b)_\infty} \prod_{i,j=1}^r \frac{(qc_jx_i/x_j, qx_i/c_ix_j)_\infty}{(qc_jx_i/c_ix_j, qx_i/x_j)_\infty} \\ & \times \prod_{i=1}^r \frac{(ax_iq/b, bq/ax_i, ax_iq/c_i, c_iq/ax_i)_\infty}{(ax_iq/bc_i, bc_iq/ax_i, ax_iq, q/ax_i)_\infty} \\ & \times \prod_{1\le i<j\le r} \frac{c_iq^{n_i}/x_i - c_jq^{n_j}/x_j}{c_i/x_i - c_j/x_j} \prod_{i=1}^r \frac{1-bc_iq^{n_i+|\mathbf{n}|}/ax_i}{1-bc_i/ax_i} \\ & \times (b)_{|\mathbf{n}|+|\mathbf{k}|} \prod_{i,j=1}^r \frac{1}{(qc_jx_i/x_j)_{n_j+k_i}} \prod_{i=1}^r \frac{(ax_i/c_i)_{|\mathbf{k}|-n_i}}{(ax_iq/b)_{k_i-|\mathbf{n}|}} \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned} g_{\mathbf{kl}} = {} & \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i=1}^r \frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \\ & \times \frac{1}{(bq)_{|\mathbf{k}|+|\mathbf{l}|}} \prod_{i,j=1}^r (c_jx_i/x_j)_{k_i+l_j} \prod_{i=1}^r \frac{(ax_i/b)_{k_i-|\mathbf{l}|}}{(ax_iq/c_i)_{|\mathbf{k}|-l_i}} \cdot q^{|\mathbf{k}|-r|\mathbf{l}|}. \end{aligned} \tag{3.5b}$$

Proof. We show that the inverse matrices (3.5a)/(3.5b) satisfy the orthogonality relation (3.1). (An analogous computation reveals that the inverse matrices (3.5a)/(3.5b) also satisfy the dual orthogonality relation (3.2).) Writing out the sum $\sum_{\mathbf{k}\in\mathbb{Z}^r} f_{\mathbf{nk}} g_{\mathbf{kl}}$ with the above choices of $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ we observe that the multiple series can be summed by an application of the A_r very-well-poised ${}_6\psi_6$ summation in Proposition 2.6. The specializations needed there are

$$b \mapsto bq^{|\mathbf{n}|}, \quad c_i \mapsto c_iq^{l_i}, \quad d \mapsto aq^{-|\mathbf{l}|}/b, \quad e_i \mapsto aq^{-n_i}/c_i, \quad i=1,\dots,r. \tag{3.6}$$

The summation formula gives us a product containing the factors

$$(q^{1+|\mathbf{l}|-|\mathbf{n}|})_\infty \prod_{i,j=1}^r (q^{1+n_j-l_i} c_j x_i/c_i x_j)_\infty. \tag{3.7}$$

Since (3.7) vanishes for all r-tuples of integers $\mathbf{n}$ and $\mathbf{l}$ with $\mathbf{n} \neq \mathbf{l}$, we can simplify the product (setting $\mathbf{n} = \mathbf{l}$, the only non-zero case) and readily determine that the sum indeed boils down to $\delta_{\mathbf{nl}}$. The details are as follows:

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathbb{Z}^r} f_{\mathbf{nk}} g_{\mathbf{kl}} = & \sum_{k_1,\dots,k_r=-\infty}^{\infty} \frac{(bq,q/b)_\infty}{(bq/C,Cq/b)_\infty} \prod_{i,j=1}^r \frac{(qc_jx_i/x_j, qx_i/c_ix_j)_\infty}{(qc_jx_i/c_ix_j, qx_i/x_j)_\infty} \\
& \times \prod_{i=1}^r \frac{(ax_iq/b, bq/ax_i, ax_iq/c_i, c_iq/ax_i)_\infty}{(ax_iq/bc_i, bc_iq/ax_i, ax_iq, q/ax_i)_\infty} \\
& \times \prod_{1\le i<j\le r} \frac{c_iq^{n_i}/x_i - c_jq^{n_j}/x_j}{c_i/x_i - c_j/x_j} \prod_{i=1}^r \frac{1-bc_iq^{n_i+|\mathbf{n}|}/ax_i}{1-bc_i/ax_i} \\
& \times (b)_{|\mathbf{n}|+|\mathbf{k}|} \prod_{i,j=1}^r \frac{1}{(qc_jx_i/x_j)_{n_j+k_i}} \prod_{i=1}^r \frac{(ax_i/c_i)_{|\mathbf{k}|-n_i}}{(ax_iq/b)_{k_i-|\mathbf{n}|}} \\
& \times \prod_{1\le i<j\le r} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i=1}^r \frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \\
& \times \frac{1}{(bq)_{|\mathbf{k}|+|\mathbf{l}|}} \prod_{i,j=1}^r (c_jx_i/x_j)_{k_i+l_j} \prod_{i=1}^r \frac{(ax_i/b)_{k_i-|\mathbf{l}|}}{(ax_iq/c_i)_{|\mathbf{k}|-l_i}} \cdot q^{|\mathbf{k}|-r|\mathbf{l}|} \\
= & \frac{(bq,q/b)_\infty}{(bq/C,Cq/b)_\infty} \prod_{i,j=1}^r \frac{(qc_jx_i/x_j, qx_i/c_ix_j)_\infty}{(qc_jx_i/c_ix_j, qx_i/x_j)_\infty} \\
& \times \prod_{i=1}^r \frac{(ax_iq/b, bq/ax_i, ax_iq/c_i, c_iq/ax_i)_\infty}{(ax_iq/bc_i, bc_iq/ax_i, ax_iq, q/ax_i)_\infty} \\
& \times \prod_{1\le i<j\le r} \frac{c_iq^{n_i}/x_i - c_jq^{n_j}/x_j}{c_i/x_i - c_j/x_j} \prod_{i=1}^r \frac{1-bc_iq^{n_i+|\mathbf{n}|}/ax_i}{1-bc_i/ax_i} \\
& \times (b)_{|\mathbf{n}|} \prod_{i,j=1}^r \frac{1}{(qc_jx_i/x_j)_{n_j}} \prod_{i=1}^r \frac{(ax_i/c_i)_{-n_i}}{(ax_iq/b)_{-|\mathbf{n}|}} \\
& \times \frac{1}{(bq)_{|\mathbf{l}|}} \prod_{i,j=1}^r (c_jx_i/x_j)_{l_j} \prod_{i=1}^r \frac{(ax_i/b)_{-|\mathbf{l}|}}{(ax_iq/c_i)_{-l_i}} \cdot q^{-r|\mathbf{l}|}
\end{aligned}$$

$$\times \sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i=1}^{r} \frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \prod_{i,j=1}^{r} \frac{(q^{l_j}c_jx_i/x_j)_{k_i}}{(q^{1+n_j}c_jx_i/x_j)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(ax_iq^{-n_i}/c_i)_{|\mathbf{k}|}\,(ax_iq^{-|\mathbf{l}|}/b)_{k_i}}{(ax_iq^{1-l_i}/c_i)_{|\mathbf{k}|}\,(ax_iq^{1-|\mathbf{n}|}/b)_{k_i}} \cdot \frac{(bq^{|\mathbf{n}|})_{|\mathbf{k}|}}{(bq^{1+|\mathbf{l}|})_{|\mathbf{k}|}}\, q^{|\mathbf{k}|}$$

$$= \frac{(bq,q/b)_\infty}{(bq/C,Cq/b)_\infty} \prod_{i,j=1}^{r} \frac{(qc_jx_i/x_j,qx_i/c_ix_j)_\infty}{(qc_jx_i/c_ix_j,qx_i/x_j)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{(ax_iq/b,bq/ax_i,ax_iq/c_i,c_iq/ax_i)_\infty}{(ax_iq/bc_i,bc_iq/ax_i,ax_iq,q/ax_i)_\infty}$$

$$\times \prod_{1\le i<j\le r} \frac{c_iq^{n_i}/x_i-c_jq^{n_j}/x_j}{c_i/x_i-c_j/x_j} \prod_{i=1}^{r} \frac{1-bc_iq^{n_i+|\mathbf{n}|}/ax_i}{1-bc_i/ax_i}$$

$$\times \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j)_{l_j}}{(qc_jx_i/x_j)_{n_j}} \prod_{i=1}^{r} \frac{(ax_i/c_i)_{-n_i}\,(ax_i/b)_{-|\mathbf{l}|}}{(ax_iq/c_i)_{-l_i}\,(ax_iq/b)_{-|\mathbf{n}|}} \cdot \frac{(b)_{|\mathbf{n}|}}{(bq)_{|\mathbf{l}|}} q^{-r|\mathbf{l}|}$$

$$\times \frac{(q^{1+|\mathbf{l}|-|\mathbf{n}|},Cq/b,bq/C)_\infty}{(q,bq^{1+|\mathbf{l}|},q^{1-|\mathbf{n}|}/b)_\infty} \prod_{i,j=1}^{r} \frac{(q^{1+n_j-l_i}c_jx_i/c_ix_j,qx_i/x_j)_\infty}{(q^{1-l_i}x_i/c_ix_j,q^{1+n_j}c_jx_i/x_j)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{(ax_iq^{1-l_i-|\mathbf{n}|}/bc_i,bc_iq^{1+n_i+|\mathbf{l}|}/ax_i,ax_iq,q/ax_i)_\infty}{(ax_iq^{1-|\mathbf{n}|}/b,ax_iq^{1-l_i}/c_i,bq^{1+|\mathbf{l}|}/ax_i,c_iq^{1+n_i}/ax_i)_\infty}. \tag{3.8}$$

Now we set $\mathbf{n}=\mathbf{l}$, apply several elementary identities from [10, App. I] and apply the $n\mapsto r$, $x_i\mapsto c_iq^{-n_i}/x_i$, $y_i\mapsto n_i$, $i=1,\dots,r$, case of [18, Lem. 3.12], specifically

$$\prod_{i,j=1}^{r} (q^{1+n_j-n_i}c_jx_i/c_ix_j)_{n_i-n_j} = (-1)^{(r-1)|\mathbf{n}|}\, q^{\binom{|\mathbf{n}|}{2}-r\sum_{i=1}^{r}\binom{n_i}{2}}$$

$$\times \prod_{i=1}^{r}\left(\frac{c_i}{x_i}\right)^{|\mathbf{n}|-rn_i} \prod_{1\le i<j\le r} \frac{c_i/x_i-c_j/x_j}{c_iq^{n_i}/x_i-c_jq^{n_j}/x_j}, \tag{3.9}$$

to transform the last expression obtained in (3.8) to $\delta_{\mathbf{nl}}$. □

Remark 3.2. The $c_i\to 1$, $i=1,\dots,r$, case of Theorem 3.1 can be reduced to Milne's [18, Thm. 3.41] A_r extension of Bressoud's matrix inverse [7]. In particular, since $1/(q)_{n+k}=0$ for $n+k<0$, and $(1)_{k+l}=0$ for $k+l>0$, the orthogonality relation (3.1) then reduces to

$$\sum_{\substack{-n_i\le k_i\le -l_i\\ i=1,\dots,r}} f_{\mathbf{nk}}g_{\mathbf{kl}} = \delta_{\mathbf{nl}},$$

i.e. (after replacing $\mathbf{k}$ by $-\mathbf{k}$),

$$\sum_{\substack{l_i\le k_i\le n_i\\ i=1,\dots,r}} f_{\mathbf{n},-\mathbf{k}}\,g_{-\mathbf{k},\mathbf{l}}=\delta_{\mathbf{n}\mathbf{l}}.$$

The r-dimensional matrices $(f_{\mathbf{n},-\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{-\mathbf{k},\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are thus mutually inverse *lower-triangular* matrices.

Theorem 3.3 (Another A_r multilateral matrix inverse)**.** *Let a, b, $c_1,\dots,c_r$ and $x_1,\dots,x_r$ be indeterminate. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are* inverses *of each other where*

$$\begin{aligned}
f_{\mathbf{nk}}=&\frac{(aq/C,Cq/a)_\infty}{(aq,q/a)_\infty}\prod_{i,j=1}^r\frac{(qc_jx_i/x_j,qx_i/c_ix_j)_\infty}{(qc_jx_i/c_ix_j,qx_i/x_j)_\infty}\\
&\times\prod_{i=1}^r\frac{(ax_iq/b,bq/ax_i,bc_iq/Cx_i,Cx_iq/bc_i)_\infty}{(ax_iq/bc_i,bc_iq/ax_i,Cx_iq/b,bq/Cx_i)_\infty}\\
&\times\prod_{1\le i<j\le r}\frac{c_iq^{n_i}/x_i-c_jq^{n_j}/x_j}{c_i/x_i-c_j/x_j}\prod_{i=1}^r\frac{1-bc_iq^{n_i+|\mathbf{n}|}/ax_i}{1-bc_i/ax_i}\\
&\times(a/C)_{|\mathbf{k}|-|\mathbf{n}|}\prod_{i,j=1}^r\frac{1}{(qc_jx_i/x_j)_{n_j+k_i}}\prod_{i=1}^r\frac{(bc_i/Cx_i)_{n_i+|\mathbf{k}|}}{(ax_iq/b)_{k_i-|\mathbf{n}|}}
\end{aligned}\tag{3.10a}$$

and

$$\begin{aligned}
g_{\mathbf{kl}}=&\frac{(1-aq^{2|\mathbf{k}|})}{(1-a)}\prod_{1\le i<j\le r}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i=1}^r\frac{1-bq^{|\mathbf{k}|-k_i}/Cx_i}{1-b/Cx_i}\\
&\times\frac{1}{(aq/C)_{|\mathbf{k}|-|\mathbf{l}|}}\prod_{i,j=1}^r(c_jx_i/x_j)_{k_i+l_j}\prod_{i=1}^r\frac{(ax_i/b)_{k_i-|\mathbf{l}|}}{(bc_iq/Cx_i)_{|\mathbf{k}|+l_i}}\cdot q^{|\mathbf{k}|-r|\mathbf{l}|}.
\end{aligned}\tag{3.10b}$$

Proof. The proof is similar to the proof of Theorem 3.1 but utilizes Proposition 2.7 in addition to Proposition 2.6.

We first show that the inverse matrices (3.10a)/(3.10b) satisfy the orthogonality relation (3.1). Writing out the sum $\sum_{\mathbf{k}\in\mathbb{Z}^r}f_{\mathbf{nk}}g_{\mathbf{kl}}$ with the above choices of $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ we observe that the multiple series can be summed by an application of the A_r very-well-poised ${}_6\psi_6$ summation in Proposition 2.7. The specializations needed there are exactly the same as in Equation (3.6). Again, the summation formula leads to a product containing the factors in (3.7). We can thus simplify the product (setting $\mathbf{n}=\mathbf{l}$, the only non-zero case) and readily determine, by applying several elementary identities for q-shifted factorials, including (3.9), that the sum indeed boils down to $\delta_{\mathbf{nl}}$.

An analogous computation reveals that the inverse matrices (3.10a)/(3.10b) also satisfy the dual orthogonality relation (3.2). Writing out the sum $\sum_{\mathbf{k}\in\mathbb{Z}^r}g_{\mathbf{nk}}f_{\mathbf{kl}}$

with the above choices of $g_{\mathbf{nk}}$ and $f_{\mathbf{kl}}$ we observe that the multiple series can be summed by an application of the A_r very-well-poised ${}_6\psi_6$ summation in Proposition 2.6. The specializations needed there are

$$a \mapsto b/a, \qquad b \mapsto Cq^{-|\mathbf{n}|}/a, \qquad d \mapsto bq^{|\mathbf{l}|}/C,$$
$$c_i \mapsto c_i q^{n_i}, \qquad e_i \mapsto bq^{-l_i}/ac_i, \qquad x_i \mapsto c_i/x_i, \qquad i = 1, \dots, r.$$

The summation formula gives us a product containing the factors

$$(q^{1-|\mathbf{l}|+|\mathbf{n}|})_\infty \prod_{i,j=1}^{r} (q^{1+l_j-n_i} c_i x_j / c_j x_i)_\infty. \tag{3.11}$$

Since (3.11) vanishes for all r-tuples of integers $\mathbf{n}$ and $\mathbf{l}$ with $\mathbf{n} \neq \mathbf{l}$, we can simplify the product (setting $\mathbf{n} = \mathbf{l}$, the only non-zero case) and readily determine, by applying several elementary identities for q-shifted factorials, that the sum indeed boils down to $\delta_{\mathbf{nl}}$. We omit the details, being similar to those as in the proof of Theorem 3.1. □

Remark 3.4. The $c_i \to 1$, $i = 1, \dots, r$, case of Theorem 3.3 can be reduced to the author's [25, Cor. 3.2] A_r extension of Bressoud's matrix inverse [7], which can also be obtained by specializing Bhatnagar and Milne's matrrix inverse [5, Thm. 3.48]. As in Remark 3.2, the r-dimensional matrices $(f_{\mathbf{n},-\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{-\mathbf{k},\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are then mutually inverse *lower-triangular* matrices.

The following matrix inverse serves as a bridge between C_r series and D_r series.

Theorem 3.5 (A C_r/D_r multilateral matrix inverse)**.** *Let a, b, $c_1, \dots, c_r$ and $x_1, \dots, x_r$ be indeterminate. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ is the* left-inverse *of $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ where*

$$\begin{aligned} f_{\mathbf{nk}} = {} & \prod_{i=1}^{r} \frac{(ax_iq/b, bq/ax_i, bx_iq, q/bx_i)_\infty}{(ax_iq/bc_i, bc_iq/ax_i, bx_iq/c_i, c_iq/bx_i)_\infty} \\ & \times \frac{\prod_{i,j=1}^{r}(qc_jx_i/x_j, qx_i/c_ix_j, ax_ix_jq/c_i, c_jq/ax_ix_j)_\infty}{\prod_{i,j=1}^{r}(qc_jx_i/c_ix_j, qx_i/x_j)_\infty \prod_{1\le i\le j\le r}(ax_ix_jq, q/ax_ix_j)_\infty} \\ & \times \prod\nolimits_{1\le i<j\le r} (ax_ix_jq/c_ic_j, c_ic_jq/ax_ix_j)_\infty^{-1} \\ & \times \prod_{1\le i<j\le r} \frac{c_iq^{n_i}/x_i - c_jq^{n_j}/x_j}{c_i/x_i - c_j/x_j} \prod_{i,j=1}^{r} \frac{(ax_ix_j/c_j)_{k_i-n_j}}{(qc_jx_i/x_j)_{k_i+n_j}} \\ & \times \prod_{i=1}^{r} \frac{(1-bc_iq^{n_i+|\mathbf{n}|}/ax_i)(1-bx_iq^{|\mathbf{n}|-n_i}/c_i)\,(bx_i)_{k_i+|\mathbf{n}|}}{(1-bc_i/ax_i)(1-bx_i/c_i)\,(ax_iq/b)_{k_i-|\mathbf{n}|}} \end{aligned} \tag{3.12a}$$

and

$$g_{\mathbf{kl}} = \prod_{1\le i<j\le r} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{1\le i\le j\le r} \frac{1 - ax_ix_jq^{k_i+k_j}}{1 - ax_ix_j}$$

$$\times \prod_{i,j=1}^r \frac{(c_jx_i/x_j)_{k_i+l_j}}{(ax_ix_jq/c_j)_{k_i-l_j}} \prod_{i=1}^r \frac{(ax_i/b)_{k_i-|\mathbf{l}|}}{(bx_iq)_{k_i+|\mathbf{l}|}} \cdot q^{|\mathbf{k}|+(1-2r)|\mathbf{l}|}. \quad (3.12b)$$

Proof. The proof is completely analogous to the proofs of Theorems 3.1 and 3.3, with the only difference that the orthogonality relation (3.1) of the inverse matrices (3.12a)/(3.12b) is now established by using Proposition 2.8. The specializations (3.6) again lead to a product containing the factors in (3.7). □

Remark 3.6. Theorem 3.5 states that $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ is the *left-inverse* of $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$. This is because the infinite multiple sum $\sum_{\mathbf{k}\in\mathbb{Z}^r} f_{\mathbf{nk}}g_{\mathbf{kl}}$ converges for all $r = 1, 2\dots$, and evaluates to $\delta_{\mathbf{nl}}$. On the contrary, $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ is *not* the *right-inverse* of $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ unless $r = 1$. The infinite multiple sum $\sum_{\mathbf{k}\in\mathbb{Z}^r} g_{\mathbf{nk}}f_{\mathbf{kl}}$ converges for all r but for $r > 1$ in general does *not* evaluate to $\delta_{\mathbf{nl}}$ (which we find quite surprising). In particular, we do *not* have a closed form evaluation for the convergent multilateral sum

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \frac{c_iq^{k_i}/x_i - c_jq^{k_j}/x_j}{c_i/x_i - c_j/x_j} \prod_{i=1}^r \frac{(1-bc_iq^{k_i+|\mathbf{k}|}/ax_i)(1-bx_iq^{|\mathbf{k}|-k_i}/c_i)}{(1-bc_i/ax_i)(1-bx_i/c_i)}$$
$$\times \prod_{i,j=1}^r \frac{(q^{n_i}c_jx_i/x_j, c_jq^{-n_i}/ax_ix_j)_{k_j}}{(q^{1+l_i}c_jx_i/x_j, c_jq^{1-l_i}/ax_ix_j)_{k_j}} \prod_{i=1}^r \frac{(bx_iq^{l_i}, bq^{-l_i}/ax_i)_{|\mathbf{k}|}}{(bx_iq^{1+n_i}, bq^{1-n_i}/ax_i)_{|\mathbf{k}|}} \cdot q^{|\mathbf{k}|}.$$

To see what can go wrong, when one applies inverse relations in an incorrect way, see the Appendix.

Remark 3.7. The $c_i \mapsto ax_i^2$, $i = 1, \dots, r$, case of Theorem 3.5 reduces to a multidimensional matrix inverse involving two *lower-triangular* matrices being mutually inverse (a C_r/D_r extension of Bressoud's matrix inverse [7]), a result first derived in [22, Thm. 5.10]. The combination of this inversion with Denis–Gustafson's/Milne–Lilly's [8, 20] C_r terminating balanced very-well-poised ${}_8\phi_7$ summation, stated here as Proposition 2.4, led in [22, Thm. 5.14] to a D_r terminating balanced very-well-poised ${}_8\phi_7$ summation, equivalent (by a polynomial argument) to the D_r summation stated here as Proposition 2.5.

4. Multivariable balanced very-well-poised ${}_8\psi_8$ summations

As applications of the multilateral matrix inverses of Section 3, we provide three multidimensional extensions of the ${}_8\psi_8$ summation formula in Proposition 2.1.

Theorem 4.1 (An A_r balanced very-well-poised ${}_8\psi_8$ summation formula)**.** *Let a, b, $c_1, \dots, c_r$, d and $x_1, \dots, x_r$ be indeterminate, let $k_1, \dots, k_r$ be integers, and let M be a nonnegative integer. Then*

$$\sum_{n_1,\dots,n_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \frac{x_iq^{n_i} - x_jq^{n_j}}{x_i - x_j} \prod_{i=1}^r \frac{1-ax_iq^{n_i+|\mathbf{n}|}}{1-ax_i} \prod_{i,j=1}^r \frac{(c_jx_i/x_j)_{n_i}}{(q^{1+k_j}c_jx_i/x_j)_{n_i}}$$

$$\times \prod_{i=1}^{r} \frac{(ax_iq^{-k_i}/c_i)_{|\mathbf{n}|}\,(bx_i, ax_iq^{-M}/d)_{n_i}}{(ax_iq/c_i)_{|\mathbf{n}|}\,(bx_iq^{-M}, ax_iq^{1-|\mathbf{k}|}/d)_{n_i}} \cdot \frac{(dq^{|\mathbf{k}|}, aq^{1+M}/b)_{|\mathbf{n}|}}{(dq^{1+M}, aq/b)_{|\mathbf{n}|}}\, q^{|\mathbf{n}|}$$

$$= \prod_{i,j=1}^{r} \frac{(qc_jx_i/c_ix_j, qx_i/x_j)_\infty}{(qc_jx_i/x_j, qx_i/c_ix_j)_\infty} \prod_{i=1}^{r} \frac{(ax_iq, q/ax_i, ax_iq/c_id, c_idq/ax_i)_\infty}{(ax_iq/c_i, c_iq/ax_i, ax_iq/d, dq/ax_i)_\infty}$$

$$\times \frac{(dq/C, Cq/d)_\infty}{(dq, q/d)_\infty} \frac{(dq, aq/bC)_M}{(aq/b, dq/C)_M} \prod_{i=1}^{r} \frac{(c_iq/bx_i, dq/ax_i)_M}{(c_idq/ax_i, q/bx_i)_M} \prod_{i,j=1}^{r} \frac{(qc_jx_i/x_j)_{k_i}}{(qc_jx_i/c_ix_j)_{k_i}}$$

$$\times \frac{(bd/a, q^{-M})_{|\mathbf{k}|}}{(d, bCq^{-M}/a)_{|\mathbf{k}|}} \prod_{i=1}^{r} \frac{(c_id/ax_i)_{|\mathbf{k}|}\,(c_iq/ax_i, c_idq^{1+M}/bCx_i)_{k_i}}{(d/ax_i)_{|\mathbf{k}|}\,(c_iq/bx_i, c_idq^{1+M}/ax_i)_{k_i}}. \quad (4.1)$$

Proof. We combine the multilateral matrix inverse in Theorem 3.1 with the A_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation in Proposition 2.2, using the inverse relations (3.4). (Alternatively, we may also use the inverse relations (3.3), with a similar analysis as in the proof of Theorem 4.3.)

In particular, we have (3.4b), with $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ as in (3.5b),

$$a_{\mathbf{l}} = \frac{(bq/d, bq/C)_M}{(bq, bq/Cd)_M} \prod_{i=1}^{r} \frac{(ax_iq, ax_iq/c_id)_M}{(ax_iq/d, ax_iq/c_i)_M} \prod_{i,j=1}^{r} (c_jx_i/x_j)_{l_j}$$

$$\times \frac{(bq^{1+M}/d)_{|\mathbf{l}|}}{(bq^{1+M}, bq/d)_{|\mathbf{l}|}} \prod_{i=1}^{r} \frac{(c_iq^{-M}/ax_i, c_id/ax_i)_{l_i}}{(c_idq^{-M}/ax_i)_{l_i}\,(bq/ax_i)_{|\mathbf{l}|}}$$

$$\times (-1)^{(r-1)|\mathbf{l}|} a^{(1-r)|\mathbf{l}|} b^{r|\mathbf{l}|} q^{r\binom{|\mathbf{l}|}{2} - \sum_{i=1}^{r}\binom{l_i}{2}} \prod_{i=1}^{r} c_i^{-l_i} x_i^{l_i-|\mathbf{l}|},$$

and

$$b_{\mathbf{k}} = \frac{(d, q^{-M})_{|\mathbf{k}|}}{(Cdq^{-M}/b)_{|\mathbf{k}|}} \prod_{i=1}^{r} \frac{(abx_iq^{1+M}/Cd)_{k_i}\,(ax_i)_{|\mathbf{k}|}}{(ax_iq/d, ax_iq^{1+M})_{k_i}} \prod_{i,j=1}^{r} \frac{1}{(qx_i/x_j)_{k_i}},$$

by the $b \mapsto d$, $c_i \mapsto c_iq^{l_i}$, $d \mapsto aq^{-|\mathbf{l}|}/b$, $i = 1, \dots, r$, case of Proposition 2.2. Therefore we must have (3.4a), with $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ as in (3.5a), and the above sequences $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After simplifications and the substitutions $a \mapsto d/a$, $b \mapsto d$, $d \mapsto bd/a$, $x_i \mapsto c_i/x_i$, $i = 1, \dots, r$, we arrive at (4.1). □

Alternative proof of Theorem 4.1. We combine the multilateral matrix inverse in Theorem 3.3 with the A_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation in Proposition 2.3, using the inverse relations (3.4).

In particular, we have (3.4b), with $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ as in (3.10b),

$$a_{\mathbf{l}} = \frac{(aq, bq/d)_M}{(aq/C, bq/Cd)_M} \prod_{i=1}^{r} \frac{(bq/Cx_i, ax_iq/c_id)_M}{(ax_iq/d, bc_iq/Cx_i)_M} \prod_{i,j=1}^{r} (c_jx_i/x_j)_{l_j}$$

$$\times \frac{(bq^{1+M}/d, Cq^{-M}/a)_{|\mathbf{l}|}}{(bq/d)_{|\mathbf{l}|}} \prod_{i=1}^{r} \frac{(c_i d/ax_i)_{l_i}\, x_i^{|\mathbf{l}|}}{(bc_i q^{1+M}/Cx_i, c_i dq^{-M}/ax_i)_{l_i}\, (bx_i q/a)_{|\mathbf{l}|}}$$
$$\times (-1)^{(r-1)|\mathbf{l}|} a^{(1-r)|\mathbf{l}|} b^{r|\mathbf{l}|} C^{-|\mathbf{l}|} q^{(r-1)\binom{|\mathbf{l}|}{2}},$$

and

$$b_{\mathbf{k}} = \frac{(a, abq^{1+M}/Cd, q^{-M})_{|\mathbf{k}|}}{(Cdq^{-M}/b, aq^{1+M})_{|\mathbf{k}|}} \prod_{i=1}^{r} \frac{(b/Cx_i)_{|\mathbf{k}|-k_i}\, (d/x_i)_{|\mathbf{k}|}}{(d/x_i)_{|\mathbf{k}|-k_i}\, (ax_i q/d)_{k_i}} \prod_{i,j=1}^{r} \frac{1}{(qx_i/x_j)_{k_i}},$$

by the $b \mapsto d$, $c_i \mapsto c_i q^{l_i}$, $d \mapsto aq^{-|\mathbf{l}|}/b$, $i = 1, \ldots, r$, case of Proposition 2.3. Therefore we must have (3.4a), with $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ as in (3.10a), and the above sequences $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After simplifications and the substitutions $a \mapsto d/a$, $b \mapsto d$, $d \mapsto bd/a$, $x_i \mapsto c_i/x_i$, $i = 1, \ldots, r$, we arrive at (4.1). □

Remark 4.2. Two special cases of Theorem 4.1 are of particular interest:

(1) If $c_i = q^{-k_i}$, for $i = 1, \ldots, r$, then the multilateral series in (4.1) gets truncated from below and from above so that the multiple sum is finite. By a polynomial argument, we can replace q^M by bc/aq. If we then perform the substitution $d \mapsto dq^{-|\mathbf{k}|}$ and replace k_i by m_i, $i = 1, \ldots, r$, we obtain an A_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation (cf. [17, Thm. 6.14]) which, via a polynomial argument, is equivalent to Proposition 2.2.

(2) If, in (4.1), we let $M \to \infty$ and perform the substitution $d \mapsto dq^{-|\mathbf{k}|}$, we can repeatedly apply analytic continuation to replace q^{k_i} by $a/c_i e_i$ for $i = 1, \ldots, r$ (in order to relax the integrality condition of the k_i's), where $e_1, \ldots, e_r$ are new complex parameters. We then obtain the A_r extension of Bailey's very-well-poised ${}_6\psi_6$ summation in Proposition 2.6.

Theorem 4.3 (An A_r balanced very-well-poised ${}_8\psi_8$ summation formula)**.** *Let a, b, $c_1, \ldots, c_r$, d and $x_1, \ldots, x_r$ be indeterminate, let $k_1, \ldots, k_r$ be integers, and let M be a nonnegative integer. Then*

$$\sum_{n_1,\ldots,n_r=-\infty}^{\infty} \frac{(1-aq^{2|\mathbf{n}|})}{(1-a)} \frac{(b, aq^{1+M}/b, aq^{-|\mathbf{k}|}/C)_{|\mathbf{n}|}}{(bq^{-M}, aq/b, aq/C)_{|\mathbf{n}|}} q^{|\mathbf{n}|} \prod_{1\le i<j\le r} \frac{x_i q^{n_i} - x_j q^{n_j}}{x_i - x_j}$$
$$\times \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j)_{n_i}}{(q^{1+k_j} c_j x_i/x_j)_{n_i}} \prod_{i=1}^{r} \frac{(dq^{1+M}/Cx_i)_{|\mathbf{n}|-n_i}\, (c_i dq^{k_i}/Cx_i)_{|\mathbf{n}|}\, (ax_i q^{-M}/d)_{n_i}}{(dq/Cx_i)_{|\mathbf{n}|-n_i}\, (c_i dq^{1+M}/Cx_i)_{|\mathbf{n}|}\, (ax_i q^{1-|\mathbf{k}|}/d)_{n_i}}$$
$$= \prod_{i,j=1}^{r} \frac{(qc_j x_i/c_i x_j, qx_i/x_j)_\infty}{(qc_j x_i/x_j, qx_i/c_i x_j)_\infty} \prod_{i=1}^{r} \frac{(ax_i q/c_i d, c_i dq/ax_i, Cx_i q/d, dq/Cx_i)_\infty}{(ax_i q/d, dq/ax_i, c_i dq/Cx_i, Cx_i q/c_i d)_\infty}$$
$$\times \frac{(aq, q/a)_\infty}{(aq/C, Cq/a)_\infty} \frac{(Cq/b, aq/bC)_M}{(aq/b, q/b)_M} \prod_{i=1}^{r} \frac{(c_i dq/Cx_i, dq/ax_i)_M}{(c_i dq/ax_i, dq/Cx_i)_M} \prod_{i,j=1}^{r} \frac{(qc_j x_i/x_j)_{k_i}}{(qc_j x_i/c_i x_j)_{k_i}}$$

$$\times \frac{(Cq/a,q^{-M})_{|\mathbf{k}|}}{(Cq/b,bCq^{-M}/a)_{|\mathbf{k}|}} \prod_{i=1}^{r} \frac{(c_i d/ax_i)_{|\mathbf{k}|}\,(bc_i d/aCx_i, c_i dq^{1+M}/bCx_i)_{k_i}}{(d/ax_i)_{|\mathbf{k}|}\,(c_i d/Cx_i, c_i dq^{1+M}/ax_i)_{k_i}}. \quad (4.2)$$

Proof. We combine the multilateral matrix inverse in Theorem 3.3 with the A_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation in Proposition 2.2, using the inverse relations (3.3).

In particular, we have (3.3b), with $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ as in (3.10b),

$$a_{\mathbf{k}} = \frac{(bq/d,bq/ad)_M}{(bCq/ad,bq/Cd)_M} \prod_{i=1}^{r} \frac{(bc_iq/ax_i, bq/Cx_i)_M}{(bc_iq/Cx_i, bq/ax_i)_M} \prod_{i,j=1}^{r} (c_j x_i/x_j)_{k_i}$$
$$\times \prod_{1\le i<j\le r} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i=1}^{r} \frac{(bq^{1+M}/Cx_i)_{|\mathbf{k}|-k_i}\,(ax_iq^{-M}/b)_{k_i}}{(b/Cx_i)_{|\mathbf{k}|-k_i}\,(bc_iq^{1+M}/Cx_i)_{|\mathbf{k}|}}$$
$$\times \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \frac{(ad/b,bq^{1+M}/d)_{|\mathbf{k}|}}{(adq^{-M}/b,bq/d)_{|\mathbf{k}|}} q^{|\mathbf{k}|},$$

and

$$b_{\mathbf{l}} = \prod_{1\le i<j\le r} \frac{c_iq^{l_i}/x_i - c_jq^{l_j}/x_j}{c_i/x_i-c_j/x_j} \prod_{i=1}^{r} \frac{1-bc_iq^{l_i+|\mathbf{l}|}/ax_i}{1-bc_i/ax_i} \prod_{i,j=1}^{r} \frac{1}{(qc_ix_j/c_jx_i)_{l_i}}$$
$$\times \frac{(q^{-M})_{|\mathbf{l}|}}{(bCq/ad,Cdq^{-M}/b)_{|\mathbf{l}|}} \prod_{i=1}^{r} \frac{(bc_i/ax_i)_{|\mathbf{l}|}\,(c_id/Cx_i, b^2c_iq^{1+M}/aCdx_i)_{l_i}}{(bc_iq^{1+M}/ax_i)_{l_i}} x_i^{|\mathbf{l}|}$$
$$\times (-1)^{(r-1)|\mathbf{l}|} a^{(r-1)|\mathbf{l}|} b^{-r|\mathbf{l}|} C^{|\mathbf{l}|} q^{|\mathbf{l}|+(1-r)\binom{|\mathbf{l}|}{2}},$$

by the $k_i \mapsto l_i$, $a \mapsto b/a$, $c_i \mapsto c_iq^{k_i}$, $d \mapsto q^{1-|\mathbf{k}|}/a$, $x_i \mapsto c_i/x_i$, $i=1,\dots,r$, case of Proposition 2.2. Therefore we must have (3.3a), with $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ as in (3.10a), and the above sequences $a_{\mathbf{k}}$ and $b_{\mathbf{n}}$. After simplifications and the simultaneous substitutions $b \mapsto d$, $d \mapsto bd/a$, $k_i \mapsto n_i$, $n_i \mapsto k_i$, $i=1,\dots,r$, we arrive at (4.2). □

Remark 4.4. Two special cases of Theorem 4.3 are of particular interest:

(1) If $c_i = q^{-k_i}$, for $i=1,\dots,r$, then the multilateral series in (4.1) gets truncated from below and from above so that the multiple sum is finite. By a polynomial argument, we can replace q^M by bc/aq. If we then perform the substitution $d \mapsto dq^{-|\mathbf{k}|}$ and replace k_i by m_i, $i=1,\dots,r$, we obtain an A_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation (cf. [25, Thm. 4.1]) which, via a polynomial argument, is equivalent to Proposition 2.3.

(2) If, in (4.1), we let $b\to\infty$, we can apply analytic continuation to replace q^M by bd/a. If we then perform the substitutions $d \mapsto dq^{-|\mathbf{k}|}$ and $c_i \mapsto c_iq^{-k_i}$, for $i=1,\dots,r$, we can repeatedly apply analytic continuation to replace

q^{k_i} by c_i/e_i, for $i=1,\dots,r$. After subsequent relabelling of parameters $b\mapsto d$, $c_i\mapsto a/e_i$, $d\mapsto b$, $e_i\mapsto c_i$, for $i=1,\dots,r$, we obtain exactly the A_r extension of Bailey's very-well-poised ${}_6\psi_6$ summation in Proposition 2.7.

Theorem 4.5 (A C_r balanced very-well-poised ${}_8\psi_8$ summation formula)**.** *Let a, b, $c_1,\dots,c_r$, d and $x_1,\dots,x_r$ be indeterminate, let $k_1,\dots,k_r$ be integers, and let M be a nonnegative integer. Then*

$$\sum_{n_1,\dots,n_r=-\infty}^{\infty} q^{|\mathbf{n}|}\prod_{1\le i<j\le r}\frac{x_iq^{n_i}-x_jq^{n_j}}{x_i-x_j}\prod_{1\le i\le j\le r}\frac{1-ax_ix_jq^{n_i+n_j}}{1-ax_ix_j}$$
$$\times\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j,ax_ix_jq^{-k_j}/c_j)_{n_i}}{(ax_ix_jq/c_i,q^{1+k_j}c_jx_i/x_j)_{n_i}}\prod_{i=1}^{r}\frac{(bx_i,dx_iq^{|\mathbf{k}|},ax_iq^{1+M}/b,ax_iq^{-M}/d)_{n_i}}{(ax_iq/b,ax_iq^{1-|\mathbf{k}|}/d,bx_iq^{-M},dx_iq^{1+M})_{n_i}}$$
$$=\frac{\prod_{i,j=1}^{r}(qc_jx_i/c_ix_j,qx_i/x_j)_\infty\prod_{1\le i\le j\le r}(ax_ix_jq,q/ax_ix_j)_\infty}{\prod_{i,j=1}^{r}(qc_jx_i/x_j,qx_i/c_ix_j,ax_ix_jq/c_i,c_jq/ax_ix_j)_\infty}$$
$$\times\prod_{1\le i<j\le r}(ax_ix_jq/c_ic_j,c_ic_jq/ax_ix_j)_\infty\prod_{i=1}^{r}\frac{(ax_iq/c_id,c_idq/ax_i,dx_iq/c_i,c_iq/dx_i)_\infty}{(ax_iq/d,dq/ax_i,dx_iq,q/dx_i)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(ax_iq/bc_i,c_iq/bx_i,dx_iq,dq/ax_i)_M\,(c_id/ax_i)_{|\mathbf{k}|}}{(c_idq/ax_i,dx_iq/c_i,q/bx_i,ax_iq/b)_M\,(dx_i,d/ax_i)_{|\mathbf{k}|}}\prod_{1\le i<j\le r}(c_ic_j/ax_ix_j)^{-1}_{k_i+k_j}$$
$$\times\frac{(bd/a,dq^{1+M}/b,q^{-M})_{|\mathbf{k}|}\prod_{i=1}^{r}(dx_i/c_i)_{|\mathbf{k}|-k_i}}{\prod_{i=1}^{r}(c_iq/bx_i,bc_iq^{-M}/ax_i,c_idq^{1+M}/ax_i)_{k_i}}\prod_{i,j=1}^{r}\frac{(qc_jx_i/x_j,c_jq/ax_ix_j)_{k_j}}{(qc_jx_i/c_ix_j)_{k_j}}.\tag{4.3}$$

Proof. We combine the multilateral matrix inverse in Theorem 3.5 with the D_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation in Proposition 2.5, using the inverse relations (3.3).

In particular, we have (3.3b), with $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ as in (3.12b),

$$a_{\mathbf{k}}=\prod_{i=1}^{r}\frac{(bc_iq/ax_i,bx_iq/c_i,bq/adx_i,bx_iq/d)_M\,(adx_i/b,bx_iq^{1+M}/d,ax_iq^{-M}/b)_{k_i}}{(bx_iq/c_id,bc_iq/adx_i,bx_iq,bq/ax_i)_M\,(adx_iq^{1-M}/b,bx_iq^{1+M},bx_iq/d)_{k_i}}$$
$$\times q^{|\mathbf{k}|}\prod_{1\le i<j\le r}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{1\le i\le j\le r}\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j)_{k_i}}{(ax_ix_jq/c_j)_{k_i}},$$

and

$$b_{\mathbf{l}}=\prod_{1\le i<j\le r}\frac{c_iq^{l_i}/x_i-c_jq^{l_j}/x_j}{c_i/x_i-c_j/x_j}\prod_{i=1}^{r}\frac{1-bc_iq^{l_i+|\mathbf{l}|}/ax_i}{1-bc_i/ax_i}\prod_{i,j=1}^{r}\frac{1}{(qc_ix_j/c_jx_i)_{l_i}}$$
$$\times\prod_{1\le i<j\le r}\frac{1}{(c_ic_j/ax_ix_j)_{l_i+l_j}}\prod_{i=1}^{r}(bc_i/ax_i)_{|\mathbf{l}|}(bx_iq/c_i)_{|\mathbf{l}|-l_i}c_i^{rl_i}x_i^{-rl_i}$$

$$\times \frac{(d, b^2q^{1+M}/ad, q^{-M})_{|\mathbf{l}|}}{\prod_{i=1}^r (bc_iq/adx_i, dc_iq^{-M}/bx_i, bc_iq^{1+M}/ax_i)_{l_i}} b^{-r|\mathbf{l}|} q^{-r\binom{|\mathbf{l}|}{2}+r\sum_{i=1}^r \binom{l_i+1}{2}},$$

by the $k_i \mapsto l_i$, $a \mapsto b/a$, $b \mapsto d$, $c_i \mapsto c_iq^{k_i}$, $cd \mapsto 1/a$, $x_i \mapsto c_i/x_i$, $i = 1, \dots, r$, case of Proposition 2.5. Therefore we must have (3.3a), with $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ as in (3.12a), and the above sequences $a_{\mathbf{k}}$ and $b_{\mathbf{n}}$. After simplifications and the simultaneous substitutions $b \mapsto d$, $d \mapsto bd/a$, $k_i \mapsto n_i$, $n_i \mapsto k_i$, $i = 1, \dots, r$, we arrive at (4.3). □

Remark 4.6. Two special cases of Theorem 4.5 are of particular interest:

(1) If $c_i = q^{-k_i}$, for $i = 1, \dots, r$, then the multilateral series in (4.3) gets truncated from below and from above so that the multiple sum is finite. By a polynomial argument, we can replace q^M by bc/aq. If we then perform the substitution $d \mapsto dq^{-|\mathbf{k}|}$ and replace k_i by m_i, $i = 1, \dots, r$, we obtain the C_r extension of Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation in Proposition 2.4.

(2) If, in (4.3), we let $M \to \infty$ and perform the substitution $d \mapsto dq^{-|\mathbf{k}|}$, we can repeatedly apply analytic continuation to replace q^{k_i} by a/c_ie_i for $i = 1, \dots, r$ (in order to relax the integrality condition of the k_i's), where $e_1, \dots, e_r$ are new complex parameters. We then obtain the C_r extension of Bailey's very-well-poised ${}_6\psi_6$ summation in Proposition 2.8.

Appendix A

It is indeed quite interesting that concerning the multilateral matrix inversion result in Theorem 3.5, $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ is the left-inverse of $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$, but *not* the right-inverse (unless in special cases, e.g., when both matrices are upper- or lower-triangular). Let us assume, for a moment, that $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ would also be the right-inverse of $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ (which cannot be justified, see also Remark 3.6). By combining these matrices with the C_r very-well-poised ${}_6\psi_6$ summation formula in Proposition 2.8, by virtue of multidimensional inverse relations one would be able to deduce a "new" D_r very-well-poised ${}_6\psi_6$ summation formula which, however, turns out to be *false* for $r > 1$, as the series does not converge.

In particular, we have (3.4b) with $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ as in (3.12b),

$$a_{\mathbf{l}} = \prod_{1\le i<j\le r} (ax_ix_jq/c_ic_j, aq/e_ie_jx_ix_j)_\infty \prod_{1\le i\le j\le r} (ax_ix_jq, q/ax_ix_j)_\infty$$
$$\times \frac{(bq/d)_\infty}{(a^{r+1}bq/CdE)_\infty} \prod_{i,j=1}^r \frac{(ax_iq/c_ie_jx_j, qx_i/x_j)_\infty}{(ax_iq/e_jx_j, q/e_jx_ix_j, ax_ix_jq/c_i, qx_i/c_ix_j)_\infty}$$
$$\times \prod_{i=1}^r \frac{(ax_iq/c_id, aq/de_ix_i, bx_iq/c_i, bq/e_ix_i)_\infty}{(bx_iq, bq/ax_i, ax_iq/d, q/dx_i)_\infty}$$

$$\times \prod_{i,j=1}^{r} (c_ie_jx_j/ax_i)_{l_i} \prod_{1\le i<j\le r} (c_ic_j/ax_ix_j)_{l_i+l_j} \prod_{i=1}^{r} \frac{(c_id/ax_i)_{l_i}}{(bq/e_ix_i)_{|\mathbf{l}|}\,(bx_iq/c_i)_{|\mathbf{l}|-l_i}}$$

$$\times \frac{1}{(bq/d)_{|\mathbf{l}|}} \left(\frac{a^rb^r}{CdE}\right)^{|\mathbf{l}|} q^{(r-1)\left(\binom{|\mathbf{l}|}{2}-\sum_{i=1}^{r}\binom{l_i+1}{2}\right)} \prod_{i=1}^{r} \left(\frac{x_i}{c_i}\right)^{(r-1)l_i}$$

and

$$b_{\mathbf{k}} = \prod_{i,j=1}^{r} \frac{(e_jx_ix_j)_{k_i}}{(ax_iq/e_jx_j)_{k_i}} \prod_{i=1}^{r} \frac{(dx_i)_{k_i}}{(ax_iq/d)_{k_i}} \cdot \left(\frac{a^rb}{CdE}\right)^{|\mathbf{k}|},$$

by the $b \mapsto aq^{-|\mathbf{l}|}/b$, $c_i \mapsto c_iq^{l_i}$, $i = 1, \dots, r$, case of Proposition 2.8. As we are (erroneously) assuming that $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ is the right-inverse of $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$, we must have (3.4a), with $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ as in (3.5a), and the above sequences $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After simplifications and the simultaneous substitutions $a \mapsto bc/a^2$, $b \mapsto bc/a$, $c_i \mapsto aq^{-k_i}/c_i$, $d \mapsto cd/a$, $e_i \mapsto bcc_ie_iq^{k_i}/a^3$, $x_i \mapsto aq^{-k_i}/c_ix_i$, $i = 1, \dots, r$, we can get rid of the k_i and would arrive at the following identity:

$$\sum_{n_1,\dots,n_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \frac{x_iq^{n_i}-x_jq^{n_j}}{x_i-x_j} \prod_{i=1}^{r} \frac{1-ax_iq^{n_i+|\mathbf{n}|}}{1-ax_i} \prod_{1\le i<j\le r} (a^2x_ix_j/bc)_{n_i+n_j}$$

$$\times \prod_{i,j=1}^{r} \frac{(e_jx_i/x_j)_{n_i}}{(ax_iq/c_jx_j, ac_ix_ix_jq/bc)_{n_i}} \prod_{i=1}^{r} \frac{(bc/c_ix_i, c_ix_i)_{|\mathbf{n}|}\,(dx_i)_{n_i}}{(ax_iq/e_i)_{|\mathbf{n}|}\,(bc/ax_i)_{|\mathbf{n}|-n_i}}\, x_i^{n_i}$$

$$\times \frac{1}{(aq/d)_{|\mathbf{n}|}} \left(\frac{a^2q}{bcdE}\right)^{|\mathbf{n}|} q^{-e_2(\mathbf{n})}$$

$$= \prod_{1\le i<j\le r} \frac{(a^2x_ix_jq/bc)_\infty}{(a^2x_ix_jq/bce_ie_j)_\infty} \prod_{i,j=1}^{r} \frac{(ax_iq/c_je_ix_j, ac_ix_ix_jq/bce_j, qx_i/x_j)_\infty}{(ax_iq/c_jx_j, ac_ix_ix_jq/bc, qx_i/e_ix_j)_\infty}$$

$$\times \frac{(aq/dE)_\infty}{(aq/d)_\infty} \prod_{i=1}^{r} \frac{(ax_iq, q/ax_i, ax_iq/bc, aq/c_idx_i, ac_ix_iq/bcd)_\infty}{(ax_iq/e_i, c_ix_iq/bc, q/c_ix_i, q/dx_i, a^2x_iq/bcde_i)_\infty}, \quad \text{(A.1)}$$

where $e_2(\mathbf{n})$ is the second elementary symmetric function of $(n_1, \dots, n_r)$.

Now, due to the factor $q^{-e_2(\mathbf{n})}$ appearing in the summand, the series in (A.1) does *not* converge for $r \ge 2$. Therefore, the identity as stated is false. However, it is valid for $r \ge 2$ whenever the series terminates. For instance, when $c_i = a$ and $e_i = q^{-m_i}$, for $i = 1, \dots, r$, (A.1) reduces to Bhatnagar's D_r terminating very-well-poised ${}_6\phi_5$ summation, derived in [4, Thm. 2].

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