SUMMATIONS AND TRANSFORMATIONS FOR MULTIPLE BASIC AND ELLIPTIC HYPERGEOMETRIC SERIES BY DETERMINANT EVALUATIONS

HJALMAR ROSENGREN AND MICHAEL SCHLOSSER*

Dedicated to Tom Koornwinder

ABSTRACT. Using multiple q-integrals and a determinant evaluation, we establish a multivariable extension of Bailey's nonterminating ${}_{10}\phi_9$ transformation. From this result, we deduce new multivariable terminating ${}_{10}\phi_9$ transformations, ${}_8\phi_7$ summations and other identities. We also use similar methods to derive new multivariable ${}_1\psi_1$ summations. Some of our results are extended to the case of elliptic hypergeometric series.

1. INTRODUCTION

Basic hypergeometric series have various applications in combinatorics, number theory, representation theory, statistics, and physics, see Andrews [1], [2]. For a general account of the importance of basic hypergeometric series in the theory of special functions see Andrews, Askey, and Roy [3].

There are different types of multivariable basic hypergeometric series extending the classical one-dimensional theory [11]. Several recent multivariable extensions can be associated to *root systems* or, equivalently, to *Lie algebras*. The specific series we consider in this paper have the advantage that they can be conveniently studied from a purely analytical point of view. In this respect, we understand the root system terminology used in this paper (such as "A_{r-1} series", or "C_r series") simply as a classification of certain multiple series according to specific factors (such as a Vandermonde determinant) appearing in the summand. We omit giving a precise definition here, but refer instead to papers of Bhatnagar [6] or Milne [18, Sec. 5]. We mention that, although these series first arose (in the limit $q \to 1$) in the representation theory of compact Lie groups [14], many questions remain about this connection. In particular, there is no known (quantum) group interpretation of the type of series that we will study.

In this paper, we give a multivariable nonterminating ${}_{10}\phi_9$ transformation for the root system C_r (or, equivalently, the symplectic group $Sp(r)$), see Corollary 4.1. Our result extends a C_r nonterminating ${}_8\phi_7$ summation recently found by one of us in [24]. To our knowledge, our new transformation formula is the first multivariable generalization of Bailey's nonterminating ${}_{10}\phi_9$ transformation (see (2.4)

Date: first version: April 17, 2003; revised: June 16, 2003.

2000 *Mathematics Subject Classification.* Primary 33D67; Secondary 05A30, 33D05.

Key words and phrases. ${}_{10}\phi_9$ transformations, ${}_8\phi_7$ summations, ${}_1\psi_1$ summations, basic hypergeometric series, multiple q-integrals, C_r series, A_{r-1} series, elliptic hypergeometric series.

*The second author was supported by an APART grant of the Austrian Academy of Sciences.

below) that has appeared in the literature. We deduce this result from an equivalent multiple q-integral transformation, Theorem 3.1. In our proof of the latter we utilize a simple determinant method, essentially the same which was introduced by Gustafson and Krattenthaler [12] and which was further exploited by one of us in [23] and [24] to derive a number of identities for multiple basic hypergeometric series. This method was also recently employed by Spiridonov [22, Th. 3] who used it to derive a C_r elliptic Nasrallah–Rahman integral. Here, as in [24], we apply the determinant method to *q-integrals* to derive our main result. We also derive some new multivariable extensions of Ramanujan's ${}_1\psi_1$ summation.

In the final section we briefly discuss *elliptic* extensions of some of our identities. Elliptic (or modular) hypergeometric series is a recently introduced extension of basic hypergeometric series, which was motivated by certain models in statistical mechanics [10]. Warnaar [27] used the determinant method to give an elliptic analogue of a C_r Jackson summation from [23]. This identity was used by one of us [21] to prove a second elliptic C_r Jackson summation conjectured by Warnaar (the basic case of this identity was proved by van Diejen and Spiridonov [8]; cf. [20] for a third proof), as well as to prove an elliptic analogue of a third C_r Jackson summation due to Denis and Gustafson [7] and Milne and Newcomb [19]; cf. [22]. (Spiridonov and van Diejen showed that the second and third summation also follow from certain conjectured multiple integral evaluations [8], [9].) Note that although all three Jackson summations are connected to the root system C_r, they are different in nature. We want to stress here that the first summation, which is related to the present work, is apparently the simplest but may be used to prove the other two. We hope that our new identities will also be useful to study different, apparently more complicated, types of multivariable series.

Our paper is organized as follows: In Section 2, we review some basics in the theory of basic hypergeometric series. We also note a determinant lemma which we need as an ingredient in proving our results. In Section 3, we derive a multiple q-integral transformation, Theorem 3.1, which in Section 4 is used to explicitly write out a nonterminating ${}_{10}\phi_9$ transformation for the root system C_r, see Corollary 4.1. In Section 5 we specialize this identity to obtain new terminating C_r ${}_{10}\phi_9$ transformations. In Section 6 we derive new multivariable extensions of Ramanujan's ${}_1\psi_1$ summation. In Section 7 we give a number of special and limit cases of our multivariable ${}_{10}\phi_9$ transformations. Finally, in Section 8 we prove that our terminating C_r ${}_{10}\phi_9$ transformations and ${}_8\phi_7$ summations extend to the case of elliptic hypergeometric series.

2. Basic hypergeometric series and a determinant lemma

Here we recall some standard notation for basic hypergeometric series (cf. [11]).

Let q be a complex number (called the "base") such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{and} \qquad (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

Since we are working with the same base q throughout this article, we omit writing it out explicitly, i.e., we use

$$(a)_k := (a;q)_k, \tag{2.1}$$

where k is an integer or infinity. Further, we employ the condensed notation

$$(a_1,\dots,a_m)_k \equiv (a_1)_k\dots(a_m)_k,$$

where k is an integer or infinity. We denote the *basic hypergeometric ${}_s\phi_{s-1}$ series* by

$$ {}_s\phi_{s-1}\left[\begin{matrix}a_1,a_2,\dots,a_s\\ b_1,b_2,\dots,b_{s-1}\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_s)_k}{(q,b_1,\dots,b_{s-1})_k}z^k, \tag{2.2}$$

and the *bilateral basic hypergeometric ${}_s\psi_s$ series* by

$$ {}_s\psi_s\left[\begin{matrix}a_1,a_2,\dots,a_s\\ b_1,b_2,\dots,b_s\end{matrix};q,z\right] := \sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\dots,a_s)_k}{(b_1,b_2,\dots,b_s)_k}z^k, \tag{2.3}$$

respectively. See [11, p. 25 and p. 125] for the criteria of when these series terminate, or, if not, when they converge.

The classical theory of basic hypergeometric series contains numerous summation and transformation formulae involving ${}_s\phi_{s-1}$ or ${}_s\psi_s$ series. Many of these require that the parameters satisfy the condition of being either balanced and/or very-well-poised. An ${}_s\phi_{s-1}$ basic hypergeometric series is called *balanced* if $b_1\cdots b_{s-1} = a_1\cdots a_s q$ and $z=q$. An ${}_s\phi_{s-1}$ series is *well-poised* if $a_1q = a_2b_1 = \cdots = a_sb_{s-1}$ and *very-well-poised* if it is well-poised and if $a_2 = -a_3 = q\sqrt{a_1}$. Note that the factor

$$\frac{1-a_1q^{2k}}{1-a_1}$$

appears in a very-well-poised series. Similarly, a bilateral ${}_s\psi_s$ basic hypergeometric series is well-poised if $a_1b_1 = a_2b_2\cdots = a_sb_s$ and very-well-poised if, in addition, $a_1 = -a_2 = qb_1 = -qb_2$.

The standard reference for basic hypergeometric series is Gasper and Rahman's text [11]. In our computations throughout this paper we frequently use some elementary identities of q-shifted factorials, listed in [11, Appendix I].

In the following we display some identities which we utilize in the subsequent sections.

Bailey's [5, Eq. (7.2)] nonterminating very-well-poised ${}_{10}\phi_9$ summation,

$$\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix}a,\ qa^{\frac12},-qa^{\frac12},b,c,d,e,f,g,h\\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h\end{matrix};q,q\right]\\
&\qquad+\frac{(aq,b/a,c,d,e,f,g,h)_\infty}{(b^2q/a,a/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h)_\infty}\\
&\qquad\times\frac{(bq/c,bq/d,bq/e,bq/f,bq/g,bq/h)_\infty}{(bc/a,bd/a,be/a,bf/a,bg/a,bh/a)_\infty}\\
&\quad\times{}_{10}\phi_9\left[\begin{matrix}b^2/a,\ qba^{-\frac12},-qba^{-\frac12},b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a\\ ba^{-\frac12},-ba^{-\frac12},bq/a,bq/c,bq/d,bq/e,bq/f,bq/g,bq/h\end{matrix};q,q\right]\\
&\qquad=\frac{(aq,b/a,\lambda q/f,\lambda q/g,\lambda q/h,bf/\lambda,bg/\lambda,bh/\lambda)_\infty}{(\lambda q,b/\lambda,aq/f,aq/g,aq/h,bf/a,bg/a,bh/a)_\infty}\\
&\quad\times{}_{10}\phi_9\left[\begin{matrix}\lambda,\ q\lambda^{\frac12},-q\lambda^{\frac12},b,\lambda c/a,\lambda d/a,\lambda e/a,f,g,h\\ \lambda^{\frac12},-\lambda^{\frac12},\lambda q/b,aq/c,aq/d,aq/e,\lambda q/f,\lambda q/g,\lambda q/h\end{matrix};q,q\right]\\
&\qquad+\frac{(aq,b/a,f,g,h,bq/f,bq/g,bq/h)_\infty}{(b^2q/\lambda,\lambda/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h)_\infty}
\end{aligned}$$

$$\times \frac{(\lambda c/a, \lambda d/a, \lambda e/a, abq/\lambda c, abq/\lambda d, abq/\lambda e)_\infty}{(bc/a, bd/a, be/a, bf/a, bg/a, bh/a)_\infty}$$
$$\times {}_{10}\phi_9\left[\begin{matrix} b^2/\lambda,\ qb\lambda^{-\frac{1}{2}}, -qb\lambda^{-\frac{1}{2}}, b, bc/a, bd/a, be/a, bf/\lambda, bg/\lambda, bh/\lambda \\ b\lambda^{-\frac{1}{2}}, -b\lambda^{-\frac{1}{2}}, bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, bq/f, bq/g, bq/h \end{matrix}; q, q\right], \quad (2.4)$$

where $\lambda = a^2q/cde$ and $a^3q^2 = bcdefgh$ (cf. [11, Eq. (2.12.9)]), stands on the top of the classical hierarchy of transformation formulae for basic hypergeometric series. If $h = q^{-n}$, where $n = 0, 1, 2, \dots$, it reduces to Bailey's terminating very-well-poised ${}_{10}\phi_9$ transformation (cf. [11, Eq. (2.9.1)]):

$$ {}_{10}\phi_9\left[\begin{matrix} a,\ qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, g, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq^{1+n} \end{matrix}; q, q\right]$$
$$= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q)_n}$$
$$\times {}_{10}\phi_9\left[\begin{matrix} \lambda,\ q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, g, q^{-n} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, \lambda q/g, \lambda q^{1+n} \end{matrix}; q, q\right], \quad (2.5)$$

where $\lambda = a^2q/bcd$ and $a^3q^{3+n} = bcdefg$. For $cd = aq$ (hence $b\lambda = a$) Bailey's transformation reduces to Jackson's [16] terminating very-well-poised balanced ${}_8\phi_7$ summation which, after substitution of variables ($f \mapsto c$, $g \mapsto d$) can be written as (cf. [11, Eq. (2.6.2)])

$$ {}_8\phi_7\left[\begin{matrix} a,\ qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{1+n} \end{matrix}; q, q\right]$$
$$= \frac{(aq, aq/bc, aq/bd, aq/cd)_n}{(aq/b, aq/c, aq/d, aq/bcd)_n}, \quad (2.6)$$

where $a^2q^{1+n} = bcde$. On the other hand, if $cd = aq$ (i.e., $e\lambda = a$) (2.4) reduces to Bailey's nonterminating two-term ${}_8\phi_7$ summation. These identities contain many other important summation and transformation formulae for basic hypergeometric series. For a sequence of derivations leading up to the nonterminating ${}_{10}\phi_9$ transformation, see the exposition in Gasper and Rahman [11, Section 2].

For studying nonterminating basic hypergeometric series it is often convenient to utilize Jackson's [15] q-integral notation, defined by

$$\int_a^b f(t)\, d_qt = \int_0^b f(t)\, d_qt - \int_0^a f(t)\, d_qt, \quad (2.7)$$

where

$$\int_0^a f(t)\, d_qt = a(1-q)\sum_{k=0}^{\infty} f(aq^k)q^k. \quad (2.8)$$

If f is continuous on $[0, a]$, then it is easily seen that

$$\lim_{q\to 1^-}\int_0^a f(t)\, d_qt = \int_0^a f(t)\, dt,$$

see [11, Eq. (1.11.6)].

Using the above q-integral notation, the nonterminating ${}_{10}\phi_9$ transformation (2.4) can be expressed as

$$\int_a^b \frac{(1-t^2/a)(qt/a, qt/b, qt/c, qt/d, qt/e, qt/f, qt/g, qt/h)_\infty}{(t, bt/a, ct/a, dt/a, et/a, ft/a, gt/a, ht/a)_\infty}\, d_qt$$

$$= \frac{a}{\lambda} \frac{(b/a, aq/b, \lambda c/a, \lambda d/a, \lambda e/a, bf/\lambda, bg/\lambda, bh/\lambda)_\infty}{(b/\lambda, \lambda q/b, c, d, e, bf/a, bg/a, bh/a)_\infty}$$
$$\times \int_\lambda^b \frac{(1-t^2/\lambda)(qt/\lambda, qt/b, aqt/c\lambda, aqt/d\lambda, aqt/e\lambda, qt/f, qt/g, qt/h)_\infty}{(t, bt/\lambda, ct/a, dt/a, et/a, ft/\lambda, gt/\lambda, ht/\lambda)_\infty} d_q t, \quad (2.9)$$

where $\lambda = a^2q/cde$ and $a^3q^2 = bcdefgh$ (cf. [11, Eq. (2.12.10)]).

One of the most important summmation theorems for bilateral basic hypergeometric series is Ramanujan's ${}_1\psi_1$ summation [13, Eq. (12.12.2)], which reads

$${}_1\psi_1\left[\begin{matrix}a\\b\end{matrix}; q, z\right] = \frac{(q, az, q/az, b/a)_\infty}{(b, z, b/az, q/a)_\infty}, \quad (2.10)$$

where $|b/a| < |z| < 1$ (cf. [11, Eq. (5.2.1)]). This beautiful formula contains several important identities as special cases, see Gasper and Rahman [11].

We conclude this section with a determinant evaluation, which will be our main tool. It was given explicitly in [23, Lemma A.1], as a special case of a determinant lemma by Krattenthaler [17, Lemma 34].

Lemma 2.1. *Let $X_1, \dots, X_r$, A, B, and C be indeterminate. Then there holds*

$$\det_{1\le i,j\le r}\left(\frac{(AX_i, AC/X_i)_{r-j}}{(BX_i, BC/X_i)_{r-j}}\right) = \prod_{1\le i<j\le r}(X_j - X_i)(1 - C/X_iX_j)$$
$$\times A^{\binom{r}{2}} q^{\binom{r}{3}} \prod_{i=1}^r \frac{(B/A, ABCq^{2r-2i})_{i-1}}{(BX_i, BC/X_i)_{r-1}}. \quad (2.11)$$

The above determinant evaluation was generalized to the elliptic case (more precisely, to an evaluation involving Jacobi theta functions) by Warnaar [27, Cor. 5.4]. We make use of the elliptic version of Lemma 2.1 in Section 8.

3. A multiple q-integral transformation

By iteration, the extension of (2.8) to *multiple* q-integrals is straightforward:

$$\int_0^{a_1} \dots \int_0^{a_r} f(t_1, \dots, t_r)\, d_qt_r \dots d_qt_1$$
$$= a_1 \dots a_r (1-q)^r \sum_{k_1,\dots,k_r=0}^\infty f(a_1q^{k_1}, \dots, a_rq^{k_r}) q^{k_1+\dots+k_r}. \quad (3.1)$$

Similarly, the extension of (2.7) is

$$\int_{a_1}^{b_1} \dots \int_{a_r}^{b_r} f(t_1, \dots, t_r)\, d_qt_r \dots d_qt_1$$
$$= \sum_{S\subseteq\{1,2,\dots,r\}} \left(\prod_{i\in S}(-a_i)\right)\left(\prod_{i\notin S} b_i\right)(1-q)^r$$
$$\times \sum_{k_1,\dots,k_r=0}^\infty f(c_1(S)q^{k_1}, \dots, c_r(S)q^{k_r}) q^{k_1+\dots+k_r}, \quad (3.2)$$

where the outer sum runs over all 2^r subsets S of $\{1, 2, \dots, r\}$, and where $c_i(S) = a_i$ if $i \in S$ and $c_i(S) = b_i$ if $i \notin S$, for $i = 1, \dots, r$.

We give our main result, a C_r extension of (2.9):

Theorem 3.1. *Let $a^3q^{3-r}=bc_id_ie_ix_ifgh$ and $\lambda=a^2q/c_id_ie_ix_i$ for $i=1,\dots,r$. Then there holds*

$$\begin{aligned}
&\int_{ax_1}^{b}\cdots\int_{ax_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)\prod_{i=1}^{r}(1-t_i^2/a)\frac{(qt_i/ax_i,qt_i/b)_\infty}{(t_ix_i,bt_i/a)_\infty}\\
&\quad\times\prod_{i=1}^{r}\frac{(qt_i/c_i,qt_i/d_i,qt_i/e_i,qt_i/f,qt_i/g,qt_i/h)_\infty}{(c_it_i/a,d_it_i/a,e_it_i/a,ft_i/a,gt_i/a,ht_i/a)_\infty}\,d_qt_r\dots d_qt_1\\
&\quad=\left(\frac{a}{\lambda}\right)^{\binom{r+1}{2}}\prod_{i=1}^{r}\frac{(b/ax_i,ax_iq/b,\lambda c_ix_i/a,\lambda d_ix_i/a,\lambda e_ix_i/a)_\infty}{(b/\lambda x_i,\lambda x_iq/b,c_ix_i,d_ix_i,e_ix_i)_\infty}\\
&\qquad\times\prod_{i=1}^{r}\frac{(bfq^{i-1}/\lambda,bgq^{i-1}/\lambda,bhq^{i-1}/\lambda)_\infty}{(bfq^{i-1}/a,bgq^{i-1}/a,bhq^{i-1}/a)_\infty}\\
&\times\int_{\lambda x_1}^{b}\cdots\int_{\lambda x_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/\lambda)\prod_{i=1}^{r}(1-t_i^2/\lambda)\frac{(qt_i/\lambda x_i,qt_i/b)_\infty}{(t_ix_i,bt_i/\lambda)_\infty}\\
&\times\prod_{i=1}^{r}\frac{(aqt_i/c_i\lambda,aqt_i/d_i\lambda,aqt_i/e_i\lambda,qt_i/f,qt_i/g,qt_i/h)_\infty}{(c_it_i/a,d_it_i/a,e_it_i/a,ft_i/\lambda,gt_i/\lambda,ht_i/\lambda)_\infty}\,d_qt_r\dots d_qt_1. \qquad (3.3)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)&=\prod_{i=1}^{r}\frac{(q^{2-r}t_i/g,aq^{2-r}/gt_i)_{r-1}}{(aq^{2-r}/fg,fq^{2+r-2i}/g)_{i-1}}t_i^{r-1}\\
&\quad\times f^{-\binom{r}{2}}q^{-\binom{r}{3}}\det_{1\le i,j\le r}\left(\frac{(ft_i/a,f/t_i)_{r-j}}{(q^{2-r}t_i/g,aq^{2-r}/gt_i)_{r-j}}\right),
\end{aligned}$$

due to the $X_i\mapsto t_i$, $A\mapsto f/a$, $B\mapsto q^{2-r}/g$, and $C\mapsto a$ case of Lemma 2.1. Hence, using some elementary identities from [11, Appendix I], we may write the left-hand side of (3.3) as

$$\begin{aligned}
&\left(\frac{a}{g}\right)^{\binom{r}{2}}q^{-\binom{r}{3}}\prod_{i=1}^{r}(aq^{2-r}/fg,fq^{2+r-2i}/g)_{i-1}^{-1}\\
&\qquad\times\det_{1\le i,j\le r}\left(\int_{ax_i}^{b}\frac{(1-t_i^2/a)(t_iq/ax_i,t_iq/b)_\infty}{(t_ix_i,bt_i/a)_\infty}\right.\\
&\qquad\left.\times\frac{(t_iq/c_i,t_iq/d_i,t_iq/e_i,t_iq^{1-r+j}/f,t_iq^{2-j}/g,t_iq/h)_\infty}{(c_it_i/a,d_it_i/a,e_it_i/a,ft_iq^{r-j}/a,gt_iq^{j-1}/a,ht_i/a)_\infty}\,d_qt_i\right).
\end{aligned}$$

Now, to the integral inside the determinant we apply the q-integral transformation (2.9), with the substitution $t\mapsto t_ix_i$, and the replacements $a\mapsto ax_i^2$, $b\mapsto bx_i$, $c\mapsto c_ix_i$, $d\mapsto d_ix_i$, $e\mapsto e_ix_i$, $f\mapsto fq^{r-j}x_i$, $g\mapsto gq^{j-1}x_i$, and $h\mapsto hx_i$. Thus we obtain

$$\begin{aligned}
&\left(\frac{a}{g}\right)^{\binom{r}{2}}q^{-\binom{r}{3}}\prod_{i=1}^{r}(aq^{2-r}/fg,fq^{2+r-2i}/g)_{i-1}^{-1}\\
&\qquad\times\det_{1\le i,j\le r}\left(\frac{a}{\lambda}\frac{(b/ax_i,ax_iq/b,\lambda c_ix_i/a,\lambda d_ix_i/a,\lambda e_ix_i/a)_\infty}{(b/\lambda x_i,\lambda x_iq/b,c_ix_i,d_ix_i,e_ix_i)_\infty}\right.
\end{aligned}$$

$$\times \frac{(bfq^{r-j}/\lambda, bgq^{j-1}/\lambda, bh/\lambda)_\infty}{(bfq^{r-j}/a, bgq^{j-1}/a, bh/a)_\infty} \int_{\lambda x_i}^{b} \frac{(1-t_i^2/a)(t_iq/\lambda x_i, t_iq/b)_\infty}{(t_ix_i, bt_i/\lambda)_\infty}$$
$$\times \frac{(at_iq/c_i\lambda, at_iq/d_i\lambda, at_iq/e_i\lambda, t_iq^{1-r+j}/f, t_iq^{2-j}/g, t_iq/h)_\infty}{(c_it_i/a, d_it_i/a, e_it_i/a, ft_iq^{r-j}/\lambda, gt_iq^{j-1}/\lambda, ht_i/\lambda)_\infty} \, d_qt_i \Bigg).$$

Now, by using linearity of the determinant with respect to rows and columns, we take some factors out of the determinant and obtain

$$\left(\frac{a}{f\lambda}\right)^{\binom{r}{2}} q^{-\binom{r}{3}} \left(\frac{a}{\lambda}\right)^r \prod_{i=1}^{r} (aq^{2-r}/fg, fq^{2+r-2i}/g)_{i-1}$$
$$\times \prod_{i=1}^{r} \frac{(b/ax_i, ax_iq/b, \lambda c_ix_i/a, \lambda d_ix_i/a, \lambda e_ix_i/a, bfq^{i-1}/\lambda, bgq^{i-1}/\lambda, bh/\lambda)_\infty}{(b/\lambda x_i, \lambda x_iq/b, c_ix_i, d_ix_i, e_ix_i, bfq^{i-1}/a, bgq^{i-1}/a, bh/a)_\infty}$$
$$\times \int_{\lambda x_1}^{b} \dots \int_{\lambda x_r}^{b} \prod_{i=1}^{r} (1-t_i^2/\lambda) \frac{(qt_i/\lambda x_i, qt_i/b)_\infty}{(t_ix_i, bt_i/\lambda)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(aqt_i/c_i\lambda, aqt_i/d_i\lambda, aqt_i/e_i\lambda, qt_i/f, qt_i/g, qt_i/h)_\infty \, t_i^{r-1}}{(c_it_i/a, d_it_i/a, e_it_i/a, ft_i/\lambda, gt_i/\lambda, ht_i/\lambda)_\infty (t_iq^{1+r-j/f}, \lambda q^{2-r}/gt_i)_{r-1}}$$
$$\times \det_{1\le i,j\le r} \left(\frac{(ft_i/\lambda, f/t_i)_{r-j}}{(q^{2-r}t_i/g, \lambda q^{2-r}/gt_i)_{r-j}} \right) d_qt_r \dots d_qt_1. \quad (3.4)$$

The determinant can be evaluated by means of Lemma 2.1 with $X_i \mapsto t_i$, $A \mapsto f/\lambda$, $B \mapsto q^{2-r}/g$, and $C \mapsto \lambda$; specifically

$$\det_{1\le i,j\le r} \left(\frac{(ft_i/\lambda, f/t_i)_{r-j}}{(q^{2-r}t_i/g, \lambda q^{2-r}/gt_i)_{r-j}} \right)$$
$$= f^{\binom{r}{2}} q^{\binom{r}{3}} \prod_{1\le i<j\le r} (t_i - t_j)(1 - t_it_j/\lambda) \prod_{i=1}^{r} \frac{(\lambda q^{2-r}/fg, fq^{2+r-2i}/g)_{i-1}}{(q^{2-r}t_i/g, \lambda q^{2-r}/gt_i)_{r-1}} t_i^{1-r}.$$

Now note that $bh/\lambda = aq^{2-r}/fg$ and $bh/a = \lambda q^{2-r}/fg$ whence we have

$$\prod_{i=1}^{r} \frac{(bh/\lambda)_\infty (\lambda q^{2-r}/fg)_{i-1}}{(bh/a)_\infty (aq^{2-r}/fg)_{i-1}} = \prod_{i=1}^{r} \frac{(bhq^{i-1}/\lambda)_\infty}{(bhq^{i-1}/a)_\infty}.$$

Substituting these calculations into (3.4) we arrive at the right-hand side of (3.3). □

Remark 3.2. Note that in (3.3), the multiple q-integrals converge absolutely since, when writing the left- and right-hand sides as determinants of single q-integrals as in the above proof, we have absolute convergence in each entry of the respective determinants.

We can iterate the transformation in Theorem 3.1 to obtain a different transformation formula. We need to permute some variables first, else we would get back the original identity. In order to proceed, we specialize the variables first and let $c_i = c$, $d_i = d$ and $e_i = e/x_i$, for $i = 1, \dots, r$. For convenience, we further interchange the variables e and g. In this first step, this gives the transformation

$$\int_{ax_1}^{b}\cdots\int_{ax_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)\prod_{i=1}^{r}(1-t_i^2/a)\frac{(qt_i/ax_i,qt_i/b)_\infty}{(t_ix_i,bt_i/a)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(qt_i/c,qt_i/d,qt_i/e,qt_i/f,qt_ix_i/g,qt_i/h)_\infty}{(ct_i/a,dt_i/a,et_i/a,ft_i/a,gt_i/ax_i,ht_i/a)_\infty}\,d_qt_r\ldots d_qt_1$$
$$=\left(\frac{a}{\lambda}\right)^{\binom{r+1}{2}}\prod_{i=1}^{r}\frac{(b/ax_i,ax_iq/b,\lambda cx_i/a,\lambda dx_i/a,\lambda g/a)_\infty}{(b/\lambda x_i,\lambda x_iq/b,cx_i,dx_i,g)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(beq^{i-1}/\lambda,bfq^{i-1}/\lambda,bhq^{i-1}/\lambda)_\infty}{(beq^{i-1}/a,bfq^{i-1}/a,bhq^{i-1}/a)_\infty}$$
$$\times\int_{\lambda x_1}^{b}\cdots\int_{\lambda x_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/\lambda)\prod_{i=1}^{r}(1-t_i^2/\lambda)\frac{(qt_i/\lambda x_i,qt_i/b)_\infty}{(t_ix_i,bt_i/\lambda)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(aqt_i/c\lambda,aqt_i/d\lambda,aqt_ix_i/g\lambda,qt_i/e,qt_i/f,qt_i/h)_\infty}{(ct_i/a,dt_i/a,gt_i/ax_i,et_i/\lambda,ft_i/\lambda,ht_i/\lambda)_\infty}\,d_qt_r\ldots d_qt_1,\quad(3.5)$$

where $a^3q^{3-r}=bcdefgh$ and $\lambda=a^2q/cdg$. Now, we iterate (3.5), in the second step with $a\mapsto a^2q/cdg$, $c\mapsto e$, $d\mapsto f$, $e\mapsto aq/dg$, $f\mapsto aq/cg$, and $g\mapsto aq/cd$. The result is the following.

Corollary 3.3. *Let $a^3q^{3-r}=bcdefgh$. Then there holds*

$$\int_{ax_1}^{b}\cdots\int_{ax_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)\prod_{i=1}^{r}(1-t_i^2/a)\frac{(qt_i/ax_i,qt_i/b)_\infty}{(t_ix_i,bt_i/a)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(qt_i/c,qt_i/d,qt_i/e,qt_i/f,qt_ix_i/g,qt_i/h)_\infty}{(ct_i/a,dt_i/a,et_i/a,ft_i/a,gt_i/ax_i,ht_i/a)_\infty}\,d_qt_r\ldots d_qt_1$$
$$=\prod_{i=1}^{r}\frac{(b/ax_i,ax_iq/b,ax_iq/cg,ax_iq/dg,ax_iq/eg,ax_iq/fg,bhq^{r-1}/a)_\infty}{(cx_i,dx_i,ex_i,fx_i,g,gq^{1-r}/hx_i,hq^rx_i/g)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(gq^{1-i},aq^{2-i}/ch,aq^{2-i}/dh,aq^{2-i}/eh,aq^{2-i}/fh)_\infty}{(bcq^{i-1}/a,bdq^{i-1}/a,beq^{i-1}/a,bfq^{i-1}/a,bhq^{i-1}/a)_\infty}\cdot\left(\frac{agq^{1-r}}{bh}\right)^{\binom{r+1}{2}}$$
$$\times\int_{bhq^{r-1}x_1/g}^{b}\cdots\int_{bhq^{r-1}x_r/g}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-gq^{1-r}t_it_j/bh)$$
$$\times\prod_{i=1}^{r}(1-gq^{1-r}t_i^2/bh)\frac{(gq^{2-r}t_i/bhx_i,qt_i/b,cgt_i/a,dgt_i/a)_\infty}{(t_ix_i,gq^{1-r}t_i/h,at_iq^{2-r}/bch,at_iq^{2-r}/bdh)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(egt_i/a,fgt_i/a,aq^{2-r}t_ix_i/bh,qt_i/h)_\infty}{(at_iq^{2-r}/beh,at_iq^{2-r}/bfh,gt_i/ax_i,gq^{1-r}t_i/b)_\infty}\,d_qt_r\ldots d_qt_1.\quad(3.6)$$

4. Multivariable nonterminating ${}_{10}\phi_9$ transformations

We write out (3.2) explicitly for the integral on the left-hand side of (3.3):

$$\int_{ax_1}^{b}\cdots\int_{ax_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)\prod_{i=1}^{r}(1-t_i^2/a)\frac{(qt_i/ax_i,qt_i/b)_\infty}{(t_ix_i,bt_i/a)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{(qt_i/c_i, qt_i/d_i, qt_i/e_i, qt_i/f, qt_i/g, qt_i/h)_\infty}{(c_it_i/a, d_it_i/a, e_it_i/a, ft_i/a, gt_i/a, ht_i/a)_\infty} \, d_qt_r \ldots d_qt_1$$

$$= \sum_{S\subseteq\{1,2,\ldots,r\}} (-1)^{|S|}a^{|S|}b^{r-|S|}(1-q)^r a^{\binom{|S|}{2}} b^{\binom{r-|S|}{2}} \prod_{i\in S} x_i$$

$$\times \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} (x_iq^{k_i} - x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j}) \prod_{i\in S}(1-ax_i^2q^{2k_i})$$

$$\times \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} (q^{k_i} - q^{k_j})(1-b^2q^{k_i+k_j}/a) \prod_{i\notin S}(1-b^2q^{2k_i}/a)$$

$$\times \prod_{i\in S, j\notin S} (ax_iq^{k_i} - bq^{k_j})(1-bx_iq^{k_i+k_j})(-1)^{\chi(i>j)}$$

$$\times \prod_{i\in S} \frac{(q^{1+k_i}, ax_iq^{1+k_i}/b, ax_iq^{1+k_i}/c_i, ax_iq^{1+k_i}/d_i, ax_iq^{1+k_i}/e_i)_\infty}{(ax_i^2q^{k_i}, bx_iq^{k_i}, c_ix_iq^{k_i}, d_ix_iq^{k_i}, e_ix_iq^{k_i})_\infty}$$

$$\times \prod_{i\in S} \frac{(ax_iq^{1+k_i}/f, ax_iq^{1+k_i}/g, ax_iq^{1+k_i}/h)_\infty}{(fx_iq^{k_i}, gx_iq^{k_i}, hx_iq^{k_i})_\infty} \prod_{i\notin S} \frac{(bq^{1+k_i}/ax_i, q^{1+k_i})_\infty}{(bx_iq^{k_i}, b^2q^{k_i}/a)_\infty}$$

$$\times \prod_{i\notin S} \frac{(bq^{1+k_i}/c_i, bq^{1+k_i}/d_i, bq^{1+k_i}/e_i, bq^{1+k_i}/f, bq^{1+k_i}/g, bq^{1+k_i}/h)_\infty}{(bc_iq^{k_i}/a, bd_iq^{k_i}/a, be_iq^{k_i}/a, bfq^{k_i}/a, bgq^{k_i}/a, bhq^{k_i}/a)_\infty} \cdot q^{\sum_{i=1}^r k_i},$$

where $|S|$ denotes the number of elements of S, and χ is the truth function (which evaluates to one if the argument is true and to zero otherwise). A similar expression is obtained for the right-hand side of (3.3). Now, if we divide both sides by

$$(-1)^r a^{\binom{r+1}{2}} x_1 \ldots x_r (1-q)^r \prod_{1\le i<j\le r} (x_i - x_j)(1-ax_ix_j)$$

$$\times \prod_{i=1}^{r} \frac{(q, ax_iq/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i, ax_iq/f, ax_iq/g, ax_iq/h)_\infty}{(ax_i^2q, bx_i, c_ix_i, d_ix_i, e_ix_i, fx_i, gx_i, hx_i)_\infty}$$

and simplify, we obtain the following result which reduces to (2.4) when $r = 1$.

Corollary 4.1 (A C_r nonterminating ${}_{10}\phi_9$ transformation)**.** *Let*

$$a^3q^{3-r} = bc_id_ie_ix_ifgh$$

and $\lambda = a^2q/c_id_ie_ix_i$ *for* $i = 1, \ldots, r$. *Then there holds*

$$\sum_{S\subseteq\{1,2,\ldots,r\}} \left(\frac{b}{a}\right)^{\binom{r-|S|}{2}} \prod_{i\notin S} \frac{(ax_i^2q, c_ix_i, d_ix_i, e_ix_i)_\infty}{(ax_i/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i)_\infty}$$

$$\times \prod_{i\notin S} \frac{(fx_i, gx_i, hx_i, b/ax_i, bq/c_i, bq/d_i, bq/e_i, bq/f, bq/g, bq/h)_\infty}{(ax_iq/f, ax_iq/g, ax_iq/h, b^2q/a, bc_i/a, bd_i/a, be_i/a, bf/a, bg/a, bh/a)_\infty}$$

$$\times \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} \frac{(x_iq^{k_i} - x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)} \prod_{i\in S} \frac{(1-ax_i^2q^{2k_i})}{(1-ax_i^2)}$$

$$\times \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} \frac{(q^{k_i}-q^{k_j})(1-b^2q^{k_i+k_j}/a)}{(x_i-x_j)(1-ax_ix_j)} \prod_{i\notin S} \frac{(1-b^2q^{2k_i}/a)}{(1-b^2/a)}$$

$$\times \prod_{i\in S, j\notin S} \frac{(x_iq^{k_i}-bq^{k_j}/a)(1-bx_iq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}$$

$$\times \prod_{i\in S} \frac{(ax_i^2, bx_i, c_ix_i, d_ix_i, e_ix_i, fx_i, gx_i, hx_i)_{k_i}}{(q, ax_iq/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i, ax_iq/f, ax_iq/g, ax_iq/h)_{k_i}}$$

$$\times \prod_{i\notin S} \frac{(b^2/a, bx_i, bc_i/a, bd_i/a, be_i/a, bf/a, bg/a, bh/a)_{k_i}}{(q, bq/ax_i, bq/c_i, bq/d_i, bq/e_i, bq/f, bq/g, bq/h)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}$$

$$= \prod_{i=1}^r \frac{(ax_i^2q, b/ax_i, \lambda x_iq/f, \lambda x_iq/g, \lambda x_iq/h, bfq^{i-1}/\lambda, bgq^{i-1}/\lambda, bhq^{i-1}/\lambda)_\infty}{(\lambda x_i^2q, b/\lambda x_i, ax_iq/f, ax_iq/g, ax_iq/h, bfq^{i-1}/a, bgq^{i-1}/a, bhq^{i-1}/a)_\infty}$$

$$\times \prod_{1\le i<j\le r} \frac{(1-\lambda x_ix_j)}{(1-ax_ix_j)} \sum_{S\subseteq\{1,2,\dots,r\}} \left(\frac{b}{\lambda}\right)^{\binom{r-|S|}{2}}$$

$$\times \prod_{i\notin S} \frac{(\lambda x_i^2q, \lambda c_ix_i/a, \lambda d_ix_i/a, \lambda e_ix_i/a, fx_i, gx_i, hx_i)_\infty}{(\lambda x_i/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i, \lambda x_iq/f, \lambda x_iq/g, \lambda x_iq/h)_\infty}$$

$$\times \prod_{i\notin S} \frac{(b/\lambda x_i, abq/c_i\lambda, abq/d_i\lambda, abq/e_i\lambda, bq/f, bq/g, bq/h)_\infty}{(b^2q/\lambda, bc_i/a, bd_i/a, be_i/a, bf/\lambda, bg/\lambda, bh/\lambda)_\infty}$$

$$\times \sum_{k_1,\dots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-\lambda x_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-\lambda x_ix_j)} \prod_{i\in S} \frac{(1-\lambda x_i^2q^{2k_i})}{(1-\lambda x_i^2)}$$

$$\times \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} \frac{(q^{k_i}-q^{k_j})(1-b^2q^{k_i+k_j}/\lambda)}{(x_i-x_j)(1-\lambda x_ix_j)} \prod_{i\notin S} \frac{(1-b^2q^{2k_i}/\lambda)}{(1-b^2/\lambda)}$$

$$\times \prod_{i\in S, j\notin S} \frac{(x_iq^{k_i}-bq^{k_j}/\lambda)(1-bx_iq^{k_i+k_j})}{(x_i-x_j)(1-\lambda x_ix_j)}$$

$$\times \prod_{i\in S} \frac{(\lambda x_i^2, bx_i, \lambda c_ix_i/a, \lambda d_ix_i/a, \lambda e_ix_i/a, fx_i, gx_i, hx_i)_{k_i}}{(q, \lambda x_iq/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i, \lambda x_iq/f, \lambda x_iq/g, \lambda x_iq/h)_{k_i}}$$

$$\times \prod_{i\notin S} \frac{(b^2/\lambda, bx_i, bc_i/a, bd_i/a, be_i/a, bf/\lambda, bg/\lambda, bh/\lambda)_{k_i}}{(q, bq/\lambda x_i, abq/c_i\lambda, abq/d_i\lambda, abq/e_i\lambda, bq/f, bq/g, bq/h)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}. \tag{4.1}$$

For the absolute convergence of the multiple series in (4.1), see Remark 3.2.

Similarly, we write out the transformation of sums resulting from Corollary 3.3. For $r=1$, the following Corollary reduces to Bailey's transformation formula in [4, Eq. (8.1)] (cf. [11, Ex. 2.30]).

Corollary 4.2 (A C_r nonterminating ${}_{10}\phi_9$ transformation). *Let $a^3q^{3-r}=bcdefgh$. Then there holds*

$$\sum_{S\subseteq\{1,2,\dots,r\}} \left(\frac{b}{a}\right)^{\binom{r-|S|}{2}} \prod_{i\notin S} \frac{(ax_i^2q, cx_i, dx_i, ex_i, fx_i)_\infty}{(ax_i/b, ax_iq/c, ax_iq/d, ax_iq/e, ax_iq/f)_\infty}$$

$$\times \prod_{i\notin S} \frac{(g, hx_i, b/ax_i, bq/c, bq/d, bq/e, bq/f, bx_iq/g, bq/h)_\infty}{(ax_i^2q/g, ax_iq/h, b^2q/a, bc/a, bd/a, be/a, bf/a, bg/ax_i, bh/a)_\infty}$$

$$\times \sum_{k_1,\dots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)} \prod_{i\in S} \frac{(1-ax_i^2q^{2k_i})}{(1-ax_i^2)}$$

$$\times \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} \frac{(q^{k_i}-q^{k_j})(1-b^2q^{k_i+k_j}/a)}{(x_i-x_j)(1-ax_ix_j)} \prod_{i\notin S} \frac{(1-b^2q^{2k_i}/a)}{(1-b^2/a)}$$

$$\times \prod_{i\in S, j\notin S} \frac{(x_iq^{k_i}-bq^{k_j}/a)(1-bx_iq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}$$

$$\times \prod_{i\in S} \frac{(ax_i^2, bx_i, cx_i, dx_i, ex_i, fx_i, g, hx_i)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d, ax_iq/e, ax_iq/f, ax_i^2q/g, ax_iq/h)_{k_i}}$$

$$\times \prod_{i\notin S} \frac{(b^2/a, bx_i, bc/a, bd/a, be/a, bf/a, bg/ax_i, bh/a)_{k_i}}{(q, bq/ax_i, bq/c, bq/d, bq/e, bq/f, bx_iq/g, bq/h)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}$$

$$= \prod_{i=1}^{r} \frac{(ax_i^2q, b/ax_i, bx_iq^r/g, bchq^{r-1}x_i/a, bdhq^{r-1}x_i/a, behq^{r-1}x_i/a)_\infty}{(bhx_i^2q^r/g, gq^{1-r}/hx_i, ax_iq/c, ax_iq/d, ax_iq/e, ax_iq/f)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{(bfhq^{r-1}x_i/a, gq^{1-i}, aq^{2-i}/ch, aq^{2-i}/dh, aq^{2-i}/eh, aq^{2-i}/fh)_\infty}{(ax_iq/h, bcq^{i-1}/a, bdq^{i-1}/a, beq^{i-1}/a, bfq^{i-1}/a, bhq^{i-1}/a)_\infty}$$

$$\times \prod_{1\le i<j\le r} \frac{(1-bhq^{r-1}x_ix_j/g)}{(1-ax_ix_j)} \sum_{S\subseteq\{1,2,\dots,r\}} \left(\frac{gq^{1-r}}{h}\right)^{\binom{r-|S|}{2}}$$

$$\times \prod_{i\notin S} \frac{(bhx_i^2q^r/g, ax_iq/cg, ax_iq/dg)_\infty}{(hx_iq^{r-1}/g, bchq^{r-1}x_i/a, bdhq^{r-1}x_i/a)_\infty}$$

$$\times \prod_{i\notin S} \frac{(ax_iq/eg, ax_iq/fg, bhq^{r-1}/a, hx_i, gq^{1-r}/hx_i)_\infty}{(behq^{r-1}x_i/a, bfhq^{r-1}x_i/a, ax_i^2q/g, bx_iq^r/g, bgq^{2-r}/h)_\infty}$$

$$\times \prod_{i\notin S} \frac{(bcg/a, bdg/a, beg/a, bfg/a, ax_iq^{2-r}/h, bq/h)_\infty}{(aq^{2-r}/ch, aq^{2-r}/dh, aq^{2-r}/eh, aq^{2-r}/fh, bg/ax_i, gq^{1-r})_\infty}$$

$$\times \sum_{k_1,\dots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-bhx_ix_jq^{r-1+k_i+k_j}/g)}{(x_i-x_j)(1-bhx_ix_jq^{r-1}/g)}$$

$$\times \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} \frac{(q^{k_i}-q^{k_j})(1-bgq^{1-r+k_i+k_j}/h)}{(x_i-x_j)(1-bhx_ix_jq^{r-1}/g)} \prod_{i\notin S} \frac{(1-bgq^{1-r+2k_i}/h)}{(1-bgq^{1-r}/h)}$$

$$\times \prod_{i\in S, j\notin S} \frac{(x_iq^{k_i}-gq^{1-r+k_j}/h)(1-bx_iq^{k_i+k_j})}{(x_i-x_j)(1-bhx_ix_jq^{1-r}/g)} \prod_{i\in S} \frac{(1-bhx_i^2q^{r-1+2k_i}/g)}{(1-bhx_i^2q^{r-1}/g)}$$

$$\times \prod_{i\in S} \frac{(bhx_i^2q^{r-1}/g, bx_i, ax_iq/cg, ax_iq/dg, ax_iq/eg, ax_iq/fg)_{k_i}}{(q, hx_iq^r/b, bchq^{r-1}x_i/a, bdhq^{r-1}x_i/a, behq^{r-1}x_i/a, bfhq^{r-1}x_i/a)_{k_i}}$$

$$\times \prod_{i\in S} \frac{(bhq^{r-1}/a, hx_i)_{k_i}}{(ax_i^2q/g, bx_iq^r/g)_{k_i}} \prod_{i\notin S} \frac{(bgq^{1-r}/h, bx_i)_{k_i}}{(q, gq^{2-r}/hx_i)_{k_i}}$$
$$\times \prod_{i\notin S} \frac{(aq^{2-r}/ch, aq^{2-r}/dh, aq^{2-r}/eh, aq^{2-r}/fh, bg/ax_i, gq^{1-r})_{k_i}}{(bcg/a, bdg/a, beg/a, bfg/a, ax_iq^{2-r}/h, bq/h)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}. \tag{4.2}$$

5. Terminating C_r ${}_{10}\phi_9$ transformations

We will now specialize Corollary 4.1 to obtain terminating multivariable ${}_{10}\phi_9$ transformations. We accomplish this by multiplying both sides of the identity with $\prod_{i=1}^r (bd_i/a)_\infty$ and then letting $bd_i/a = q^{-n_i}$. Then only the terms corresponding to $S = \emptyset$ are non-zero. After a change of variables this gives the following transformation:

Corollary 5.1 (A C_r terminating ${}_{10}\phi_9$ transformation). *Let $a^3q^{3-r+n_i} = bcde_if_ig_i$, for $i = 1,\dots,r$, and $\lambda = a^2q^{2-r}/bcd$. Then there holds*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j})(1 - aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a, b, c, d, e_i, f_i, g_i, q^{-n_i})_{k_i}}{(1-a)(q, aq/b, aq/c, aq/d, aq/e_i, aq/f_i, aq/g_i, aq^{1+n_i})_{k_i}} q^{k_i}$$
$$= \left(\frac{\lambda}{a}\right)^{\binom{r}{2}} \prod_{i=1}^{r-1} \frac{(b,c,d)_i}{(b\lambda/a, c\lambda/a, d\lambda/a)_i} \prod_{i=1}^r \frac{(aq, \lambda q/e_i, \lambda q/f_i, aq/e_if_i)_{n_i}}{(\lambda q, aq/e_i, aq/f_i, \lambda q/e_if_i)_{n_i}}$$
$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j})(1 - \lambda q^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-\lambda q^{2k_i})(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e_i, f_i, g_i, q^{-n_i})_{k_i}}{(1-\lambda)(q, aq/b, aq/c, aq/d, \lambda q/e_i, \lambda q/f_i, \lambda q/g_i, \lambda q^{1+n_i})_{k_i}} q^{k_i}. \tag{5.1}$$

Note that in (5.1) only terms with all k_i distinct are non-zero. Corollary 5.1 reduces to Bailey's [4] transformation formula for $r = 1$ (cf. [11, Eq. (2.9.1)]).

Similarly to the derivation of Corollary 4.2 from Corollary 4.1, we can obtain another transformation from (5.1) by iteration. For this, we first specialize the variables letting $f_i = f$, $g_i = g$, for $i = 1,\dots,r$, and write $e_i = eq^{n_i}$ for convenience. We then iterate the relation with $a \mapsto a^2q^{2-r}/bcd$, $b \mapsto aq^{2-r}/cd$, $c \mapsto g$, $d \mapsto f$, $f \mapsto aq^{2-r}/bc$, and $g \mapsto aq^{2-r}/bd$. We obtain the following result which reduces to [11, Ex. 2.19] (with e replaced by eq^n) for $r = 1$.

Corollary 5.2 (A C_r terminating ${}_{10}\phi_9$ transformation). *Let $bcdefg = a^3q^{3-r}$. Then there holds*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j})(1 - aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a, b, c, d, eq^{n_i}, f, g, q^{-n_i})_{k_i}}{(1-a)(q, aq/b, aq/c, aq/d, aq^{1-n_i}/e, aq/f, aq/g, aq^{1+n_i})_{k_i}} q^{\sum_{i=1}^r k_i}$$

$$
\begin{aligned}
&= \left(\frac{a^2q^{4-2r}}{b^2cdfg}\right)^{\binom{r}{2}} \prod_{i=1}^{r-1} \frac{(b,c,d,f,g)_i}{(aq^{2-r}/bc, aq^{2-r}/bd, aq^{2-r}/bf, aq^{2-r}/bg, a^2q^{4-2r}/bcdfg)_i} \\
&\times \prod_{i=1}^{r} \frac{(aq, aq^{1-n_i}/ce, aq^{1-n_i}/de, aq^{1-n_i}/ef, aq^{1-n_i}/eg, bq^{r-1})_{n_i}}{(aq/c, aq/d, aq^{1-n_i}/e, aq/f, aq/g, bq^{r-1-n_i}/e)_{n_i}} (eq^{n_i})^{n_i} \\
&\times \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-eq^{1-r+k_i+k_j}/b) \\
&\times \prod_{i=1}^{r} \frac{(1-eq^{1-r+2k_i}/b)(eq^{1-r}/b, eq^{1-r}/a, aq^{2-r}/bf, aq^{2-r}/bg)_{k_i}}{(1-eq^{1-r}/b)(q, aq/b, ef/a, eg/a)_{k_i}} \\
&\times \prod_{i=1}^{r} \frac{(eq^{n_i}, aq^{2-r}/bc, aq^{2-r}/bd, q^{-n_i})_{k_i}}{(q^{2-r-n_i}/b, ec/a, ed/a, eq^{2-r+n_i}/b)_{k_i}} q^{\sum_{i=1}^r k_i}. \qquad (5.2)
\end{aligned}
$$

Again, only terms with all k_i distinct are non-zero.

Finally, we give another multivariable extension of [11, Ex. 2.19]. We first let $g = q^{-N}$ in Corollary 4.2 and then do the simultaneous replacements $b \mapsto e$, $e \mapsto g$, and $h \mapsto b$.

Corollary 5.3 (A C_r terminating ${}_{10}\phi_9$ transformation). *Let $a^3q^{3-r+N} = bcdefg$. Then there holds*

$$
\begin{aligned}
&\sum_{k_1,\dots,k_r=0}^{N} \prod_{1\le i<j\le r} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)} \prod_{i=1}^{r} \frac{(1-ax_i^2q^{2k_i})}{(1-ax_i^2)} q^{k_i} \\
&\times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, dx_i, ex_i, fx_i, gx_i, q^{-N})_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d, ax_iq/e, ax_iq/e, ax_iq/f, ax_iq/g, ax_i^2q^{1+N})_{k_i}} \\
&= q^{\binom{r}{3}} \left(\frac{q^{N+1}}{b}\right)^{\binom{r}{2}} \frac{1}{(b/e)^r_{N+r-1}} \prod_{1\le i<j\le r} \frac{1}{(x_i-x_j)(1-ax_ix_j)} \\
&\times \prod_{i=1}^{r} \frac{(bx_i)^N (ax_i^2q)_N (bx_i)_{N+r-1} (q^{2-r}ax_i/b)_{r-1}}{(q^{N+r+1-i}, aq^{2-r}/be, aq^{2-r}/bc, aq^{2-r}/bd, aq^{2-r}/bf, aq^{2-r}/bg)_{i-1}} \\
&\times \prod_{i=1}^{r} \frac{(q^{2-i}a/ce, q^{2-i}a/de, q^{2-i}a/ef, q^{2-i}a/eg)_{N+i-1}}{(ax_iq/c, ax_iq/d, ax_iq/e, ax_iq/f, ax_iq/g)_N} \\
&\times \sum_{k_1,\dots,k_r=0}^{N+r-1} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-eq^{1-r-N+k_i+k_j}/b) \\
&\times \prod_{i=1}^{r} \frac{(1-eq^{1-r-N+2k_i}/b)(eq^{1-r-N}/b, ex_i, eq^{-N}/ax_i, q^{1-r-N})_{k_i}}{(1-eq^{1-r-N}/b)(q, q^{2-r-N}/bx_i, ax_iq^{2-r}/b, eq/b)_{k_i}} \\
&\times \prod_{i=1}^{r} \frac{(aq^{2-r}/bc, aq^{2-r}/bd, aq^{2-r}/bf, aq^{2-r}/bg)_{k_i}}{(ceq^{-N}/a, deq^{-N}/a, efq^{-N}/a, egq^{-N}/a)_{k_i}} q^{k_i}. \qquad (5.3)
\end{aligned}
$$

Note that the sum on the right lives on a larger hypercube, though this is to some extent "compensated" by the fact that, on the right but not on the left, only terms with all k_i distinct are non-zero.

6. A_{r-1} ${}_1\psi_1$ SUMMATIONS

Previously, one of us [23, Theorems 2.1 and 3.3] utilized determinant evaluations to derive (among other summations) several A_{r-1} extensions of Ramanujan's ${}_1\psi_1$ summation. Here we provide more multivariable extensions of Ramanujan's ${}_1\psi_1$ summation by the same method. The difference is that, like for most sums in Section 5, the terms of our bilateral series are now zero unless all summation indices k_i are different.

We note that we were not able to give new C_r extensions of Bailey's ${}_6\psi_6$ summation (cf. [11, Eq. (II.33)]) by the same method. All such obtained ${}_6\psi_6$ summations would be in fact just special cases of an identity given earlier by one of us, namely the identity obtained from equating Eq. (3.4) with Eq. (3.5) from [23].

The following result reduces to Ramanujan's ${}_1\psi_1$ summation (cf. [11, Eq. (5.2.1)]) for $r = 1$.

Proposition 6.1 (An A_{r-1} ${}_1\psi_1$ summation). *We have*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j}) \prod_{i=1}^{r} \frac{(a_i)_{k_i}}{(b_i)_{k_i}} z_i^{k_i}$$
$$= q^{\binom{r}{2}} \det_{1\le i,j\le r}\left(b_i^{j-r} \frac{(z_i)_{r-j}}{(a_i z_i q/b_i)_{r-j}} \right) \prod_{i=1}^{r} \frac{(q, a_i z_i, q/a_i z_i, b_i/a_i)_\infty}{(b_i, z_i, b_i/a_i z_i, q/a_i)_\infty}, \tag{6.1}$$

provided $|b_i q^{1-r}/a_i| < |z_i| < 1$, *for* $i = 1, \dots, r$.

Proof. We have

$$\prod_{1\le i<j\le r} (q^{k_i}-q^{k_j}) = \det_{1\le i,j\le r}\left(\left(q^{k_i}\right)^{r-j} \right),$$

due to the classical Vandermonde determinant evaluation. Hence we may write the left-hand side of (6.1) as

$$\det_{1\le i,j\le r}\left(\sum_{k_i=-\infty}^{\infty} \frac{(a_i)_{k_i}}{(b_i)_{k_i}} \left(z_i q^{r-j}\right)^{k_i} \right).$$

Now, to the sum inside the determinant we apply Ramanujan's ${}_1\psi_1$ summation (2.10), with $a \mapsto a_i$, $b \mapsto b_i$, and $z \mapsto z_i q^{r-j}$. Thus we obtain

$$\det_{1\le s,t\le r}\left(\frac{(q, a_i z_i q^{r-j}, q^{1+j-r}/a_i z_i, b_i/a_i)_\infty}{(b_i, z_i q^{r-j}, b_i q^{j-r}/a_i z_i, q/a_i)_\infty} \right).$$

Now, by using linearity of the determinant with respect to rows, we take some factors out of the determinant and readily obtain the right-hand side of (6.1). $\square$

We now consider different specializations of the parameters a_i, b_i, and z_i, for $i = 1, \dots, r$, for which the determinant in Proposition 6.1 can be reduced to a product by means of Lemma 2.1.

A simple choice would be $b_i = b$ and $z_i = z$, for $i = 1, \dots, r$. In this case we recover the special case of Eq. (2.2) of [23, Th. 2.1] where in the latter identity both sides are multiplied by $\prod_{1\le i<j\le r}(1 - x_i/x_j)$ and then the specializations $x_i = 1$, for $i = 1, \dots, r$, are being made. Another simple choice would be $a_i = a$ and $b_i = b$, for $i = 1, \dots, r$. In this case we similarly recover a special case of Eq. (2.3) of [23, Th. 2.1].

We also recover a previous result if in Proposition 6.1 we choose the specializations $a_i = a/x_i$, $b_i = b$, and $z_i = zx_i$, for $i = 1, \dots, r$. This yields a special case of Eq. (2.10) of [23]. Similarly, if we choose $a_i = a$, $b_i = bx_i$, and $z_i = z$, for $i = 1, \dots, r$, the determinant can be evaluated by (6.3) and we obtain a special case of Eq. (2.13) of [23].

Nevertheless, we are able to give two different multivariable ${}_1\psi_1$ summations resulting from special cases of Proposition 6.1 which are not covered by previous results.

First, we choose $a_i = ax_i$, $b_i = bx_i$, and $z_i = z$, for $i = 1, \dots, r$. In this case we obtain the following result.

Corollary 6.2 (An A_{r-1} ${}_1\psi_1$ summation). *We have*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j}) \prod_{i=1}^{r} \frac{(ax_i)_{k_i}}{(bx_i)_{k_i}} z^{k_i}$$
$$= (az)^{-\binom{r}{2}} q^{-\binom{r}{3}} \prod_{1\le i<j\le r} (1/x_j - 1/x_i) \prod_{i=1}^{r} \frac{(q, azx_i, q/azx_i, b/a)_\infty}{(bx_i, zq^{i-1}, bq^{1-i}/az, q/ax_i)_\infty}, \tag{6.2}$$

provided $|bq^{1-r}/a| < |z| < 1$.

Next, we choose $a_i = a/x_i$, $b_i = bx_i$, and $z_i = zx_i$, for $i = 1, \dots, r$. We now require a particular limit case of the determinant evaluation in Lemma 2.1, which can be generally stated as

$$\det_{1\le i,j\le r} \left(X_i^{r-j} \frac{(A/X_i)_{r-j}}{(BX_i)_{r-j}} \right) = \prod_{1\le i<j\le r} (X_i - X_j) \prod_{i=1}^{r} \frac{(ABq^{2r-2i})_{i-1}}{(BX_i)_{r-1}}. \tag{6.3}$$

The result is the following.

Corollary 6.3 (An A_{r-1} ${}_1\psi_1$ summation). *We have*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j}) \prod_{i=1}^{r} \frac{(a/x_i)_{k_i}}{(bx_i)_{k_i}} (zx_i)^{k_i}$$
$$= a^{-\binom{r}{2}} \prod_{1\le i<j\le r} (x_i - x_j) \prod_{i=1}^{r} (bq^{1-2r+i}/az^2)_{i-1} \frac{(q, az, q/az, bx_i^2/a)_\infty}{(bx_i, zx_i, bx_iq^{2-r}/az, x_iq/a)_\infty}, \tag{6.4}$$

provided $|bx_i^2 q^{1-r}/a| < |zx_i| < 1$ *for* $i = 1, \dots, r$.

7. Specializations

We will now point out some interesting special cases and consequences of our new C_r ${}_{10}\phi_9$ transformations.

7.1. **A Watson transformation.** In Corollary 5.1, we first remove the dependency of the parameters by replacing g_i by $a^3 q^{3-r+n_i}/bcde_i f_i$. If we now let $d \to \infty$ and relabel $f_i \mapsto d_i$, for $i = 1, \dots, r$, we obtain the following multivariable generalization of Watson's transformation (cf. [11, Eq. (2.5.1)]):

Corollary 7.1 (A multivariable Watson transformation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times\prod_{i=1}^r\frac{(1-aq^{2k_i})(a,b,c,d_i,e_i,q^{-n_i})_{k_i}}{(1-a)(q,aq/b,aq/c,aq/d_i,aq/e_i,aq^{1+n_i})_{k_i}}\left(\frac{a^2q^{3-r+n_i}}{bcd_ie_i}\right)^{k_i}$$
$$=\left(-\frac{a}{bc}\right)^{\binom{r}{2}}q^{-2\binom{r}{3}}\prod_{i=1}^{r-1}\frac{(b,c)_i}{(aq^{2-r}/bc)_i}\prod_{i=1}^r\frac{(aq,aq/d_ie_i)_{n_i}}{(aq/d_i,aq/e_i)_{n_i}}$$
$$\times\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^r\frac{(aq^{2-r}/bc,d_i,e_i,q^{-n_i})_{k_i}}{(q,aq/b,aq/c,d_ie_iq^{-n_i}/a)_{k_i}}q^{k_i}. \tag{7.1}$$

In the special case when $bc=aq$, the right-hand side can be written as a multiple of a determinant; cf. the proof of Corollary 7.3. This gives the C_r terminating ${}_6\phi_5$ summation stated as Corollary 7.10 below. Similarly, the case $b=q^{1-r}$ gives the A_{r-1} terminating ${}_3\phi_2$ summation in Corollary 7.15 below. As a matter of fact, we can even extract a C_r Jackson summation from our Watson transformation in Corollary 7.1. We specialize the parameters so that the series on the left-hand side becomes balanced, i.e., we set $a^3q^{2-r+n_i}=bcd_ie_i$, for $i=1,\dots,r$. Note that in this case the multivariable ${}_4\phi_3$ on the right-hand side of (7.1) reduces to a multivariable ${}_3\phi_2$ which can be transformed into a multiple of a determinant by Corollary 7.15. (Here we can assume that Corollary 7.15 is already known – after all, we just pointed out that it follows from Corollary 7.1.) However, the result we would obtain by this procedure is only a special case of Corollary 7.3 below.

We also remark that on the right-hand side, the "type C" Vandermonde determinant

$$\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})\prod_{i=1}^r(1-aq^{2k_i}) \tag{7.2}$$

got reduced to the classical "type A" Vandermonde determinant. Moreover, the very-well-poised condition of the parameters got lost, while the series remains balanced. The left-hand side retains the factor (7.2) and is very-well-poised but not balanced. Dealing here with r-dimensional series, we simply mean by these terms that the respective series are very-well-poised and/or balanced when $r=1$.

7.2. **An A_{r-1} Sears transformation.** To obtain a multivariable extension of Sears' transformation (cf. [11, Eq. (3.2.1)]) from Corollary 5.1, replace b by aq/b and e_i by aq/e_i, for $i=1,\dots,r$, and then take the limit $a\to 0$. After relabeling of parameters, $b\mapsto d$, $d\mapsto b$, $f_i\mapsto a_i$, for $i=1,\dots,r$, we obtain

Corollary 7.2 (An A_{r-1} Sears transformation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^r\frac{(a_i,b,c,q^{-n_i})_{k_i}}{(q,d,e_i,a_ibcq^{r-n_i}/de_i)_{k_i}}q^{k_i}$$
$$=\left(\frac{dq^{1-r}}{bc}\right)^{\binom{r}{2}}\prod_{i=1}^{r-1}\frac{(b,c)_i}{(dq^{1-r}/b,dq^{1-r}/c)_i}\prod_{i=1}^r\frac{(de_iq^{-r}/bc,e_i/a_i)_{n_i}}{(e_i,de_iq^{-r}/a_ibc)_{n_i}}$$

$$\times \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j}) \prod_{i=1}^{r} \frac{(a_i, dq^{1-r}/b, dq^{1-r}/c, q^{-n_i})_{k_i}}{(q, d, de_i q^{-r}/bc, a_i q^{1-n_i}/e_i)_{k_i}} q^{k_i}. \tag{7.3}$$

When $d=b$, the right-hand side can be expressed as a determinant. This gives a special case of Corollary 7.15 below.

7.3. $\boldsymbol{C_r}$ **Jackson summations.** We start with the following identity, which reduces to Jackson's summation formula (2.6) when $r=1$.

Corollary 7.3 (A C_r Jackson summation). *Let* $bc_id_ie_i = a^2q^{2-r+n_i}$ *for* $i = 1,\dots,r$. *Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times \prod_{i=1}^{r} \frac{(1-aq^{2k_i})(a,b,c_i,d_i,e_i,q^{-n_i})_{k_i}}{(1-a)(q,aq/b,aq/c_i,aq/d_i,aq/e_i,aq^{1+n_i})_{k_i}} q^{k_i}$$
$$= (-b)^{-\binom{r}{2}} q^{-2\binom{r}{3}} \prod_{i=1}^{r} \frac{(aq^{2-r}/b)_{r-1}(b)_{i-1}}{(aq^{2+r-2i}/b)_{i-1}} \frac{(aq, aq/c_id_i, aq/c_ie_i, aq/d_ie_i)_{n_i}}{(aq/c_i, aq/d_i, aq/e_i, aq/c_id_ie_i)_{n_i}}$$
$$\times \det_{1\le i,j\le r}\left(\frac{(c_i,d_i,e_i,q^{-n_i})_{r-j}}{(aq^{2-r}/bc_i, aq^{2-r}/bd_i, aq^{2-r}/be_i, aq^{2-r+n_i}/b)_{r-j}}\right). \tag{7.4}$$

Proof. To obtain the left-hand side, we let $cd=aq$ in Corollary 5.1 and make the change of variables $(e_i,f_i,g_i)\mapsto(c_i,d_i,e_i)$. On the right-hand side we observe that, since $\lambda b/a = q^{1-r}$, only terms with $k_i\le r-1$ are non-zero. Since we may also assume that the k_i are distinct, $(k_1,\dots k_r)$ must be a permutation of $(0,\dots,r-1)$. This allows us to pull out some factors from the sum, for instance

$$\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j}) = \prod_{0\le i<j\le r-1}(q^i-q^j)\operatorname{sgn}(k),$$

where sgn denotes the sign of the permutation. After some cancellation, we obtain the expression

$$\left(\frac{\lambda q}{a}\right)^{\binom{r}{2}} \prod_{i=1}^{r-1} \frac{(1-\lambda q^{2i})(b,\lambda)_i}{(1-\lambda)(q,aq/b)_i} \prod_{i=1}^{r} \frac{(aq,\lambda q/c_i,\lambda q/d_i,aq/c_id_i)_{n_i}}{(\lambda q,aq/c_i,aq/d_i,\lambda q/c_id_i)_{n_i}}$$
$$\times \prod_{0\le i<j\le r-1}(q^i-q^j)(1-\lambda q^{i+j}) \sum_k \operatorname{sgn}(k) \prod_{i=1}^{r} \frac{(c_i,d_i,e_i,q^{-n_i})_{k_i}}{(\lambda q/c_i,\lambda q/d_i,\lambda q/e_i,\lambda q^{1+n_i})_{k_i}} \tag{7.5}$$

for the right-hand side. Up to a factor $(-1)^{\binom{r}{2}}$, obtained from inverting the order of the columns, the sum in k equals the determinant in (7.4). After some manipulations of q-shifted factorials, including the easily verified identities

$$\prod_{0\le i<j\le r-1}(q^i-q^j)\prod_{i=1}^{r-1}\frac{1}{(q)_i} = q^{\binom{r}{3}}$$

and (recall that $\lambda = aq^{1-r}/b$)

$$\prod_{0\le i<j\le r-1}(1-\lambda q^{i+j})\prod_{i=1}^{r-1}\frac{(1-\lambda q^{2i})(\lambda)_i}{(1-\lambda)(aq/b)_i} = (aq^{2-r}/b)_{r-1}^{r}\prod_{i=1}^{r-1}\frac{1}{(aq^{r-2i}/b)_i},$$

we arrive at (7.4). □

Next we consider two cases of Corollary 7.3 when the determinant on the right-hand side is computed by Lemma 2.1. For the first case we put $c_i = c$, $d_i = d$ and write $e_i = eq^{n_i}$.

Corollary 7.4 (A C_r Jackson summation). *If $bcde = a^2q^{2-r}$, then*

$$\begin{aligned}\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}&\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})\\ &\times\prod_{i=1}^{r}\frac{(1-aq^{2k_i})(a,b,c,d,eq^{n_i},q^{-n_i})_{k_i}}{(1-a)(q,aq/b,aq/c,aq/d,aq^{1-n_i}/e,aq^{1+n_i})_{k_i}}q^{k_i}\\ &= q^{-\binom{r}{3}}\left(\frac{e}{a}\right)^{\binom{r}{2}}\prod_{i=1}^{r-1}\frac{(b,c,d)_i}{(aq^{2-r}/bc,aq^{2-r}/bd,aq^{2-r}/cd)_i}\\ \times\prod_{1\le i<j\le r}&(q^{n_i}-q^{n_j})(1-eq^{n_i+n_j})\prod_{i=1}^{r}\frac{(aq,aq^{2-r}/bc,aq^{2-r}/bd,aq^{2-r}/cd)_{n_i}}{(aq/b,aq/c,aq/d,aq^{2-r}/bcd)_{n_i}}.\end{aligned}\tag{7.6}$$

Next we give the case of Corollary 7.3 when $c_i = c$ and $n_i = N$ for all i, and we write $d_i = dx_i$, $e_i = e/x_i$. We have used some manipulations to write the result so that the symmetry between b and c is exhibited. An alternative way to obtain this identity, which gives it in the form we want immediately, is to put $bh/a = q^{-N}$ and $d_ie_i = aq$ in Corollary 4.1.

Corollary 7.5 (A C_r Jackson summation). *If $bcde = a^2q^{2-r+N}$, then*

$$\begin{aligned}\sum_{\substack{0\le k_i\le N\\ i=1,\dots,r}}&\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})\\ &\times\prod_{i=1}^{r}\frac{(1-aq^{2k_i})(a,b,c,dx_i,e/x_i,q^{-N})_{k_i}}{(1-a)(q,aq/b,aq/c,aq/dx_i,aqx_i/e,aq^{1+N})_{k_i}}q^{k_i}\\ &=\left(\frac{e^2}{ad}\right)^{\binom{r}{2}}\prod_{1\le i<j\le r}(x_i-x_j)(1-dx_ix_j/e)\prod_{i=1}^{r-1}\frac{(b,c,q^{-N})_i}{(aq^{2-r}/bc)_i}\\ &\times\prod_{i=1}^{r}\frac{(aq,aq^{2-r}/bc)_N(aq^{2-r}/bdx_i,aq^{2-r}/cdx_i)_{N+1-r}}{x_i^{2(r-1)}(aq^{2-i}/b,aq^{2-i}/c)_{N+1-i}(aq/dx_i,aq^{2-r}/bcdx_i)_N}.\end{aligned}\tag{7.7}$$

The above C_r Jackson summations all contain the factor (7.2) in the summand. We now turn our attention to C_r Jackson summations of a different type, namely, of the type encountered in [23, Th. 4.2]. These also follow naturally from our C_r nonterminating ${}_{10}\phi_9$ transformation in Corollary 4.1. For this case we put $gh = aq$, and $e_i = q^{-n_i}/x_i$, for $i = 1,\dots,r$, in (4.1). On the right-hand side, because of $bf/\lambda = q^{1-r}$, only the term with $S = \emptyset$ is non-zero. This term, similar as in the proof of Corollary 7.3, is a determinant. The result is the following (where we have replaced f by e).

Corollary 7.6 (A C_r Jackson summation). *If* $bc_id_ie = a^2q^{2-r+n_i}$, *then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)} \prod_{i=1}^r \frac{(1-ax_i^2q^{2k_i})}{(1-ax_i^2)}$$
$$\times \prod_{i=1}^r \frac{(ax_i^2, bx_i, c_ix_i, d_ix_i, ex_i, q^{-n_i})_{k_i}}{(q, ax_iq/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e, ax_i^2q^{1+n_i})_{k_i}} q^{k_i}$$
$$= (-1)^{\binom{r}{2}} b^{-\binom{r}{2}} q^{-2\binom{r}{3}} \prod_{1\le i<j\le r} \frac{1}{(x_i-x_j)(1-ax_ix_j)}$$
$$\times \prod_{i=1}^r \frac{(aqx_i^2, aq^{2-r}/bc_i, aq^{2-r}/bd_i)_{n_i}(aq/c_id_i)_{n_i+i-r}(bx_i)_{r-1}}{(aqx_i/c_i, aqx_i/d_i, aq^{2-r}/bc_id_ix_i)_{n_i}(aqx_i/b)_{n_i+1-r}(eq^{2+r-2i}/b)_{i-1}}$$
$$\times \det_{1\le i,j\le r}\left(\frac{(ex_i, ec_i/a, ed_i/a, eq^{-n_i}/ax_i)_{r-j}}{(q^{2-r}/bx_i, q^{-n_i}ed_i/a, q^{-n_i}ec_i/a, aq^{2-r+n_i}x_i/b)_{r-j}}\right). \tag{7.8}$$

Analogously to Corollary 7.3 we have two special cases where the determinant evaluates as a product of linear factors, giving rise to two different explicit Jackson summations. The first one is $c_i = c$, $d_i = d$, $n_i = N$, $i = 1, \dots, r$, which gives the Tejasi sum in [23, Th. 4.2]. The case $x_i = x$, $d_i = d$, $c_i = cq^{n_i}$, $i = 1, \dots, r$, gives back Corollary 7.4.

7.4. **C_r nonterminating ${}_6\phi_5$ summations.** We work out the various new C_r extensions of Rogers' nonterminating ${}_6\phi_5$ summation (cf. [11, Eq. (2.7.1)]) following from our results in Section 7.3.

First, we give the special case of Corollary 7.3 arising from formally replacing b by $a^2q^{2-r+n_i}/c_id_ie_i$ (note that this is independent of i), and then letting $n_i \to \infty$, for $i = 1, \dots, r$. After subsequently relabeling $e_i \mapsto b_i$, for $i = 1, \dots, r$, we have the following result.

Corollary 7.7 (A C_r nonterminating ${}_6\phi_5$ summation). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a, b_i, c_i, d_i)_{k_i}}{(1-a)(q, aq/b_i, aq/c_i, aq/d_i)_{k_i}} \left(\frac{a^2q^{2-r}}{b_ic_id_i}\right)^{k_i}$$
$$= q^{-\binom{r}{3}} \det_{1\le i,j\le r}\left(\frac{(b_i, c_i, d_i)_{r-j}}{(b_ic_id_i/a)_{r-j}}\right) \prod_{i=1}^r \frac{(aq, aq/b_ic_i, aq/b_id_i, aq/c_id_i)_\infty}{(aq/b_i, aq/c_i, aq/d_i, aq/b_ic_id_i)_\infty}, \tag{7.9}$$

provided $|aq^{2-r}/b_ic_id_i| < 1$, *for* $i = 1, \dots, r$.

The following two results give cases where the determinant on the right-hand side of (7.9) can be factored by virtue of Lemma 2.1.

For the first case, we specialize Corollary 7.7 by putting $b_i = b$ and $c_i = c$, for $i = 1, \dots, r$.

Corollary 7.8 (A C_r nonterminating ${}_6\phi_5$ summation). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j}) \\ \times \prod_{i=1}^{r} \frac{(1-aq^{2k_i})(a,b,c,d_i)_{k_i}}{(1-a)(q,aq/b,aq/c,aq/d_i)_{k_i}} \left(\frac{a^2q^{2-r}}{bcd_i}\right)^{k_i} \\ = \prod_{1\le i<j\le r}(d_j-d_i) \prod_{i=1}^{r} \frac{(b,c,bc/a)_{i-1}}{(bcd_i/a)_{r-1}} \frac{(aq,aq/bc,aq/bd_i,aq/cd_i)_\infty}{(aq/b,aq/c,aq/d_i,aq/bcd_i)_\infty}, \tag{7.10}$$

provided $|aq^{2-r}/bcd_i|<1$, *for* $i=1,\dots,r$.

The other case results from choosing $b_i=b$, $c_i=cx_i$, and $d_i=d/x_i$, for $i=1,\dots,r$, in Corollary 7.7.

Corollary 7.9 (A C_r nonterminating ${}_6\phi_5$ summation). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j}) \\ \times \prod_{i=1}^{r} \frac{(1-aq^{2k_i})(a,b,cx_i,d/x_i)_{k_i}}{(1-a)(q,aq/b,aq/cx_i,ax_iq/d)_{k_i}} \left(\frac{a^2q^{2-r}}{bcd}\right)^{k_i} \\ = c^{\binom{r}{2}} \prod_{1\le i<j\le r}(x_j-x_i)(1-d/cx_ix_j) \\ \times \prod_{i=1}^{r} \frac{(b)_{i-1}}{(bcd/a)_{i-1}} \frac{(aq,aq/bcx_i,ax_iq/bd,aq/cd)_\infty}{(aq/b,aq/cx_i,ax_iq/d,aq/bcd)_\infty}, \tag{7.11}$$

provided $|aq^{2-r}/bcd|<1$, *for* $i=1,\dots,r$.

7.5. $\boldsymbol{C_r}$ **terminating** $\boldsymbol{{}_6\phi_5}$ **summations.** We now list the terminating versions of the above multivariable ${}_6\phi_5$ summations. These reduce to [11, Eq. (2.4.2)] for $r=1$.

We start with the $d_i=q^{-n_i}$, $i=1,\dots,r$, case of Corollary 7.7.

Corollary 7.10 (A C_r terminating ${}_6\phi_5$ summation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j}) \\ \times \prod_{i=1}^{r} \frac{(1-aq^{2k_i})(a,b_i,c_i,q^{-n_i})_{k_i}}{(1-a)(q,aq/b_i,aq/c_i,aq^{1+n_i})_{k_i}} \left(\frac{a^2q^{2-r+n_i}}{b_ic_i}\right)^{k_i} \\ = q^{-\binom{r}{3}} \det_{1\le i,j\le r}\left(\frac{(b_i,c_i,q^{-n_i})_{r-j}}{(b_ic_iq^{-n_i}/a)_{r-j}}\right) \prod_{i=1}^{r} \frac{(aq,aq/b_ic_i)_{n_i}}{(aq/b_i,aq/c_i)_{n_i}}. \tag{7.12}$$

We give four different special cases of Corollary 7.10, in each case for which the determinant factors by virtue of Lemma 2.1.

For the first case we put $b_i=b$ and $c_i=c$, for $i=1,\dots,r$.

Corollary 7.11 (A C_r terminating ${}_6\phi_5$ summation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a,b,c,q^{-n_i})_{k_i}}{(1-a)(q,aq/b,aq/c,aq^{1+n_i})_{k_i}} \left(\frac{a^2q^{2-r+n_i}}{bc}\right)^{k_i}$$
$$= \prod_{1\le i<j\le r} (q^{n_i}-q^{n_j}) \prod_{i=1}^r \frac{(b,c,bc/a)_{i-1}}{(bc/a)_{r-1}} \frac{(aq,aq^{2-r}/bc)_{n_i}}{(aq/b,aq/c)_{n_i}}. \quad (7.13)$$

For the second case we put $b_i=b$ and $d_i=q^{-N}$, for $i=1,\dots,r$.

Corollary 7.12 (A C_r terminating ${}_6\phi_5$ summation)**.** *We have*

$$\sum_{\substack{0\le k_i\le N\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a,b,c_i,q^{-N})_{k_i}}{(1-a)(q,aq/b,aq/c_i,aq^{1+N})_{k_i}} \left(\frac{a^2q^{2-r+N}}{bc_i}\right)^{k_i}$$
$$= \prod_{1\le i<j\le r} (c_j-c_i) \prod_{i=1}^r \frac{(b,q^{-N},bq^{-N}/a)_{i-1}}{(bc_iq^{-N}/a)_{r-1}} \frac{(aq,aq/bc_i)_N}{(aq/b,aq/c_i)_N}. \quad (7.14)$$

Note that Corollaries 7.11 and 7.12 are equivalent since they can be derived from each other by applying a standard polynomial argument; cf. [18, Th. 4.2]. The same situation appears with Corollaries 7.13 and 7.14 below.

We now specialize Corollary 7.10 by putting $b_i=bx_i$, $c_i=c/x_i$, and $d_i=q^{-N}$, for $i=1,\dots,r$.

Corollary 7.13 (A C_r terminating ${}_6\phi_5$ summation)**.** *We have*

$$\sum_{\substack{0\le k_i\le N\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a,bx_i,c/x_i,q^{-N})_{k_i}}{(1-a)(q,aq/bx_i,ax_iq/c,aq^{1+N})_{k_i}} \left(\frac{a^2q^{2-r+N}}{bc}\right)^{k_i}$$
$$= b^{\binom{r}{2}} \prod_{1\le i<j\le r} (x_j-x_i)(1-c/bx_ix_j) \prod_{i=1}^r \frac{(q^{-N})_{i-1}}{(bcq^{-N}/a)_{i-1}} \frac{(aq,aq/bc)_N}{(aq/bx_i,ax_iq/c)_N}. \quad (7.15)$$

The last case, which extends the $c\mapsto cq^n$ version of [11, Eq. (2.4.2)] comes from putting $b_i=b$, $c_i=cq^{n_i}$, and $d_i=q^{-n_i}$, for $i=1,\dots,r$, in Corollary 7.10.

Corollary 7.14 (A C_r terminating ${}_6\phi_5$ summation)**.** *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \prod_{1\le i<j\le r} (q^{k_i}-q^{k_j})(1-aq^{k_i+k_j})$$
$$\times \prod_{i=1}^r \frac{(1-aq^{2k_i})(a,b,cq^{n_i},q^{-n_i})_{k_i}}{(1-a)(q,aq/b,aq^{1-n_i}/c,aq^{1+n_i})_{k_i}} \left(\frac{a^2q^{2-r}}{bc}\right)^{k_i}$$

$$= \prod_{1\le i<j\le r}(q^{-n_j}-q^{-n_i})(1-cq^{n_i+n_j})\prod_{i=1}^r \frac{(b)_{i-1}}{(bc/a)_{i-1}}\frac{(aq,aq^{1-n_i}/bc)_{n_i}}{(aq/b,aq^{1-n_i}/c)_{n_i}}. \tag{7.16}$$

7.6. **A_{r-1} terminating $_3\phi_2$ summations.** In the following, we derive some multivariable extensions of the terminating balanced $_3\phi_2$ (or q-Pfaff–Saalschütz) summation (cf. [11, Eq. (1.7.2)]).

In the C_r Jackson summation in Corollary 7.3, we first remove the dependency of the parameters by replacing e_i by $a^2q^{2-r+n_i}/bc_id_i$, for $i=1,\dots,r$. Then we replace b by aq^{2-r}/b and let $a\to 0$. We perform the substitutions $c_i\mapsto a_i$, $d_i\mapsto b_i$, for $i=1,\dots,r$, and $b\mapsto c$ and obtain the following result.

Corollary 7.15 (An A_{r-1} terminating $_3\phi_2$ summation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^r\frac{(a_i,b_i,q^{-n_i})_{k_i}}{(q,cq^{r-1},a_ib_iq^{1-n_i}/c)_{k_i}}q^{k_i}$$
$$= c^{\binom{r}{2}}\prod_{i=1}^r\frac{(c)_{r-1}}{(cq^{2r-2i})_{i-1}}\frac{(c/a_i,c/b_i)_{n_i}}{(c,c/a_ib_i)_{n_i}}$$
$$\times\det_{1\le i,j\le r}\left(\left(a_ib_iq^{-n_i}\right)^{j-r}\frac{(a_i,b_i,q^{-n_i})_{r-j}}{(c/a_i,c/b_i,cq^{n_i})_{r-j}}\right). \tag{7.17}$$

Next, we give three special cases where the determinant appearing on the right-hand side of (7.17) factors.

For the first case we put $a_i=a$, $b_i=b$, for $i=1,\dots,r$. We would now require a limit case of the determinant evaluation in Lemma 2.1, the one which we explicitly stated in (6.3). Equivalently, we start with Corollary 7.4, replace e by a^2q^{2-r}/bcd, and then d by aq^{2-r}/d. We take $a\to 0$ and simultaneously substitute the variables $c\mapsto a$ and $d\mapsto c$. The result is the following.

Corollary 7.16 (An A_{r-1} terminating $_3\phi_2$ summation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^r\frac{(a,b,q^{-n_i})_{k_i}}{(q,cq^{r-1},abq^{1-n_i}/c)_{k_i}}q^{k_i}$$
$$=\left(\frac{c}{ab}\right)^{\binom{r}{2}}\prod_{1\le i<j\le r}(q^{n_i}-q^{n_j})\prod_{i=1}^r\frac{(a,b)_{i-1}}{(c/a,c/b)_{i-1}}\frac{(c/a,c/b)_{n_i}}{(cq^{r-1},c/ab)_{n_i}}. \tag{7.18}$$

For the second case we specialize Corollary 7.15 by putting $a_i=a$, $n_i=N$, for $i=1,\dots,r$. We again would require (6.3). Equivalently, we start with Corollary 7.5, replace e by a^2q^{2-r+N}/bcd, and then b by aq^{2-r}/b. We take $a\to 0$, and then simultaneously substitute the variables $c\mapsto a$, $x_i\mapsto b_i/d$, for $i=1,\dots,r$, and $b\mapsto c$. This yields the following result.

Corollary 7.17 (An A_{r-1} terminating $_3\phi_2$ summation). *We have*

$$\sum_{\substack{0\le k_i\le N\\ i=1,\dots,r}}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^r\frac{(a,b_i,q^{-N})_{k_i}}{(q,cq^{r-1},ab_iq^{1-N}/c)_{k_i}}q^{k_i}$$

$$= \left(\frac{cq^N}{a}\right)^{\binom{r}{2}} \prod_{1\le i<j\le r} (1/b_i - 1/b_j) \prod_{i=1}^{r} \frac{(c)_{r-1}(a,q^{-N})_{i-1}}{(c/b_i)_{r-1}(c/a,cq^N)_{i-1}} \frac{(c/a,c/b_i)_N}{(c,c/ab_i)_N}. \quad (7.19)$$

We could have also applied a standard polynomial argument to Corollary 7.16, in order to derive Corollary 7.17.

For the next case, we set $a_i = a$, $b_i = bq^{n_i}$, for $i = 1,\ldots,r$, in Corollary 7.15. Equivalently, we start with Corollary 7.4, replace d by a^2q^{2-r}/bce, and then b by aq^{2-r}/b. We take $a \to 0$, and then simultaneously substitute the variables $b \mapsto c$, $c \mapsto a$ and $e \mapsto b$, and arrive at the following result which reduces to the $b \mapsto bq^n$ case of [11, Eq. (1.7.2)].

Corollary 7.18 (An A_{r-1} terminating ${}_3\phi_2$ summation). *We have*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\ldots,r}} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j}) \prod_{i=1}^{r} \frac{(a,bq^{n_i},q^{-n_i})_{k_i}}{(q,cq^{r-1},abq/c)_{k_i}} q^{k_i}$$
$$= q^{-\binom{r}{3}} a^{-\binom{r}{2}} \prod_{1\le i<j\le r} (q^{-n_i} - q^{-n_j})(1 - bq^{n_i+n_j})$$
$$\times \prod_{i=1}^{r} \frac{(a)_{i-1}}{(c/a,bq^{2-r}/c)_{i-1}} \frac{(c/a,cq^{r-1-n_i}/b)_{n_i}}{(cq^{r-1},cq^{-n_i}/ab)_{n_i}}. \quad (7.20)$$

For the third case, we choose $a_i = ax_i$, $b_i = b/x_i$, and $n_i = q^{-N}$ for $i = 1,\ldots,r$. Equivalently, we start with Corollary 7.5, replace c by a^2q^{2-r+N}/bde, and then b by aq^{2-r}/b. We take $a \to 0$, and then simultaneously substitute the variables $d \mapsto a$, $e \mapsto b$ and $b \mapsto c$, and arrive at the following result.

Corollary 7.19 (An A_{r-1} terminating ${}_3\phi_2$ summation). *We have*

$$\sum_{\substack{0\le k_i\le N\\ i=1,\ldots,r}} \prod_{1\le i<j\le r} (q^{k_i} - q^{k_j}) \prod_{i=1}^{r} \frac{(ax_i,b/x_i,q^{-N})_{k_i}}{(q,cq^{r-1},abq^{1-N}/c)_{k_i}} q^{k_i}$$
$$= q^{\binom{r}{3}} \left(\frac{cq^N}{b}\right)^{\binom{r}{2}} \prod_{1\le i<j\le r} (x_j - x_i)(1 - b/ax_ix_j)$$
$$\times \prod_{i=1}^{r} \frac{(c)_{r-1}(c/ab,q^{-N})_{i-1}}{(cq^N)_{i-1}(c/ax_i,cx_i/b)_{r-1}} \frac{(c/ax_i,cx_i/b)_N}{(c,c/ab)_N}. \quad (7.21)$$

Note that Corollaries 7.18 and 7.19 are equivalent since they can be derived from each other by applying a standard polynomial argument.

7.7. **A_{r-1} nonterminating ${}_2\phi_1$ and ${}_1\phi_0$ summations.** We briefly give some A_{r-1} q-Gauß summations and A_{r-1} q-binomial theorems which follow from our results. We omit writing out the terminating versions but we have made an effort in presenting all the nonterminating ones explicitly.

We start with our A_{r-1} q-Gauß summations. If, in Corollary 7.15, we let $n_i \to \infty$, for all $i = 1,\ldots,r$, we obtain

Corollary 7.20 (An A_{r-1} q-Gauß summation). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a_i,b_i)_{k_i}}{(q,cq^{r-1})_{k_i}}\left(\frac{c}{a_ib_i}\right)^{k_i}$$
$$=(-c)^{\binom{r}{2}}q^{\binom{r}{3}}\prod_{i=1}^{r}\frac{(c)_{r-1}}{(cq^{2r-2i})_{i-1}}\frac{(c/a_i,c/b_i)_\infty}{(c,c/a_ib_i)_\infty}\det_{1\le i,j\le r}\left((a_ib_i)^{j-r}\frac{(a_i,b_i)_{r-j}}{(c/a_i,c/b_i)_{r-j}}\right), \tag{7.22}$$

where $|c/a_ib_i|<1$, *for* $i=1,\dots,r$.

To evaluate the determinant in factored form we choose different specializations of the parameters.

We can choose $a_i=ax_i$ and $b_i=b/x_i$, for $i=1,\dots,r$, and then substitute $c\mapsto cq^{1-r}$, which gives the following.

Corollary 7.21 (An A_{r-1} q-Gauß summation). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(ax_i,b/x_i)_{k_i}}{(q,c)_{k_i}}\left(\frac{cq^{1-r}}{ab}\right)^{k_i}$$
$$=\left(\frac{c}{bq}\right)^{\binom{r}{2}}q^{-\binom{r}{3}}\prod_{1\le i<j\le r}(x_i-x_j)(1-b/ax_ix_j)\prod_{i=1}^{r}\frac{(c/ax_i,cx_i/b)_\infty}{(c,cq^{1-i}/ab)_\infty}, \tag{7.23}$$

where $|cq^{1-r}|<1$.

Here is the case $a_i=a$, for $i=1,\dots,r$, of Corollary 7.20, with $c\mapsto cq^{1-r}$.

Corollary 7.22 (An A_{r-1} q-Gauß summation). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a,b_i)_{k_i}}{(q,c)_{k_i}}\left(\frac{cq^{1-r}}{ab_i}\right)^{k_i}$$
$$=\left(\frac{c}{aq}\right)^{\binom{r}{2}}q^{-2\binom{r}{3}}\prod_{1\le i<j\le r}(1/b_i-1/b_j)\prod_{i=1}^{r}(a)_{i-1}\frac{(cq^{i-1}/a,c/b_i)_\infty}{(c,cq^{1-r}/ab_i)_\infty}, \tag{7.24}$$

where $|cq^{1-r}/ab_i|<1$, *for* $i=1,\dots,r$.

Next, we give some A_{r-1} q-binomial theorems. If in Corollary 7.20 we replace b_i by c/a_iz_i, for $i=1,\dots,r$, and then let $c\to 0$, we obtain the following summation.

Corollary 7.23 (An A_{r-1} q-binomial theorem). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a_i)_{k_i}}{(q)_{k_i}}z_i^{k_i}$$
$$=(-1)^{\binom{r}{2}}q^{\binom{r}{3}}\det_{1\le i,j\le r}\left(z_i^{r-j}\frac{(a_i)_{r-j}}{(a_iz_i)_{r-j}}\right)\prod_{i=1}^{r}\frac{(a_iz_i)_\infty}{(z_i)_\infty}, \tag{7.25}$$

where $|z_i|<1$, *for* $i=1,\dots,r$.

Note that if we let $b_i=q$ for $i=1,\dots,r$ in Proposition 6.1, we alternatively get the identity

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a_i)_{k_i}}{(q)_{k_i}}z_i^{k_i}=\det_{1\le i,j\le r}\left(\frac{(z_i)_{r-j}}{(a_iz_i)_{r-j}}\right)\prod_{i=1}^{r}\frac{(a_iz_i)_\infty}{(z_i)_\infty}. \tag{7.26}$$

In fact, comparing the two right-hand sides of (7.25) and (7.26), we observe that the following transformation of determinants must hold:

$$\det_{1\le i,j\le r}\left(\frac{(z_i)_{r-j}}{(a_iz_i)_{r-j}}\right)=(-1)^{\binom{r}{2}}q^{\binom{r}{3}}\det_{1\le i,j\le r}\left(z_i^{r-j}\frac{(a_i)_{r-j}}{(a_iz_i)_{r-j}}\right). \tag{7.27}$$

This determinant transformation can also be easily proved in a more natural way. (This has been kindly communicated to us by Christian Krattenthaler.) Note that by reversing the order of columns of the matrix on the left hand side of (7.27) (by which the determinant gets multiplied with $(-1)^{\binom{r}{2}}$), we have the matrix $\big((z_i)_{j-1}/(a_iz_i)_{j-1}\big)_{1\le i,j,\le r}$. If this matrix is now being multiplied with the lower-triangular matrix $\big(q^{(j-1)(1-k)}(q^{1-k})_{k-j}/(q)_{k-j}\big)_{1\le j,k\le r}$ (which has determinant $q^{\binom{r}{2}-2\binom{r}{3}}$), we obtain, after application of the q-Chu–Vandermonde summation [11, Eq. (II.6)] the matrix $\big((-1)^{k-1}z_i^{k-1}q^{-\binom{k}{2}}(a_i)_{k-1}/(a_iz_i)_{k-1}\big)_{1\le i,k\le r}$. Changing again the order of columns and taking determinants we immediately establish (7.27).

We complete our listing of summations by giving three special cases of Corollary 7.23 for which the determinant factors by virtue of Lemma 2.1.

The first case is $a_i=a$, for $i=1,\dots,r$.

Corollary 7.24 (An A_{r-1} q-binomial theorem). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a)_{k_i}}{(q)_{k_i}}z_i^{k_i}
=q^{\binom{r}{3}}\prod_{1\le i<j\le r}(z_j-z_i)\prod_{i=1}^{r}(a)_{i-1}\frac{(az_iq^{r-1})_\infty}{(z_i)_\infty}, \tag{7.28}$$

where $|z_i|<1$, for $i=1,\dots,r$.

The second case is $a_i=a/x_i$, and $z_i=zx_i$ for $i=1,\dots,r$.

Corollary 7.25 (An A_{r-1} q-binomial theorem). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a/x_i)_{k_i}}{(q)_{k_i}}(zx_i)^{k_i}
=z^{\binom{r}{2}}q^{\binom{r}{3}}\prod_{1\le i<j\le r}(x_j-x_i)\prod_{i=1}^{r}\frac{(azq^{i-1})_\infty}{(zx_i)_\infty}, \tag{7.29}$$

where $|zx_i|<1$, for $i=1,\dots,r$.

The third case is $z_i=z$ for $i=1,\dots,r$.

Corollary 7.26 (An A_{r-1} q-binomial theorem). *We have*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{1\le i<j\le r}(q^{k_i}-q^{k_j})\prod_{i=1}^{r}\frac{(a_i)_{k_i}}{(q)_{k_i}}z^{k_i}
=z^{\binom{r}{2}}q^{2\binom{r}{3}}\prod_{1\le i<j\le r}(a_i-a_j)\prod_{i=1}^{r}\frac{(a_izq^{r-1})_\infty}{(zq^{i-1})_\infty}, \tag{7.30}$$

where $|z|<1$.

8. Elliptic extensions

In this section we extend some of our results to the case of elliptic hypergeometric series. Note that these satisfy fewer identities than basic hypergeometric series. Roughly speaking, the reason is that elliptic hypergeometric identities generalize Riemann's addition formula for theta functions, which is more complicated than the addition formulas for trigonometric functions. Moreover, infinite elliptic hypergeometric series lead to serious problems of convergence; in particular, no elliptic analogue of (2.4) is known. This means that we can only extend the results of Sections 5 and 7.3 to the elliptic case. We refer to [25] for a detailed discussion of the balanced and well-poised conditions for elliptic hypergeometric series, and their relation to modular invariance of the series.

Our elliptic extensions involve a fixed parameter p such that $|p| < 1$. We write $\theta(x) := (x, p/x; p)_\infty$ and define elliptic shifted factorials by

$$(a; q, p)_k := \prod_{j=0}^{k-1} \theta(aq^j). \tag{8.1}$$

When $p = 0$, $\theta(x) = 1 - x$ and we recover the q-shifted factorials used before. Since p and q are fixed we omit them from the notation, writing

$$(a)_k := (a; q, p)_k.$$

To use the same shorthand notation as in (2.1) might seem confusing, but has the advantage that we can almost use our previous results as they stand, just interpreting the symbol $(a)_k$ differently. The only other change we have to make is that all factors of the form $x - y$ should be replaced by $x\,\theta(y/x) = -y\,\theta(x/y)$. Thus, the ubiquitous factor

$$\prod_{1 \le i < j \le r} (t_i - t_j)(1 - at_it_j) \prod_{i=1}^{r} (1 - at_i^2)$$

is replaced by

$$\prod_{1 \le i < j \le r} t_i\,\theta(t_j/t_i)\theta(at_it_j) \prod_{i=1}^{r} \theta(at_i^2).$$

All other factors, such as q^{k_i} and and $(\lambda/a)^{\binom{r}{2}}$ in Corollary 5.1, are left untouched.

Theorem 8.1. *The following results have elliptic analogues, given as explained above:* Corollary 5.1, Corollary 5.2, Corollary 5.3, Corollary 7.3, Corollary 7.4, Corollary 7.5 *and* Corollary 7.6.

When $r = 1$, Theorem 8.1 reduces to the elliptic Jackson summation and the elliptic Bailey transformation of Frenkel and Turaev [10].

Proof. First note that to prove Corollary 5.1 we did not use the general case of (2.4). We only used the identity obtained after multiplying by $(bd/a)_\infty$ and then letting $bd/a = q^{-N}$. This is the terminating Bailey transformation (2.5), whose elliptic analogue is known to hold [10]. We also needed the determinant evaluation of Lemma 2.1, whose elliptic analogue was obtained by Warnaar [27]. Apart from these fundamental results, we only used elementary identities for q-shifted factorials that also hold in the elliptic case (indeed, they would hold if $\theta(x)$ in (8.1) was replaced by any function satisfying $\theta(1/x) = -\theta(x)/x$. Thus, the proof of Corollary 5.1 extends immediately to the elliptic case.

The derivations of Corollary 5.2, Corollary 7.3, Corollary 7.4 and Corollary 7.5 from Corollary 5.1 involve only elementary identities and Lemma 2.1, and thus extend to the elliptic case.

In the proof of Corollary 5.3 we made essential use of infinite series, so we need an alternative approach. We sketch a way to derive the elliptic case of Corollary 5.3 from the elliptic case of Corollary 5.2. We use an elliptic version of the "polynomial argument"; this idea also occurs in [22] and [27]. We start with the elliptic Corollary 5.2, multiply it by $\prod_{i=1}^r (aq^{1-n_i}/e)_{n_i}$ and then let $e \to aq^{-N}$ with N a non-negative integer. On the left-hand side we then have the factor

$$\frac{(aq^{1-n_i}/e)_{n_i}}{(aq^{1-n_i}/e)_{k_i}} \to (q^{1+k_i+N-n_i})_{n_i-k_i},$$

which vanishes for $n_i - k_i \geq N+1$. On the right-hand side, we have

$$(eq^{1-r}/a)_{k_i} = (q^{1-r-N})_{k_i}$$

which vanishes for $k_i \geq N + r$. Thus, after replacing k_i by $n_i - k_i$ on the left-hand side, we obtain a transformation of the form

$$\sum_{\substack{0 \leq k_i \leq \min(n_i, N) \\ i=1,\dots,r}} (\cdots) = \sum_{\substack{0 \leq k_i \leq \min(n_i, N+r-1) \\ i=1,\dots,r}} (\cdots).$$

A straightforward computation reveals that this is equivalent to the case $ex_i = q^{-n_i}$ of the elliptic analogue of Corollary 5.3.

Now let $f(x_1, \dots, x_n)$ denote the left-hand side minus the right-hand side of the elliptic analogue of (5.3). We have proved that f vanishes if $x_i = q^{-n_i}/e$ for all i. A computation, using that $\theta(px) = -x^{-1}\theta(x)$ and $(pa)_k = (-1)^k q^{-\binom{k}{2}} a^{-k} (a)_k$, shows that

$$f(x_1, \dots, px_i, \dots, x_n) = f(x_1, \dots, x_n).$$

Thus, f vanishes also when $x_i = p^k q^{-n_i}/e$ with k an integer. For generic values of the parameters, these zeroes have a limit point in which f is analytic, so f is identically zero by analytic continuation. Finally, analytic continuation in the remaining parameters extends the identity to non-generic situations.

It remains to treat Corollary 7.6. In this case, we only used (2.4) in the case when $e = q^{-l}$ and $bf/\lambda = q^{-m}$ for non-negative integers l, m. This is a transformation between two finite sums. If in addition $b = q^{-n}$, it reduces to a special case of (2.5). In the elliptic case, one may then use the same argument as above to prove the corresponding identity. That is, one first proves that the (known) case $b = q^{-n}$ implies the case $b = p^k q^{-n}$ for $k \in \mathbb{Z}$, and then extends it to general b by analytic continuation. Once the needed transformation formula is known, the proof of Corollary 7.6 extends immediately to the elliptic case. □

References

[1] G. E. Andrews, "Applications of basic hypergeometric functions", *SIAM Rev.* **16** (1974) 441–484.

[2] G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Regional Conference Lectures Series **66** (Amer. Math. Soc., Providence, RI, 1986).

[3] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics And Its Applications, Vol. 71, Cambridge University Press, Cambridge, 1999.
[4] W. N. Bailey, "An identity involving Heine's basic hypergeometric series", J. London Math. Soc. **4** (1929) 254–257.
[5] W. N. Bailey, "Well-poised basic hypergeometric series", *Quart. J. Math.* (Oxford) **18** (1947) 157–166.
[6] G. Bhatnagar, "D_n basic hypergeometric series", *The Ramanujan J.* **3** (1999) 175–203.
[7] R. Y. Denis and R. A. Gustafson, "An SU(n) q-beta integral transformation and multiple hypergeometric series identities", *SIAM J. Math. Anal.* **23** (1992) 552–561.
[8] J. F. van Diejen and V. P. Spiridonov, "An elliptic Macdonald-Morris conjecture and multiple modular hypergeometric sums", *Math. Res. Lett.* **7** (2000) 729–746.
[9] J. F. van Diejen and V. P. Spiridonov, "Modular hypergeometric residue sums of elliptic Selberg integrals", *Lett. Math. Phys.* **58** (2001) 223–238.
[10] I. B. Frenkel and V. G. Turaev, "Elliptic solutions of the Yang–Baxter equation and modular hypergeometric functions", in: V. I. Arnold, I. M. Gelfand, V. S. Retakh and M. Smirnov (Eds.), *The Arnold–Gelfand mathematical seminars*, Birkhäuser, Boston, 1997, pp. 171–204.
[11] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications, Vol. 35, Cambridge University Press, Cambridge, 1990.
[12] R. A. Gustafson and C. Krattenthaler, "Determinant evaluations and $U(n)$ extensions of Heine's ${}_2\phi_1$-transformations", in: M. E. H. Ismail, D. R. Masson and M. Rahman (Eds.), *Special functions, q-series and related topics*, Amer. Math. Soc., Providence, R. I., *Fields Institute Communications* **14** (1997) 83–90.
[13] G. H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge (1940); Chelsea, New York (1978).
[14] W. J. Holman, L. C. Biedenharn, and J. D. Louck, "On hypergeometric series well-poised in SU(n)", *SIAM J. Math. Anal.* **7** (1976) 529–541.
[15] F. H. Jackson, "On q-definite integrals", *Quart. J. Pure Appl. Math.* **41** (1910) 193–203.
[16] F. H. Jackson, "Summation of q-hypergeometric series", *Messenger of Math.* **57** (1921) 101–112.
[17] C. Krattenthaler, "The major counting of nonintersecting lattice paths and generating functions for tableaux", *Mem. Amer. Math. Soc.* **115**, no. 552 (1995).
[18] S. C. Milne, "Transformations of $U(n+1)$ multiple basic hypergeometric series", in: A. N. Kirillov, A. Tsuchiya, and H. Umemura (Eds.), *Physics and combinatorics: Proceedings of the Nagoya 1999 international workshop* (Nagoya University, Japan, August 23–27, 1999), World Scientific, Singapore, 2001, pp. 201–243.
[19] S. C. Milne and J. W. Newcomb, "U(n) very-well-poised ${}_{10}\phi_9$ transformations", *J. Comput. Appl. Math.* **68** (1996) 239–285.
[20] E. M. Rains, "BC_n-symmetric polynomials", preprint, arXiv:math.QA/0112035.
[21] H. Rosengren, "A proof of a multivariable elliptic summation formula conjectured by Warnaar", in: B. C. Berndt and K. Ono (Eds.), *q-Series with applications to combinatorics, number theory and physics*, Contemp. Math. 291 (2001), pp. 193–202.
[22] H. Rosengren, "Elliptic hypergeometric series on root systems", *Adv. Math.*, to appear.
[23] M. Schlosser, "Summation theorems for multidimensional basic hypergeometric series by determinant evaluations", *Discrete Math.* **210** (2000) 151–169.
[24] M. Schlosser, "A nonterminating ${}_8\phi_7$ summation for the root system C_r", *J. Comput. Appl. Math.*, to appear.
[25] V. P. Spiridonov, "Theta hypergeometric series", in: V. A. Malyshev and A. M. Vershik (Eds.), *Asymptotic combinatorics with applications to mathematical physics*, Kluwer Acad. Publ., Dordrecht, 2002, pp. 307–327.
[26] V. P. Spiridonov, "Theta hypergeometric integrals", preprint, arXiv:math.CA/0303205.
[27] S. O. Warnaar, "Summation and transformation formulas for elliptic hypergeometric series", *Constr. Approx.* **18** (2002) 479–502.

Department of Mathematics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden
E-mail address: hjalmar@math.chalmers.se
URL: http://www.math.chalmers.se/~hjalmar

Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria
E-mail address: schlosse@ap.univie.ac.at
URL: http://www.mat.univie.ac.at/~schlosse