q-ANALOGUES OF TWO PRODUCT FORMULAS OF HYPERGEOMETRIC FUNCTIONS BY BAILEY

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Dedicated to Mourad E.H. Ismail

ABSTRACT. We use Andrews' q-analogues of Watson's and Whipple's ${}_3F_2$ summation theorems to deduce two formulas for products of specific basic hypergeometric functions. These constitute q-analogues of corresponding product formulas for ordinary hypergeometric functions given by Bailey. The first formula was obtained earlier by Jain and Srivastava by a different method.

1. INTRODUCTION

We refer to Slater's text [9] for an introduction to hypergeometric series, and to Gasper and Rahman's text [5] for an introduction to basic hypergeometric series, whose notations we follow. Throughout, we assume $|q| < 1$ and $|z| < 1$.

In [1], George Andrews proved the following two theorems:

Theorem 1.

$$ {}_4\phi_3\left[\begin{matrix} a, b, c^{\frac{1}{2}}, -c^{\frac{1}{2}} \\ (abq)^{\frac{1}{2}}, -(abq)^{\frac{1}{2}}, c \end{matrix}; q, q\right] = a^{\frac{n}{2}} \frac{(aq, bq, cq/a, cq/b; q^2)_\infty}{(q, abq, cq, cq/ab; q^2)_\infty}, \tag{1.1}$$

where $b = q^{-n}$ and n is a nonnegative integer.

Theorem 2.

$$ {}_4\phi_3\left[\begin{matrix} a, q/a, c^{\frac{1}{2}}, -c^{\frac{1}{2}} \\ -q, e, cq/e \end{matrix}; q, q\right] = q^{\binom{n+1}{2}} \frac{(ea, eq/a, caq/e, cq^2/ae; q^2)_\infty}{(e, cq/e; q)_\infty}, \tag{1.2}$$

where $a = q^{-n}$ and n is a nonnegative integer.

By a standard polynomial argument (1.2) also holds when a is a complex variable but $c = q^{-2n}$ with n being a nonnegative integer. (This is the case we will make use of.)

Theorems 1 and 2 are q-analogues of Watson's and of Whipple's ${}_3F_2$ summation theorems, listed as Equations (III.23) and (III.24) in [9, p. 245], respectively.

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2. Two product formulas for basic hypergeometric functions

We now have the following two product formulas which are derived using Theorems 1 and 2. The first one in Theorem 3 was already given earlier by Jain and Srivastava [7, Equation (4.9)] (as Slobodan Damjanović has kindly pointed out to the author, after seeing an earlier version of this note), who established the result by specializing a general reduction formula for double basic hypergeometric series. The second formula in Theorem 4 appears to be new.

Theorem 3.

$$ {}_2\phi_1\left[\begin{matrix}a,-a\\ a^2\end{matrix};q,z\right]{}_2\phi_1\left[\begin{matrix}b,-b\\ b^2\end{matrix};q,-z\right]={}_4\phi_3\left[\begin{matrix}ab,-ab,abq,-abq\\ a^2q,b^2q,a^2b^2\end{matrix};q^2,z^2\right]. \tag{2.1}$$

Theorem 4.

$$ {}_2\phi_1\left[\begin{matrix}a,q/a\\ -q\end{matrix};q,z\right]{}_2\phi_1\left[\begin{matrix}b,q/b\\ -q\end{matrix};q,-z\right]=\sum_{j=0}^{\infty}\frac{(q^{2-j}/ab,aq^{1-j}/b;q^2)_j}{(q^2;q^2)_j}q^{\binom{j}{2}}(bz)^j \tag{2.2a}$$

$$ ={}_4\phi_3\left[\begin{matrix}ab,q^2/ab,aq/b,bq/a\\ -q^2,q,-q\end{matrix};q^2,z^2\right] - \frac{(a-b)(1-q/ab)}{1-q^2}\,z\,{}_4\phi_3\left[\begin{matrix}abq,q^3/ab,aq^2/b,bq^2/a\\ -q^2,q^3,-q^3\end{matrix};q^2,z^2\right]. \tag{2.2b}$$

Sketch of proofs. To prove Theorem 3, compare coefficients of z^n. The resulting identity is equivalent to Theorem 1. The proof of Theorem 4 is similar. Comparison of coefficients of z^n gives an identity which is equivalent to Theorem 2 (where in the latter theorem the restriction $a=q^{-n}$ is replaced by $c=q^{-2n}$, as mentioned). The second identity in Equation (2.2) follows from splitting the sum over j into two parts depending on the parity of j. (This is motivated by the particular numerator factors in the j-th summand.) The technical details – elementary manipulation of q-shifted factorials – are routine and thus omitted. ☐

Theorem 3 is a q-analogue of Bailey's formula in [2, p. 246, Equation (2.11)]:

$$ {}_1F_1\left[\begin{matrix}a\\ 2a\end{matrix};z\right]{}_1F_1\left[\begin{matrix}b\\ 2b\end{matrix};-z\right]={}_2F_3\left[\begin{matrix}\frac12(a+b),\frac12(a+b+1)\\ a+\frac12,b+\frac12,a+b\end{matrix};\frac14z\right]. \tag{2.3}$$

To obtain (2.3) from Theorem 3, replace (a,b,z) by $(q^a,q^b,(1-q)z/2)$, and let $q\to1$.

Similarly, Theorem 4 is a q-analogue of Bailey's formula in [2, p. 245, Equation (2.08)]:

$$ {}_2F_0\left[\begin{matrix}a,1-a\\ -\end{matrix};z\right]{}_2F_0\left[\begin{matrix}b,1-b\\ -\end{matrix};-z\right] = {}_4F_1\left[\begin{matrix}\frac12(1+a-b),\frac12(1-a+b),\frac12(a+b),\frac12(2-a-b)\\ \frac12\end{matrix};4z^2\right] - (a-b)(a+b-1)\,z \times {}_4F_1\left[\begin{matrix}\frac12(2+a-b),\frac12(2-a+b),\frac12(1+a+b),\frac12(3-a-b)\\ \frac32\end{matrix};4z^2\right]. \tag{2.4}$$

To obtain (2.4) from Theorem 4, replace (a, b, z) by $(q^a, q^b, 2z/(1-q))$ and let $q \to 1$.

3. Related results in the literature

A different product formula for basic hypergeometric functions was established by Srivastava [10, Eq. (21)] (see also [11, Eq. (3.13)]):

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ -ab \end{matrix}; q, z\right] {}_2\phi_1\left[\begin{matrix} a, b \\ -ab \end{matrix}; q, -z\right] = {}_4\phi_3\left[\begin{matrix} a^2, b^2, ab, abq \\ a^2b^2, -ab, -abq \end{matrix}; q^2, z^2\right]. \tag{3.1} $$

This formula is a q-extension of Bailey's formula in [2, p. 245, Equation (2.08)] (or, equivalently, of an identity recorded by Ramanujan [8, Ch. 13, Entry 24]).

Finally, we mention that in 1941 F.H. Jackson [6] had derived the identity

$$ {}_2\phi_1\left[\begin{matrix} a^2, b^2 \\ a^2b^2q \end{matrix}; q^2, z\right] {}_2\phi_1\left[\begin{matrix} a^2, b^2 \\ a^2b^2q \end{matrix}; q^2, qz\right] = {}_4\phi_3\left[\begin{matrix} a^2, b^2, ab, -ab \\ a^2b^2, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}} \end{matrix}; q, z\right], \tag{3.2} $$

which is a q-analogue of Clausen's formula of 1828,

$$ \left({}_2F_1\left[\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; z\right]\right)^2 = {}_3F_2\left[\begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+\frac{1}{2} \end{matrix}; z\right]. \tag{3.3} $$

Another q-analogue of Clausen's formula was delivered by Gasper in [4]. While it has the advantage that it expresses a square of a basic hypergeometric series as a basic hypergeometric series, it only holds provided the series terminate:

$$ \left({}_4\phi_3\left[\begin{matrix} a, b, aby, ab/y \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix}; q, q\right]\right)^2 = {}_5\phi_4\left[\begin{matrix} a^2, b^2, ab, aby, ab/y \\ a^2b^2, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix}; q, q\right]. \tag{3.4} $$

See [5, Sec. 8.8] for a nonterminating extension of (3.4) and related identities.

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References

[1] G.E. Andrews, "On q-analogues of the Watson and Whipple summations", SIAM J. Math. Anal. **7** (3) (1976), 332–336.

[2] W.N. Bailey, "Products of generalized hypergeometric series", Proc. London Math. Soc. **S2-28** (1) (1928), 242–254.

[3] T. Clausen, "Ueber die Fälle, wenn die Reihe von der Form ... ein Quadrat von der Form ... hat", J. reine angew. Math. **3** (1828), 89–91.

[4] G. Gasper, "q-Extensions of Clausen's formula and of the inequalities used by de Branges in his proof of the Bieberbach, Robertson and Milin conjectures", SIAM J. Math. Anal. **20** (1989), 1019–1034.

[5] G. Gasper and M. Rahman, *Basic hypergeometric series*, second edition, Encyclopedia of Mathematics and Its Applications **96**, Cambridge University Press, Cambridge, 2004.

[6] F.H. Jackson, "Certain q-identities", Quart. J. Math. (Oxford) **12** (1941), 167–172.

[7] V.K. Jain and H.M. Srivastava, "q-Series identities and reducibility of basic double hypergeometric functions", Canad. J. Math. **38** (1986), 215–231.

[8] S. Ramanujan, *Notebooks of Srinivasa Ramanujan, Vols. I and II*, Tata Institute of Fundamental Research, Bombay, 1957.

[9] L.J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

[10] H.M. Srivastava, "Some formulas of Srinivasa Ramanujan involving products of hypergeometric functions", Indian J. Math. **29** (1987), 91–100.

[11] H.M. Srivastava, "Srinivasa Ramanujan and generalized basic hypergeometric functions", Serdica Bulgariacae mathematicae publicationes **19** (1993), 191–197.

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