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# Some new applications of matrix inversions in $A_r$

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**Editor:**

**Abstract.** We apply multidimensional matrix inversions to multiple basic hypergeometric summation theorems to derive several multiple ($q$-)series identities which themselves do not belong to the hierarchy of (basic) hypergeometric series. Among these are $A_r$ terminating and nonterminating $q$-Abel and $q$-Rothe summations. Furthermore, we derive some identities of another type which appear to be new already in the one-dimensional case.

**Keywords:** multidimensional matrix inversions, multiple $q$-series associated to the root systems $A_r$ and $D_r$, $U(n+1)$ series, terminating and nonterminating $A_r$ $q$-Abel summations, terminating and nonterminating $A_r$ $q$-Rothe summations.

## 1. Introduction

Matrix inversions are very important tools in combinatorics and special functions theory. In particular, it is a widely spread and often used method to derive and prove identities for (basic) hypergeometric series with the help of so-called "inverse relations" (see Section 2), which are immediate consequences of matrix inversions.

Over the last decades, several people discovered and rediscovered useful matrix inversions. The developments started when Gould and Hsu [21] inverted a certain infinite lower-triangular matrix and Carlitz [8] found its $q$-analogue. An important special case of Carlitz's matrix inversion was rediscovered by Andrews [1], in the form of the powerful Bailey Transform [2]. In the sequel, Gessel and Stanton [17], [18], Bressoud [7], and Gasper and Rahman [14], [15] gave further important contributions to the subject.

Recently, Krattenthaler [29] achieved a great deal of unification for matrix inversions. He proved that the matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers), are inverses of each other, where

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + b_j c_k)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.1}$$

and

$$g_{kl} = \frac{(a_l + b_l c_l)}{(a_k + b_k c_k)} \frac{\prod_{j=l+1}^{k}(a_j + b_j c_k)}{\prod_{j=l}^{k-1}(c_j - c_k)}. \tag{1.2}$$

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In fact, Krattenthaler's matrix inversion contains all the inversions just mentioned as special cases.

"Multidimensional" matrix inversions (according to our terminology these are matrix inversions that arise in the theory of multiple series) *associated to root systems* were found by Milne, Lilly and Milne, and by Bhatnagar and Milne. The $A_r$ (or equivalently $U(r+1)$) and $C_r$ inversions (corresponding to the root systems $A_r$ and $C_r$, respectively) of Milne [37, Theorem 3.3], and Lilly and Milne [31], which are higher-dimensional generalizations of Andrews' Bailey transform matrices, were used to derive $A_r$ and $C_r$ extensions [37], [38] of many of the classical hypergeometric summation and transformation formulas. Bhatnagar and Milne [3, Theorem 5.7], [5, Theorem 3.48] were even able to find an $A_r$ extension of Gasper's bibasic hypergeometric matrix inversion. But none of these multidimensional matrix inversions contained Krattenthaler's inversion as a special case.

A multidimensional extension of Krattenthaler's matrix inversion (1.1)/(1.2), associated to root systems, was found by the author in [42]. Theorems 3.1 and 4.1 of [42], also stated as Theorems A.1 and A.23 in Appendix A of this article, cover all the previously discovered multidimensional matrix inversions associated to root systems [5], [31], [38] as special cases. Just recently, another multidimensional extension of Krattenthaler's matrix inverse (1.1)/(1.2) was found [30, Theorem 3.1] which covers the inversion of [10]. The matrix inverse of [30] has applications similar to those in this article although the series considered in [30] are of simpler type.

Special cases of Theorem A.1 were used in [42] to derive several summation theorems for multidimensional basic hypergeometric series. In particular, a $D_r$ ${}_8\phi_7$ summation theorem, $A_r$ and $D_r$ quadratic, and $D_r$ cubic basic hypergeometric summation theorems were derived. Moreover, the $D_r$ ${}_8\phi_7$ summation theorem of [42] lead to new $C_r$ and $D_r$ extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation in [6].

The purpose of this article is to present some new applications of the multidimensional matrix inversions of [42]. We utilize special (non-hypergeometric) cases of the multidimensional matrix inversions in Theorems A.1 and A.23 in conjunction with multiple basic hypergeometric summation theorems to derive a number of multidimensional ($q$-)series identities which themselves do not belong to the hierarchy of (basic) hypergeometric series. Moreover, to most summation theorems in this article we also provide so-called "companion summations" (for an explanation of this terminology, see Section 2).

Our article is organized as follows. In Section 2 the notion and use of inverse relations is explained, together with some standard ($q$-)series notation. The following sections contain applications of our matrix inversions in Theorems A.1 and A.23. In Section 3, we derive some $A_r$ ($q$-)Abel-type expansion formulas. Related $A_r$ ($q$-)Abel summations are given in Section 4. One of these summations (Theorem 4.4) has already been given (in an equivalent form with reversed order of summations) by Bhatnagar and Milne [5, Theorem 5.15]. These authors have also noted that by combining the different $A_r$ $q$-binomial summation theorems with different multidimensional matrix inversions one can derive several more multiple $q$-Abel summations. In this matter, Section 4 aims to give a more exhaustive treatment. In Section 5, we derive some $A_r$ ($q$-)Rothe-type expansion formulas. Related $A_r$ ($q$-)Rothe summations are given in Section 6. Section 7 is devoted to identities of an apparently new type. These identities appear to be new even in the one-dimensional case. Like the Abel- and Rothe-type identities these new identities can be derived by inverting specific (basic) hypergeometric summation theorems but they themselves do not belong to the hierarchy of (basic) hypergeometric series. In Appendix A, we state the multidimensional matrix inversions together with their specializations used in this article. In Appendix B, we list some background information needed in the proofs of our multiple summation theorems such as some $A_r$ basic hypergeometric summation theorems from Milne [37]. Finally, in Appendix C, we give the proofs of absolute convergence for the nonterminating series in this article.

## 2. Preliminaries on inverse relations and ($q$-)series notation

Here we introduce the basic concept of "inverse relations" and introduce some standard ($q$-)series notation.

Throughout this article, $r$-tuples of integers are denoted by bold letters (e.g. $\mathbf{n} = (n_1, \dots, n_r)$). Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ (as before, $\mathbb{Z}$ denotes the set of integers) be an infinite lower-triangular $r$-dimensional matrix; i.e. $f_{\mathbf{nk}} = 0$ unless $\mathbf{n} \geq \mathbf{k}$, by which we mean $n_i \geq k_i$ for all $i = 1, \dots, r$. The matrix $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ is said to be the *inverse matrix* of $F$ if and only if

$$\sum_{\mathbf{l}\leq\mathbf{k}\leq\mathbf{n}} f_{\mathbf{nk}} g_{\mathbf{kl}} = \delta_{\mathbf{nl}}$$

for all $\mathbf{n}, \mathbf{l} \in \mathbb{Z}^r$, where $\delta_{\mathbf{nl}}$ is the usual Kronecker delta.

There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1], [5], [11], [14], [15], [16, Sec. 3.8], [17], [18], [29], [31], [37], [38], [39], [40], [41], [42]). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are lower-triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} f_{\mathbf{nk}} a_{\mathbf{k}} = b_{\mathbf{n}} \tag{2.1}$$

*if and only if*

$$\sum_{\mathbf{0}\leq\mathbf{l}\leq\mathbf{k}} g_{\mathbf{kl}} b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{2.2}$$

If either (2.1) or (2.2) is known, then the other produces another summation formula. The less used dual version, the so-called "rotated inversion", can be used to derive nonterminating summations. It reads

$$\sum_{\mathbf{n}\geq\mathbf{k}} f_{\mathbf{nk}} a_{\mathbf{n}} = b_{\mathbf{k}} \tag{2.3}$$

*if and only if*

$$\sum_{\mathbf{k}\geq\mathbf{l}} g_{\mathbf{k}\mathbf{l}} b_{\mathbf{k}} = a_{\mathbf{l}}, \tag{2.4}$$

subject to suitable convergence conditions. Again, if one of (2.3) or (2.4) is known, the other produces a possibly new identity.

In the subsequent sections we use special cases of our Theorems A.1 and A.23 to derive a couple of higher dimensional summations for ordinary series and $q$-series. We find it convenient to use capital letters for the parameters appearing in our ordinary series and small letters for the parameters in our $q$-series. This convention is *not* standard, it merely aims to distinguish visibly the two types of series in this article.

Before we start to develop the applications of our Theorems, we need to recall some standard notation commonly used when considering hypergeometric, respectively basic hypergeometric series (cf. [16]). We define the *rising factorial* as

$$(A)_0 := 1, \qquad (A)_k := A(A+1)\cdots(A+k-1), \tag{2.5}$$

where $k$ is a nonnegative integer. If $A$ is not a nonpositive integer, (2.5) may also be written as the following quotient of two gamma functions,

$$(A)_k := \frac{\Gamma(A+k)}{\Gamma(A)}. \tag{2.6}$$

For $q$-series, let $q$ be a complex number such that $0 < |q| < 1$. Define

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j), \tag{2.7}$$

and the *$q$-rising factorial*,

$$(a;q)_0 := 1, \qquad (a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}), \tag{2.8}$$

where $k$ is a nonnegative integer. If $a$ is not a negative integer power of $q$, (2.8) may also be written as

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{2.9}$$

As usual, we define the *$q$-binomial coefficient* as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}},$$

for nonnegative integers $n, k$ (cf. [16, Eq. (I.39)]). For a thorough exposition on basic hypergeometric series including lists of selected summation and transformation formulas, we refer the reader to [16]. In this article, we do not make use of the compact ${}_sF_t$ and ${}_s\phi_t$ notation for hypergeometric, respectively basic hypergeometric series (cf. [16, Eqs. (1.2.16) and (1.2.22)]). For most of the series occurring in this article the latter notations cannot be applied and in the other cases we rather preferred to write the sums explicitly.

Finally, for multidimensional series, we also employ the notation $|\mathbf{k}|$ for $(k_1+\ldots+k_r)$ where $\mathbf{k}=(k_1,\ldots,k_r)$.

Concerning the nonterminating multiple series given in this article, we have stated their regions of convergence explicitly. The proofs of the absolute convergence of these series are given in Appendix C.

We have followed the convention used by previous authors [3], [5], [22], [23], [24], [32], [33], [34], [35], [36], [37], [38], [42] for naming our series as $A_r$ series. (Note that some authors also call these series $U(n)$ or $U(n+1)$ series.) According to this terminology a series is called an $A_r$ series if the summand contains the factor

$$\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)$$

in the basic case, and the factor

$$\prod_{1\le i<j\le r}\left(\frac{X_i+k_i-X_j-k_j}{X_i-X_j}\right)$$

in the ordinary case (either explicitly or "hidden" as in the "$A_r$ $q$-binomial coefficient" (B.4)). In Section 7 we come across $D_r$ series, see Remark 7.41 where we also indicate a reason for the terminology.

To most of the summation theorems in this article we also provide so-called "companion summations". Here we understand a companion to be an identity very closely (but not trivially) related to the original identity, the summations differing only by "contiguous" factors. This is best understood by looking at two such identities in question, e.g. (3.1) and (3.3), or (5.1) and (5.3). Concerning contiguous relations (and recurrence relations) in $A_r$, partial fraction decompositions serve as an adequate tool. For instructive demonstrations of the application of partial fraction decompositions in the derivation of results for $A_r$ series, see [3], [5], [23], [32], [33], [35], [36].

## 3. $A_r$ ($q$-)Abel-type expansions

A $q$-analogue of *Euler's formula*

$$e^{AZ}=\sum_{k=0}^{\infty}\frac{A(A+Bk)^{k-1}}{k!}Z^k e^{-BZk}, \tag{3.1}$$

where $\left|BZe^{1-BZ}\right|<1$ [13, p. 354] (cf. [41, Sec. 4.5]), is the expansion

$$1=\sum_{k=0}^{\infty}\frac{(a+b)(a+bq^k)^{k-1}}{(q;q)_k}\left(z(a+bq^k);q\right)_\infty z^k, \tag{3.2}$$

being valid for $|az|<1$ [30, Eq. (7.3)]. To see that (3.2) is a $q$-analogue of (3.1), do the replacements $a\mapsto 1-q^A+B$, $b\mapsto -B$, $z\mapsto Z$ and then let $q\to 1$. In this case, $\lim_{q\to 1}\frac{a+bq^k}{1-q}=A+Bk$. Also, recall that $\lim_{q\to 1}((1-q)Z;q)_\infty=e^{-Z}$.

The formula

$$\frac{e^{AZ}}{1-BZ}=\sum_{k=0}^{\infty}\frac{(A+Bk)^k}{k!}Z^k e^{-BZk}, \tag{3.3}$$

where $\left|BZe^{1-BZ}\right|<1$ (cf. [41, Sec. 4.5]), is a companion of (3.1). A $q$-analogue of (3.3) is the expansion

$$\frac{1}{1-az}=\sum_{k=0}^{\infty}\frac{(a+bq^k)^k}{(q;q)_k}\left(zq(a+bq^k);q\right)_\infty z^k, \tag{3.4}$$

being valid for $|az|<1$. (Already this identity seems to be new. We will give even more general summations in Theorems 3.9 and 3.11.) To see that (3.4) is a $q$-analogue of (3.3), do the same replacements which lead from (3.2) to (3.1).

In [9], Carlitz gave multidimensional extensions of (3.1) and (3.3) being related to MacMahon's Master Theorem. $q$-Analogues of Carlitz's Abel-type expansion formulas were derived in [30, Theorem 7.1 and Eq. (10.5)]. Instead of using the multidimensional matrix inverse of [30, Theorem 3.1] we use special cases of our multidimensional matrix inversion in Theorem A.1 and obtain multiple extensions of (3.2) and (3.4) associated to the root system $A_r$. For the convergence of the following multidimensional ($q$-)Abel expansions, see Appendix C.

**Theorem 3.5** *Let $a$, $b$, $z$, and $x_1,\dots,x_r$ be indeterminate. Then there holds*

$$\begin{aligned}1=\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(&\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)q^{\sum_{i=1}^r(i-1)k_i}\prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q;q\right)_{k_i}^{-1}\\ &\times\prod_{i=1}^{r}(a+bx_i)(a+bx_iq^{k_i})^{|\mathbf{k}|-1}x_i^{rk_i-|\mathbf{k}|}\\ &\times(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}}z^{|\mathbf{k}|}\left(z\prod_{i=1}^r(a+bx_iq^{k_i});q\right)_\infty\Bigg),\end{aligned} \tag{3.6}$$

*provided $|a^rz|<\left|q^{\frac{r-1}{2}}x_j^{-r}\prod_{i=1}^r x_i\right|$ for $j=1,\dots,r$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.5)/(A.6). Then (2.3) holds for

$$\begin{aligned}a_{\mathbf{n}}=(-1)^{(r-1)|\mathbf{n}|}q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^r\binom{n_i}{2}+\sum_{i=1}^r(i-1)n_i}z^{|\mathbf{n}|}\prod_{i=1}^{r}x_i^{rn_i-|\mathbf{n}|}\\ \times\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)\end{aligned}$$

and

$$b_{\mathbf{k}} = (-1)^{(r-1)|\mathbf{k}|}\, q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i}\, z^{|\mathbf{k}|}\prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}$$
$$\times \left(z\textstyle\prod_{i=1}^r(a+bx_iq^{k_i});q\right)_\infty \prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)$$

by the $A_r$ ${}_0\phi_0$-summation (B.7) in Theorem B.6. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (3.6). ∎

**Theorem 3.7** *Let $a$, $b$, $z$, and $x_1, \ldots, x_r$ be indeterminate. Then there holds*

$$1 = \sum_{k_1,\ldots,k_r=0}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right) q^{\sum_{i=1}^r(i-1)k_i}\prod_{i,j=1}^r\left(\frac{x_i}{x_j}q;q\right)_{k_i}^{-1}$$
$$\times(a+b)(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|-1}(-1)^{(r-1)|\mathbf{k}|} \tag{3.8}$$
$$\times\, q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}}\prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}\cdot z^{|\mathbf{k}|}\left(z(a+bq^{|\mathbf{k}|});q\right)_\infty\Bigg),$$

*provided* $|az| < \left|q^{\frac{r-1}{2}}x_j^{-r}\prod_{i=1}^r x_i\right|$ *for* $j = 1, \ldots, r$.

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.11)/(A.12). Then (2.3) holds for

$$a_{\mathbf{n}} = (-1)^{(r-1)|\mathbf{n}|}\, q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^r\binom{n_i}{2}+\sum_{i=1}^r(i-1)n_i}\, z^{|\mathbf{n}|}\prod_{i=1}^r x_i^{rn_i-|\mathbf{n}|}$$
$$\times\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)$$

and

$$b_{\mathbf{k}} = (-1)^{(r-1)|\mathbf{k}|}\, q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i}\, z^{|\mathbf{k}|}\prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}$$
$$\times\left(z(a+bq^{|\mathbf{k}|});q\right)_\infty\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)$$

by the $A_r$ ${}_0\phi_0$-summation (B.7) in Theorem B.6. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $b \mapsto bq^{-|\mathbf{l}|}$, $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (3.8). ∎

The following two theorems contain "companion identities" of Theorems 3.5 and 3.7, respectively.

**Theorem 3.9** *Let $a$, $b$, $z$, and $x_1,\dots,x_r$ be indeterminate. Then there holds*

$$\frac{1}{1-a^r z}=\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)\prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q;q\right)_{k_i}^{-1}$$
$$\times q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i}\prod_{i=1}^r(a+bx_iq^{k_i})^{|\mathbf{k}|}x_i^{rk_i-|\mathbf{k}|} \tag{3.10}$$
$$\times(-1)^{(r-1)|\mathbf{k}|}z^{|\mathbf{k}|}\left(zq\prod_{i=1}^r(a+bx_iq^{k_i});q\right)_\infty\Bigg),$$

*provided $|a^r z|<\left|q^{\frac{r-1}{2}}x_j^{-r}\prod_{i=1}^r x_i\right|$ for $j=1,\dots,r$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.5)/(A.6). Then (2.3) holds for

$$a_{\mathbf{n}}=(-1)^{(r-1)|\mathbf{n}|}q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^r\binom{n_i}{2}+\sum_{i=1}^r(i-1)n_i}z^{|\mathbf{n}|}\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)$$
$$\times\prod_{i=1}^r x_i^{rn_i-|\mathbf{n}|}(a+bx_iq^{n_i})$$

and

$$b_{\mathbf{k}}=(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}+\sum_{i=1}^r(i-1)k_i}z^{|\mathbf{k}|}\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)$$
$$\times(1-a^r z)\left(zq\prod_{i=1}^r(a+bx_iq^{k_i});q\right)_\infty\prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}(a+bx_iq^{k_i}).$$

To see that (2.3) holds with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$, we apply the $t\mapsto -b/a$, $x_i\mapsto x_iq^{k_i}$, $y_i\mapsto q^{n_i-k_i}$, $i=1,\dots,r$, case of the partial fraction decomposition (B.5),

$$\prod_{i=1}^r\frac{(a+bx_iq^{n_i})}{(a+bx_iq^{k_i})}=q^{|\mathbf{n}|-|\mathbf{k}|}+\sum_{j=1}^r\frac{a\prod_{i=1}^r(1-q^{n_i-k_j}x_i/x_j)}{(a+bx_jq^{k_j})\prod_{\substack{i=1\\ i\ne j}}^r(1-q^{k_i-k_j}x_i/x_j)},$$

and interchange summations to split $\sum_{\mathbf{n}\ge\mathbf{k}}f_{\mathbf{nk}}a_{\mathbf{n}}$ in $r+1$ sums, each of which can be evaluated by means of the $A_r$ ${}_0\phi_0$-summation (B.7) in Theorem B.6. The first sum can be evaluated directly by (B.7). In the $j$-th of the remaining $r$ sums ($j=1,\dots,r$), we first have to perform the shift $n_j\mapsto n_j+1$ before we can evaluate the sum by (B.7) (with $x_j\mapsto x_jq$). In the remaining sum of $r+1$ products, we pull out common factors and simplify the rest again by application of partial fraction decomposition (here, the $t\mapsto -b/a$, $x_i\mapsto x_iq^{k_i}$, $y_i\mapsto 0$, $i=1,\dots,r$, case of (B.5)),

$$1-z\prod_{i=1}^{r}(a+bx_iq^{k_i})\sum_{j=1}^{r}\frac{a}{(a+bx_jq^{k_j})\prod\limits_{\substack{i=1\\i\neq j}}^{r}(1-q^{k_i-k_j}x_i/x_j)}=1-a^rz.$$

Collecting our calculations, we have established (2.3), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (3.10). ■

**Theorem 3.11** *Let $a$, $b$, $z$, and $x_1, \ldots, x_r$ be indeterminate. Then there holds*

$$\begin{aligned}\frac{1}{1-az}=\sum_{k_1,\ldots,k_r=0}^{\infty}&\Bigg(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)\prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q;q\right)_{k_i}^{-1}\\&\times(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|}\,q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^{r}\binom{k_i}{2}+\sum_{i=1}^{r}(i-1)k_i}\\&\times(-1)^{(r-1)|\mathbf{k}|}\prod_{i=1}^{r}x_i^{rk_i-|\mathbf{k}|}\cdot z^{|\mathbf{k}|}\left(zq(a+bq^{|\mathbf{k}|});q\right)_\infty\Bigg),\end{aligned}\tag{3.12}$$

*provided* $|az| < \left|q^{\frac{r-1}{2}}x_j^{-r}\prod_{i=1}^{r}x_i\right|$ *for* $j = 1, \ldots, r$.

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.11)/(A.12). Then (2.3) holds for

$$\begin{aligned}a_{\mathbf{n}}=(-1)^{(r-1)|\mathbf{n}|}\,q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^{r}\binom{n_i}{2}+\sum_{i=1}^{r}(i-1)n_i}&\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)\\&\times(a+bq^{|\mathbf{n}|})\,z^{|\mathbf{n}|}\prod_{i=1}^{r}x_i^{rn_i-|\mathbf{n}|}\end{aligned}$$

and

$$\begin{aligned}b_{\mathbf{k}}=(-1)^{(r-1)|\mathbf{k}|}\,q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^{r}\binom{k_i}{2}+\sum_{i=1}^{r}(i-1)k_i}&\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)\\\times(1-az)\left(zq(a+bq^{|\mathbf{k}|});q\right)_\infty&(a+bq^{|\mathbf{k}|})\,z^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{rk_i-|\mathbf{k}|}.\end{aligned}$$

To see that (2.3) holds with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$, we apply the $t \mapsto 0$, $x_i \mapsto x_i q^{k_i}$, $y_i \mapsto q^{n_i-k_i}$, $i = 1, \ldots, r$, case of the partial fraction decomposition (B.5),

$$\begin{aligned}\frac{(a+bq^{|\mathbf{n}|})}{(a+bq^{|\mathbf{k}|})}&=q^{|\mathbf{n}|-|\mathbf{k}|}+\frac{a(1-q^{|\mathbf{n}|-|\mathbf{k}|})}{(a+bq^{|\mathbf{k}|})}\\&=q^{|\mathbf{n}|-|\mathbf{k}|}+\frac{a}{(a+bq^{|\mathbf{k}|})}\sum_{j=1}^{r}\frac{\prod\limits_{i=1}^{r}(1-q^{n_i-k_j}x_i/x_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{r}(1-q^{k_i-k_j}x_i/x_j)},\end{aligned}$$

and interchange summations to split $\sum_{\mathbf{n}\geq\mathbf{k}} f_{\mathbf{nk}} a_{\mathbf{n}}$ in $r+1$ sums, each of which can be evaluated by means of the $A_r$ ${}_0\phi_0$-summation (B.7) in Theorem B.6. The first sum can be evaluated directly by (B.7). In the $j$-th of the remaining $r$ sums ($j=1,\dots,r$), we first have to perform the shift $n_j \mapsto n_j+1$ before we can evaluate the sum by (B.7) (with $x_j \mapsto x_j q$). In the remaining sum of $r+1$ products, we pull out common factors and simplify the rest again by application of partial fraction decomposition (here, the $t\mapsto 0$, $x_i \mapsto x_i q^{k_i}$, $y_i \mapsto 0$, $i=1,\dots,r$, case of (B.5)),

$$1 - az \sum_{j=1}^{r} \frac{1}{\prod_{\substack{i=1\\ i\neq j}}^{r} (1 - q^{k_i - k_j} x_i/x_j)} = 1 - az.$$

Collecting our calculations, we have established (2.3), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i=1,\dots,r$, and the substitutions $b \mapsto bq^{-|\mathbf{l}|}$, $x_i \mapsto x_i q^{-l_i}$, $i=1,\dots,r$, we get rid of the $l_i$'s and eventually obtain (3.12). ■

We finish this section with two $q\to 1$ limiting cases of our $A_r$ $q$-Abel expansions. Note, that the expansions (3.14) and (3.15) below do not generalize (3.1) and (3.3), for here we indeed require $r\geq 2$.

**Theorem 3.13** *Let $A$, $B$, $Z$, and $X_1,\dots,X_r$ be indeterminate. Then there holds*

$$1 = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( \prod_{1\leq i<j\leq r} \left( \frac{X_i + k_i - X_j - k_j}{X_i - X_j} \right) \prod_{i,j=1}^{r} (1 + X_i - X_j)_{k_i}^{-1} \\ \times A\,(A + B|\mathbf{k}|)^{|\mathbf{k}|-1}\,(-1)^{(r-1)|\mathbf{k}|}\,Z^{|\mathbf{k}|} \Bigg), \tag{3.14}$$

$$1 = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( \prod_{1\leq i<j\leq r} \left( \frac{X_i + k_i - X_j - k_j}{X_i - X_j} \right) \prod_{i,j=1}^{r} (1 + X_i - X_j)_{k_i}^{-1} \\ \times (A + B|\mathbf{k}|)^{|\mathbf{k}|}\,(-1)^{(r-1)|\mathbf{k}|}\,Z^{|\mathbf{k}|} \Bigg), \tag{3.15}$$

*provided $r\geq 2$.*

**Proof:** In (3.8) and (3.12), respectively, do the replacements $a\mapsto 1-q^A+B$, $b\mapsto -B$, $z\mapsto Z(1-q)^{r-1}$, $x_i\mapsto q^{X_i}$, $i=1,\dots,r$, and then let $q\to 1$. In this case, $\lim_{q\to 1}\frac{a+bq^{|\mathbf{k}|}}{1-q} = A + B|\mathbf{k}|$. Also, observe that $\lim_{q\to 1}((1-q)^r Z;q)_\infty = 1$ for $r\geq 2$. ■

## 4. $A_r$ ($q$-)Abel summations

A $q$-analogue of *Abel's theorem*

$$(A+C)^n = \sum_{k=0}^{n} \binom{n}{k} A(A+Bk)^{k-1}(C-Bk)^{n-k} \tag{4.1}$$

(cf. [41, Sec. 1.5]) is the summation

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a+b)(a+bq^k)^{k-1} c^k \, (c(a+bq^k);q)_{n-k} \tag{4.2}$$

(see [26], [30, Eq. (8.1)]). To see that (4.2) is a $q$-analogue of (4.1), do the replacements $c \mapsto 1$, $a \mapsto \frac{A}{(A+C)} + \frac{B}{(A+C)(1-q)}$, $b \mapsto -\frac{B}{(A+C)(1-q)}$, and then let $q \to 1$. Another $q$-analogue of (4.1) is

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1-c(a+b))}{(1-c(a+bq^{-k}))} (c(a+bq^{-k});q)_k \, (a+bq^{-k})^{n-k} c^{n-k}. \tag{4.3}$$

To see that (4.3) is a $q$-analogue of (4.1), do the replacements $c \mapsto 1$, $a \mapsto \frac{A}{(A+C)} - \frac{B}{(A+C)(1-q)}$, $b \mapsto \frac{B}{(A+C)(1-q)}$, and then let $q \to 1$.

Other, nonsymmetric, $q$-Abel summations can be found in [27], being derived there by means of umbral calculus.

In [9], Carlitz gave a multidimensional extension of (4.1) being related to MacMahon's Master Theorem. A $q$-analogue of Carlitz's Abel summation formula was derived in [30, Theorem 8.1]. Here we use special cases of our multidimensional matrix inversion in Theorem A.1 to derive several multiple extensions of (4.2) and (4.3) associated to the root system $A_r$.

In fact, we continue the work on $A_r$ $q$-Abel summations initiated by Bhatnagar and Milne [5]. These authors have already given one $A_r$ $q$-Abel summation theorem (see Remark 4.6) which they derived by multidimensional matrix inversion. These authors have also noted that by the same method several more multiple $q$-Abel summations can be derived. We take up this matter, and provide a bunch of more $A_r$ $q$-Abel summation theorems.

**Theorem 4.4** *Let $a$, $b$, $c$, and $x_1, \dots, x_r$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$1 = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j} q;q\right)_{n_i}}{\left(\frac{x_i}{x_j} q;q\right)_{k_i} \left(\frac{x_i}{x_j} q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] c^{|\mathbf{k}|} \\ \times \left(c \textstyle\prod_{i=1}^{r}(a+bx_iq^{k_i});q\right)_{|\mathbf{n}|-|\mathbf{k}|} \displaystyle\prod_{i=1}^{r} (a+bx_i)(a+bx_iq^{k_i})^{|\mathbf{k}|-1} \Bigg). \tag{4.5}$$

*Remark 4.6.* The Abel summation (4.5), with reversed order of summations, is equivalent to Bhatnagar and Milne's Abel binomial theorem [5, Theorem 5.15].

**Proof of Theorem 4.4:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.8)/(A.9). Then (2.2) holds for

$$a_{\mathbf{k}} = \left(\frac{q}{c\prod_{i=1}^{r}(ax_iq^{k_i}+b)};q\right)_{|\mathbf{k}|}$$

and

$$b_{\mathbf{l}} = q^{\binom{|\mathbf{l}|+1}{2}}\,c^{-|\mathbf{l}|}$$

by the $A_r$ terminating $q$-binomial theorem (B.10) in Theorem B.9. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse oder of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1,\dots,r$. After performing the substitutions $x_i \mapsto q^{-n_i}/x_i$, $i = 1,\dots,r$, and $c \mapsto cq^{|\mathbf{n}|}\prod_{j=1}^{r} x_j$, we eventually obtain (4.5). ∎

*Remark 4.7.* We reversed order of summations in the proof of Theorem 4.4 because we prefer to have the sum in a form where the "contiguous" factor

$$\prod_{i=1}^{r}\frac{(a+bx_i)}{(a+bx_iq^{k_i})}$$

appearing in the summand of the series cancels when $\mathbf{k} = \mathbf{0}$ (instead of $\mathbf{k} = \mathbf{n}$). Similar reasoning holds for most of the other terminating summation theorems in this article.

**Theorem 4.8** *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$\begin{aligned}1 = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\Bigg(\prod_{i,j=1}^{r}&\left[\frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}}\right]q^{-e_2(\mathbf{k})}\,c^{|\mathbf{k}|}\\ &\times\prod_{i=1}^{r}(a+bx_i)(a+bx_iq^{k_i})^{|\mathbf{k}|-1}x_i^{k_i}\\ &\times\prod_{i=1}^{r}\left(cx_iq^{k_i-|\mathbf{k}|}\prod_{j=1}^{r}(a+bx_jq^{k_j});q\right)_{n_i-k_i}\Bigg),\end{aligned}\tag{4.9}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.8)/(A.9). Then (2.2) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r}\left(\frac{qx_i}{c\prod_{j=1}^{r}(ax_jq^{k_j}+b)};q\right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{|\mathbf{l}|+\sum_{i=1}^r \binom{l_i}{2}} c^{-|\mathbf{l}|} \prod_{i=1}^r x_i^{l_i}$$

by the $A_r$ terminating $q$-binomial theorem (B.11) in Theorem B.9. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \ldots, r$. After performing the substitutions $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \ldots, r$, and $c \mapsto c\prod_{j=1}^r x_j$, we eventually obtain (4.9). ∎

**Theorem 4.10** *Let $a$, $b$, $c$, and $x_1, \ldots, x_r$ be indeterminate, and let $n_1, \ldots, n_r$ be nonnegative integers. Then there holds*

$$\begin{aligned} 1 = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\ldots,r}} \Bigg( \prod_{i,j=1}^r \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] q^{e_2(\mathbf{k})} c^{|\mathbf{k}|} \\ \times \prod_{i=1}^r (a+bx_i)(a+bx_iq^{k_i})^{|\mathbf{k}|-1} x_i^{-k_i} \\ \times \prod_{i=1}^r \left( \frac{c}{x_i} q^{|\mathbf{n}|-n_i} \prod_{j=1}^r (a+bx_jq^{k_j});q \right)_{n_i-k_i} \Bigg), \end{aligned} \tag{4.11}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \ldots, k_r\}$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.8)/(A.9). Then (2.2) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^r \left( \frac{q^{1+|\mathbf{k}|-k_i}}{cx_i \prod_{j=1}^r (ax_jq^{k_j}+b)};q \right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{e_2(\mathbf{l})+\binom{|\mathbf{l}|+1}{2}} c^{-|\mathbf{l}|} \prod_{i=1}^r x_i^{-l_i}$$

by the $A_r$ terminating $q$-binomial theorem (B.12) in Theorem B.9. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \ldots, r$. After performing the substitutions $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \ldots, r$, and $c \mapsto cq^{2|\mathbf{n}|}\prod_{j=1}^r x_j$, we eventually obtain (4.11). ∎

**Theorem 4.12** *Let $a$, $b$, $c$, and $x_1, \ldots, x_r$ be indeterminate, and let $n_1, \ldots, n_r$ be nonnegative integers. Then there holds*

$$\begin{aligned} 1 = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\ldots,r}} \Bigg( \prod_{i,j=1}^r \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \\ \times (a+b)(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|-1} c^{|\mathbf{k}|} \left(c(a+bq^{|\mathbf{k}|});q\right)_{|\mathbf{n}|-|\mathbf{k}|} \Bigg). \end{aligned} \tag{4.13}$$

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.14)/(A.15). Then (2.2) holds for

$$a_{\mathbf{k}} = \left(\frac{q}{c(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|}$$

and

$$b_{\mathbf{l}} = q^{\binom{|\mathbf{l}|+1}{2}}\,c^{-|\mathbf{l}|}$$

by the $A_r$ terminating $q$-binomial theorem (B.10) in Theorem B.9. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1,\dots,r$. After performing the substitutions $a \mapsto aq^{-|\mathbf{n}|}$, $c \mapsto cq^{|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1,\dots,r$, we eventually obtain (4.13). ■

**Theorem 4.14** *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$1 = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left(\prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}}\right] q^{-e_2(\mathbf{k})}\,c^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{k_i}\right.$$
$$\left.\times\,(a+b)(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|-1}\prod_{i=1}^{r}\left(cx_iq^{k_i}(aq^{-|\mathbf{k}|}+b);q\right)_{n_i-k_i}\right), \tag{4.15}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.14)/(A.15). Then (2.2) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r}\left(\frac{qx_i}{c(aq^{|\mathbf{k}|}+b)};q\right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{|\mathbf{l}|+\sum_{i=1}^{r}\binom{l_i}{2}}\,c^{-|\mathbf{l}|}\prod_{i=1}^{r}x_i^{l_i}$$

by the $A_r$ terminating $q$-binomial theorem (B.11) in Theorem B.9. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1,\dots,r$. After performing the substitutions $a \mapsto aq^{-|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1,\dots,r$, we eventually obtain (4.15). ■

**Theorem 4.16** *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$1 = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j} q; q\right)_{n_i}}{\left(\frac{x_i}{x_j} q; q\right)_{k_i} \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}} \right] q^{e_2(\mathbf{k})} c^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{-k_i} \right. \\ \left. \times (a+b)(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|-1} \prod_{i=1}^{r} \left( \frac{c}{x_i} q^{|\mathbf{n}|-n_i} (a+bq^{|\mathbf{k}|}); q \right)_{n_i-k_i} \right), \tag{4.17}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \dots, k_r\}$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.14)/(A.15). Then (2.2) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \left( \frac{q^{1+|\mathbf{k}|-k_i}}{c x_i (a q^{|\mathbf{k}|} + b)}; q \right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{e_2(\mathbf{l}) + \binom{|\mathbf{l}|+1}{2}} c^{-|\mathbf{l}|} \prod_{i=1}^{r} x_i^{-l_i}$$

by the $A_r$ terminating $q$-binomial theorem (B.12) in Theorem B.9. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \dots, r$. After performing the substitutions $a \mapsto aq^{-|\mathbf{n}|}$, $c \mapsto cq^{2|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \dots, r$, we eventually obtain (4.17). ■

The following two theorems contain "companion identities" of Theorems 4.4 and 4.12, respectively.

**Theorem 4.18** *Let $a$, $b$, $c$, and $x_1, \dots, x_r$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$1 = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j} q; q\right)_{n_i}}{\left(\frac{x_i}{x_j} q; q\right)_{k_i} \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}} \right] \right. \\ \times \frac{(1 - c\prod_{i=1}^{r}(ax_i + b))}{(1 - c\prod_{i=1}^{r}(ax_i + bq^{-k_i}))} \left(c\prod_{i=1}^{r}(ax_i + bq^{-k_i}); q\right)_{|\mathbf{k}|} \\ \left. \times c^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i=1}^{r} (ax_i + bq^{-k_i})^{|\mathbf{n}|-|\mathbf{k}|} \right). \tag{4.19}$$

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.8)/(A.9). Then (2.2) holds for

$$a_{\mathbf{k}} = \left(1 - \frac{1}{c\prod_{i=1}^{r}(ax_i + b)}\right) \left( \frac{q}{c\prod_{i=1}^{r}(ax_i q^{k_i} + b)}; q \right)_{|\mathbf{k}|-1}$$

and

$$b_{\mathbf{l}} = q^{\binom{|\mathbf{l}|}{2}}\, c^{-|\mathbf{l}|} \prod_{i=1}^{r} \frac{(ax_i q^{l_i} + b)}{(ax_i + b)}.$$

To see that (2.2) holds with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$, we apply the $t \mapsto -a/b$, $x_i \mapsto x_i$, $y_i \mapsto q^{l_i}$, $i = 1, \dots, r$, case of the partial fraction decomposition (B.5),

$$\prod_{i=1}^{r} \frac{(ax_i q^{l_i} + b)}{(ax_i + b)} = q^{|\mathbf{l}|} + \sum_{j=1}^{r} \frac{b \prod\limits_{i=1}^{r} (1 - q^{l_i} x_i/x_j)}{(ax_j + b) \prod\limits_{\substack{i=1\\ i\neq j}}^{r} (1 - x_i/x_j)},$$

and interchange summations to split $\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}} b_{\mathbf{l}}$ in $r+1$ sums, each of which can be evaluated by means of the $A_r$ terminating $q$-binomial theorem (B.10) in Theorem B.9. The first sum can be evaluated directly by (B.10). In the $j$-th of the remaining $r$ sums ($j = 1, \dots, r$), we first have to perform the shift $l_j \mapsto l_j + 1$ before we can evaluate the sum by (B.10) (with $x_j \mapsto x_j q$). This gives

$$\begin{aligned}
&\left(\frac{q}{c\prod_{i=1}^{r}(ax_i q^{k_i} + b)}; q\right)_{|\mathbf{k}|} \\
&- \frac{1}{c\prod_{i=1}^{r}(ax_i q^{k_i} + b)} \sum_{j=1}^{r} \frac{b \prod\limits_{i=1}^{r} (1 - q^{k_i} x_i/x_j)}{(ax_j + b) \prod\limits_{\substack{i=1\\ i\neq j}}^{r} (1 - x_i/x_j)} \left(\frac{q}{c\prod_{i=1}^{r}(ax_i q^{k_i} + b)}; q\right)_{|\mathbf{k}|-1}.
\end{aligned}$$

Now, by application of partial fraction decomposition (here, the $t \mapsto -a/b$, $x_i \mapsto x_i$, $y_i \mapsto q^{k_i}$, $i = 1, \dots, r$, case of (B.5)), and simplification, this is transformed into $a_{\mathbf{k}}$. Thus we have established (2.2), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. We simplify the resulting identity a bit, and eventually obtain (4.19). ■

**Theorem 4.20** *Let $a$, $b$, $c$, and $x_1, \dots, x_r$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$\begin{aligned}
1 = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j} q; q\right)_{n_i}}{\left(\frac{x_i}{x_j} q; q\right)_{k_i} \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}} \right] c^{|\mathbf{n}|-|\mathbf{k}|} \\
\times \frac{(1 - c(a+b))}{(1 - c(a + bq^{-|\mathbf{k}|}))} \left(c(a + bq^{-|\mathbf{k}|}); q\right)_{|\mathbf{k}|} (a + bq^{-|\mathbf{k}|})^{|\mathbf{n}|-|\mathbf{k}|} \Bigg).
\end{aligned} \tag{4.21}$$

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.14)/(A.15). Then (2.2) holds for

$$a_{\mathbf{k}}=\left(1-\frac{1}{c(a+b)}\right)\left(\frac{q}{c(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|-1}$$

and

$$b_{\mathbf{l}}=q^{\binom{|\mathbf{l}|}{2}}\,c^{-|\mathbf{l}|}\,\frac{(aq^{|\mathbf{l}|}+b)}{(a+b)}.$$

To see that (2.2) holds with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$, we apply the $t\mapsto 0$, $x_i\mapsto x_i$, $y_i\mapsto q^{l_i}$, $i=1,\dots,r$, case of the partial fraction decomposition (B.5),

$$\frac{(aq^{|\mathbf{l}|}+b)}{(a+b)}=q^{|\mathbf{l}|}+\frac{b(1-q^{|\mathbf{l}|})}{(a+b)}=q^{|\mathbf{l}|}+\frac{b}{(a+b)}\sum_{j=1}^{r}\frac{\prod_{i=1}^{r}(1-q^{l_i}x_i/x_j)}{\prod_{\substack{i=1\\i\neq j}}^{r}(1-x_i/x_j)},$$

and interchange summations to split $\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}}g_{\mathbf{kl}}b_{\mathbf{l}}$ in $r+1$ sums, each of which can be evaluated by means of the $A_r$ terminating $q$-binomial theorem (B.10) in Theorem B.9. The first sum can be evaluated directly by (B.10). In the $j$-th of the remaining $r$ sums ($j=1,\dots,r$), we first have to perform the shift $l_j\mapsto l_j+1$ before we can evaluate the sum by (B.10) (with $x_j\mapsto x_jq$). This gives

$$\left(\frac{q}{c(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|}$$
$$-\frac{b}{(a+b)\,c(aq^{|\mathbf{k}|}+b)}\sum_{j=1}^{r}\frac{\prod_{i=1}^{r}(1-q^{k_i}x_i/x_j)}{\prod_{\substack{i=1\\i\neq j}}^{r}(1-x_i/x_j)}\left(\frac{q}{c(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|-1}.$$

Now, by application of partial fraction decomposition (here, the $t\mapsto 0$, $x_i\mapsto x_i$, $y_i\mapsto q^{k_i}$, $i=1,\dots,r$, case of (B.5)), and simplification, this is transformed into $a_{\mathbf{k}}$. Thus we have established (2.2), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. We simplify the resulting identity a bit, and eventually obtain (4.21). ■

We finish this section with two $q\to 1$ limiting cases of our $A_r$ $q$-Abel summations. These are higher-dimensional generalizations of the Abel summation (4.1).

**Theorem 4.22** *Let $A$, $B$, $C$, and $X_1,\dots,X_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$(A+C)^{|\mathbf{n}|}=\sum_{\substack{0\le k_i\le n_i\\i=1,\dots,r}}\left(\prod_{i,j=1}^{r}\left[\frac{(1+X_i-X_j)_{n_i}}{(1+X_i-X_j)_{k_i}\,(1+X_i+k_i-X_j-k_j)_{n_i-k_i}}\right]\right.$$
$$\left.\times A\,(A+B|\mathbf{k}|)^{|\mathbf{k}|-1}\,(C-B|\mathbf{k}|)^{|\mathbf{n}|-|\mathbf{k}|}\right).\tag{4.23}$$

**Proof:** In (4.13), do the replacements $a \mapsto \frac{A}{(A+C)} + \frac{B}{(A+C)(1-q)}$, $b \mapsto -\frac{B}{(A+C)(1-q)}$, $c \mapsto 1$, $x_i \mapsto q^{X_i}$, $i = 1, \dots, r$, and then let $q \to 1$. ∎

*Remark 4.24.* The same specializations as above applied to (4.15), (4.17), or (4.21) (up to relabelling), also give identity (4.23).

**Theorem 4.25** *Let $A_1, \dots, A_r$, $B$, and $C$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$C^{|\mathbf{n}|} = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(1 + \frac{A_i}{B} - \frac{A_j}{B}\right)_{n_i}}{\left(1 + \frac{A_i}{B} - \frac{A_j}{B}\right)_{k_i} \left(1 + \frac{A_i}{B} + k_i - \frac{A_j}{B} - k_j\right)_{n_i - k_i}} \right] \right. \\ \left. \times \left(C - \prod_{j=1}^{r}(A_j + Bk_j)\right)^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i=1}^{r} A_i \left(A_i + Bk_i\right)^{|\mathbf{k}|-1} \right). \tag{4.26}$$

**Proof:** In (4.5), do the replacements $a \mapsto \frac{B}{C^{1/r}(1-q)}$, $b \mapsto -\frac{B}{C^{1/r}(1-q)}$, $c \mapsto 1$, $x_i \mapsto q^{A_i/B}$, $i = 1, \dots, r$, and then let $q \to 1$. ∎

*Remark 4.27.* The summation in Theorem 4.25 is equivalent (with reversed order of summations) to [5, Corollary 5.21]. The same specializations as above applied to (4.9), (4.11), or (4.19) (up to relabelling), also give identity (4.26).

## 5. $A_r$ ($q$-)Rothe-type expansions

Another formula due to Euler (compare with (3.1)) is

$$(1+Z)^A = \sum_{k=0}^{\infty} \frac{A}{A+Bk} \binom{A+Bk}{k} Z^k (1+Z)^{-Bk}, \tag{5.1}$$

where $\left|\frac{(B-1)Z}{(1+Z)^B}\right| < 1$ [13, p. 350] (cf. [41, Sec. 4.5]). A $q$-analogue of this identity is

$$(z;q)_\infty = \sum_{k=0}^{\infty} \frac{1-(a+b)}{1-(aq^{-k}+b)} \frac{(aq^{-k}+b;q)_k}{(q;q)_k} (-1)^k q^{\binom{k}{2}} \left(z(a+bq^k);q\right)_\infty z^k, \tag{5.2}$$

being valid for $|az| < 1$ [30, Eq. (7.4)]. To see that (5.2) is a $q$-analogue of (5.1), do the replacements $a \mapsto q^A - B$, $b \mapsto B$, $z \mapsto -Z$ and then let $q \to 1$. In this case, $\lim_{q\to 1} \frac{1-(aq^{-k}+b)q^j}{1-q} = A + Bk + j - k$. Furthermore, we use $\lim_{q\to 1} \frac{(z(a+bq^k);q)_\infty}{(z;q)_\infty} = (1+Z)^{-A-Bk}$.

The formula

$$\frac{(1+Z)^A}{1 - \frac{BZ}{1+Z}} = \sum_{k=0}^{\infty} \binom{A+Bk}{k} Z^k (1+Z)^{-Bk}, \tag{5.3}$$

where $\left|\frac{(B-1)Z}{(1+Z)^B}\right| < 1$ (cf. [41, Sec. 4.5]), is a companion of (5.1). A $q$-analogue of (5.3) is the expansion

$$\frac{(zq;q)_\infty}{1-az} = \sum_{k=0}^{\infty} \frac{(aq^{-k}+b;q)_k}{(q;q)_k} (-1)^k q^{\binom{k}{2}} \left(zq(a+bq^k);q\right)_\infty q^k z^k, \tag{5.4}$$

being valid for $|az| < 1$. (Already this identity seems to be new. We will give even more general summations in Theorems 5.9 and 5.12.) To see that (5.4) is a $q$-analogue of (5.3), we can do similar replacements which lead from (5.2) to (5.1).

In [9], Carlitz gave multidimensional extensions of (5.1) and (5.3) being related to MacMahon's Master Theorem. $q$-Analogues of Carlitz's Rothe-type expansion formulas were derived in [30, Theorem 7.3 and Eq. (9.2)]. From these a (noncommutative) $q$-analogue of MacMahon's Master Theorem was deduced [30, Theorem 9.2]. Also other multiple extensions of (5.1) and (5.2), associated to the root system $A_r$, were found [30, Theorems 7.8 and 7.6]. Instead of using the multidimensional matrix inverse of [30, Theorem 3.1] we use special cases of our multidimensional matrix inversion in Theorem A.1 to derive multiple extensions of (5.2) and (5.4) associated to the root system $A_r$. For the convergence of the following multidimensional ($q$-)Rothe expansions, see Appendix C.

**Theorem 5.5** *Let $a$, $b$, $z$, and $x_1,\dots,x_r$ be indeterminate. Then there holds*

$$\begin{aligned}(z;q)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( \prod_{1\le i<j\le r} \left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right) q^{\sum_{i=1}^r (i-1)k_i} \prod_{i,j=1}^{r} \left(\frac{x_i}{x_j}q;q\right)_{k_i}^{-1} \\ \times \frac{(1-(a+b))}{(1-(aq^{-|\mathbf{k}|}+b))} (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|}\, z^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \\ \times (-1)^{r|\mathbf{k}|} q^{r\sum_{i=1}^r \binom{k_i}{2}} \left(z(a+bq^{|\mathbf{k}|});q\right)_\infty \Bigg),\end{aligned} \tag{5.6}$$

*provided $|az| < \left|q^{\frac{r-1}{2}} x_j^{-r} \prod_{i=1}^r x_i\right|$ for $j=1,\dots,r$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.17)/(A.18). Then (2.3) holds for

$$\begin{aligned}a_{\mathbf{n}} = (-1)^{(r-1)|\mathbf{n}|} q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^r \binom{n_i}{2}+\sum_{i=1}^r (i-1)n_i} \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right) \\ \times\; z^{|\mathbf{n}|} (z;q)_{|\mathbf{n}|}^{-1} \prod_{i=1}^r x_i^{rn_i-|\mathbf{n}|}\end{aligned}$$

and

$$\begin{aligned}b_{\mathbf{k}} = (-1)^{(r-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r \binom{k_i}{2}+\sum_{i=1}^r (i-1)k_i} \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right) \\ \times\; z^{|\mathbf{k}|} \frac{(z(a+bq^{|\mathbf{k}|});q)_\infty}{(z;q)_\infty} \prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}\end{aligned}$$

by the $A_r$ ${}_1\phi_1$-summation (B.15) in Theorem B.14. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $a \mapsto aq^{|\mathbf{l}|}$, $z \mapsto zq^{-|\mathbf{l}|}$, $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (5.6). ■

**Theorem 5.7** *Let $a$, $b$, $z$, and $x_1, \ldots, x_r$ be indeterminate. Then there holds*

$$(z;q)_\infty = \sum_{k_1,\ldots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) q^{\sum_{i=1}^r (i-1)k_i} \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j} q; q \right)_{k_i}^{-1} \right. \\ \times \frac{(1-(a+b))}{(1-(aq^{-|\mathbf{k}|}+b))} (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|}\, z^{|\mathbf{k}|} \\ \left. \times (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}} \left( z(a+bq^{|\mathbf{k}|});q \right)_\infty \right), \tag{5.8}$$

*provided $|az| < 1$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.17)/(A.18). Then (2.3) holds for

$$a_{\mathbf{n}} = z^{|\mathbf{n}|} (z;q)_{|\mathbf{n}|}^{-1} q^{\sum_{i=1}^r (i-1)n_i} \prod_{1\le i<j\le r} \left( 1 - \frac{x_i}{x_j} q^{n_i-n_j} \right)$$

and

$$b_{\mathbf{k}} = z^{|\mathbf{k}|} \frac{\left( z(a+bq^{|\mathbf{k}|});q \right)_\infty}{(z;q)_\infty} q^{\sum_{i=1}^r (i-1)k_i} \prod_{1\le i<j\le r} \left( 1 - \frac{x_i}{x_j} q^{k_i-k_j} \right)$$

by the $A_r$ ${}_1\phi_1$-summation (B.16) in Theorem B.14. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $a \mapsto aq^{|\mathbf{l}|}$, $z \mapsto zq^{-|\mathbf{l}|}$, $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (5.8). ■

The following two theorems contain "companion identities" of Theorems 5.5 and 5.7, respectively.

**Theorem 5.9** *Let $a$, $b$, $z$, and $x_1, \ldots, x_r$ be indeterminate. Then there holds*

$$\frac{(zq;q)_\infty}{1-az} = \sum_{k_1,\ldots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) q^{\sum_{i=1}^r (i-1)k_i} \right. \\ \times (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|} (-1)^{r|\mathbf{k}|} z^{|\mathbf{k}|} \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j} q; q \right)_{k_i}^{-1} \\ \left. \times q^{|\mathbf{k}|+r\sum_{i=1}^r \binom{k_i}{2}} \left( zq(a+bq^{|\mathbf{k}|});q \right)_\infty \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \right), \tag{5.10}$$

*provided* $|az| < \left|q^{\frac{r-1}{2}} x_j^{-r} \prod_{i=1}^r x_i\right|$ *for* $j = 1, \ldots, r$.

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.17)/(A.18). Then (2.3) holds for

$$a_{\mathbf{n}} = (-1)^{(r-1)|\mathbf{n}|} q^{-\binom{|\mathbf{n}|}{2}+r\sum_{i=1}^r \binom{n_i}{2}+\sum_{i=1}^r (i-1)n_i} \prod_{1\le i<j\le r} \left(1 - \frac{x_i}{x_j} q^{n_i-n_j}\right) \\ \times (a + (b-1)q^{|\mathbf{n}|})\, z^{|\mathbf{n}|}\, (zq;q)_{|\mathbf{n}|}^{-1} \prod_{i=1}^r x_i^{rn_i-|\mathbf{n}|}$$

and

$$b_{\mathbf{k}} = (-1)^{(r-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r \binom{k_i}{2}+\sum_{i=1}^r (i-1)k_i} \prod_{1\le i<j\le r} \left(1 - \frac{x_i}{x_j} q^{k_i-k_j}\right) \\ \times (a + (b-1)q^{|\mathbf{k}|})\, z^{|\mathbf{k}|}\, (1-az) \frac{\left(zq(a+bq^{|\mathbf{k}|});q\right)_\infty}{(zq;q)_\infty} \prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}.$$

To see that (2.3) holds with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$, we apply the $t \mapsto -(b-1)/a$, $x_i \mapsto x_i q^{k_i}$, $y_i \mapsto q^{n_i-k_i}$, $i = 1, \ldots, r$, case of the partial fraction decomposition (B.5),

$$\begin{aligned} \frac{(a + (b-1)q^{|\mathbf{n}|})}{(a + (b-1)q^{|\mathbf{k}|})} &= q^{|\mathbf{n}|-|\mathbf{k}|} + \frac{a(1-q^{|\mathbf{n}|-|\mathbf{k}|})}{(a + (b-1)q^{|\mathbf{k}|})} \\ &= q^{|\mathbf{n}|-|\mathbf{k}|} + \frac{a}{(a + (b-1)q^{|\mathbf{k}|})} \sum_{j=1}^r \frac{\prod_{i=1}^r (1 - q^{n_i-k_j} x_i/x_j)}{\prod_{i=1, i\ne j}^r (1 - q^{k_i-k_j} x_i/x_j)}, \end{aligned} \tag{5.11}$$

and interchange summations to split $\sum_{\mathbf{n}\ge\mathbf{k}} f_{\mathbf{nk}} a_{\mathbf{n}}$ in $r+1$ sums, each of which can be evaluated by means of the $A_r$ ${}_1\phi_1$-summation (B.15) in Theorem B.14. The first sum can be evaluated directly by (B.15). In the $j$-th of the remaining $r$ sums ($j = 1, \ldots, r$), we first have to perform the shift $n_j \mapsto n_j + 1$ before we can evaluate the sum by (B.15) (with $x_j \mapsto x_j q$). In the remaining sum of $r+1$ products, we pull out common factors and simplify the rest again by application of partial fraction decomposition (here, the $t \mapsto 0$, $x_i \mapsto x_i q^{k_i}$, $y_i \mapsto 0$, $i = 1, \ldots, r$, case of (B.5)),

$$1 - az \sum_{j=1}^r \frac{1}{\prod_{i=1, i\ne j}^r (1 - q^{k_i-k_j} x_i/x_j)} = 1 - az.$$

Collecting our calculations, we have established (2.3), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $a \mapsto aq^{|\mathbf{l}|}$, $z \mapsto zq^{-|\mathbf{l}|}$, $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (5.10). ∎

**Theorem 5.12** *Let $a$, $b$, $z$, and $x_1, \dots, x_r$ be indeterminate. Then there holds*

$$\frac{(zq;q)_\infty}{1-az} = \sum_{k_1,\dots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) q^{\sum_{i=1}^r (i-1)k_i} \right.$$
$$\times (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|}\,(-1)^{|\mathbf{k}|} \prod_{i,j=1}^{r} \left(\frac{x_i}{x_j}q;q\right)_{k_i}^{-1} \tag{5.13}$$
$$\left. \times z^{|\mathbf{k}|}\, q^{|\mathbf{k}|+\binom{|\mathbf{k}|}{2}} \left(zq(a+bq^{|\mathbf{k}|});q\right)_\infty \right),$$

*provided* $|az|<1$.

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.17)/(A.18). Then (2.3) holds for

$$a_{\mathbf{n}} = (a+(b-1)q^{|\mathbf{n}|})\, z^{|\mathbf{n}|}\, (z;q)_{|\mathbf{n}|}^{-1}\, q^{\sum_{i=1}^r (i-1)n_i} \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)$$

and

$$b_{\mathbf{k}} = (a+(b-1)q^{|\mathbf{k}|})\,(1-az)\, z^{|\mathbf{k}|}\, \frac{\left(zq(a+bq^{|\mathbf{k}|});q\right)_\infty}{(zq;q)_\infty}\, q^{\sum_{i=1}^r (i-1)k_i}$$
$$\times \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right).$$

To see that (2.3) holds with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$, we apply the $t\mapsto -(b-1)/a$, $x_i\mapsto x_iq^{k_i}$, $y_i\mapsto q^{n_i-k_i}$, $i=1,\dots,r$, case of the partial fraction decomposition (B.5), as in (5.11), and interchange summations to split $\sum_{\mathbf{n}\ge\mathbf{k}} f_{\mathbf{nk}}a_{\mathbf{n}}$ in $r+1$ sums, each of which can be evaluated by means of the $A_r$ ${}_1\phi_1$-summation (B.16) in Theorem B.14. The first sum can be evaluated directly by (B.16). In the $j$-th of the remaining $r$ sums ($j=1,\dots,r$), we first have to perform the shift $n_j\mapsto n_j+1$ before we can evaluate the sum by (B.16) (with $x_j\mapsto x_jq$). In the remaining sum of $r+1$ products, we pull out common factors and simplify the rest again by application of partial fraction decomposition (here, the $t\mapsto 0$, $x_i\mapsto q^{-k_i}/x_i$, $y_i\mapsto 0$, $i=1,\dots,r$, case of (B.5)),

$$1+az\sum_{j=1}^{r} \frac{(-1)^r q^{|\mathbf{k}|-rk_j} x_j^{-r} \prod_{i=1}^r x_i}{\prod_{\substack{i=1\\ i\ne j}}^{r} (1-q^{k_i-k_j}x_i/x_j)} = 1-az.$$

Collecting our calculations, we have established (2.3), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i\mapsto k_i+l_i$, $i=1,\dots,r$, and the substitutions $a\mapsto aq^{|\mathbf{l}|}$, $z\mapsto zq^{-|\mathbf{l}|}$, $x_i\mapsto x_iq^{-l_i}$, $i=1,\dots,r$, we get rid of the $l_i$'s and eventually obtain (5.13). ∎

We finish this section with two $q \to 1$ limiting cases of our $A_r$ $q$-Rothe expansions. Note, that the expansions (5.15) and (5.16) below do not generalize (5.1) and (5.3), for here we indeed require $r \geq 2$.

**Theorem 5.14** *Let $A$, $B$, $Z$, and $X_1, \ldots, X_r$ be indeterminate. Then there holds*

$$1 = \sum_{k_1,\ldots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{X_i + k_i - X_j - k_j}{X_i - X_j} \right) \prod_{i,j=1}^{r} (1 + X_i - X_j)_{k_i}^{-1} \right.$$
$$\left. \times \frac{A}{A + B|\mathbf{k}|} (A + B|\mathbf{k}|)_{|\mathbf{k}|} Z^{|\mathbf{k}|} \right), \tag{5.15}$$

$$1 = \sum_{k_1,\ldots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{X_i + k_i - X_j - k_j}{X_i - X_j} \right) \prod_{i,j=1}^{r} (1 + X_i - X_j)_{k_i}^{-1} \right.$$
$$\left. \times (A + B|\mathbf{k}|)_{|\mathbf{k}|} Z^{|\mathbf{k}|} \right), \tag{5.16}$$

*provided $r \geq 2$.*

**Proof:** In (5.8) and (5.13), respectively, do the replacements $a \mapsto q^A - B - 1$, $b \mapsto B + 1$, $z \mapsto -Z(1-q)^{r-1}$, $x_i \mapsto q^{X_i}$, $i = 1, \ldots, r$, and then let $q \to 1$. In this case, $\lim_{q\to 1} \frac{1-(aq^{-|\mathbf{k}|}+b)q^j}{1-q} = A + B|\mathbf{k}| + j$. Also, observe that $\lim_{q\to 1} \frac{(z(a+bq^{|\mathbf{k}|});q)_\infty}{(z;q)_\infty} = 1$ since $r \geq 2$. ■

*Remark 5.17.* Note, that if we specialize (5.6) and (5.10) as above, we would obtain (5.15) and (5.16), respectively, with $Z$ replaced by $(-1)^{r-1}Z$.

## 6. $A_r$ ($q$-)Rothe summations

A $q$-analogue of the *(Hagen-)Rothe summation formula* [19]

$$\binom{A+C}{n} = \sum_{k=0}^{n} \frac{A}{A+Bk} \binom{A+Bk}{k} \binom{C-Bk}{n-k} \tag{6.1}$$

is the summation

$$(c;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-(a+b)}{1-(aq^{-k}+b)} (aq^{-k}+b;q)_k$$
$$\times \; (c\,(a+bq^k);q)_{n-k}\,(-1)^k q^{\binom{k}{2}}\, c^k \tag{6.2}$$

(see [27], [30, Eq. (8.5)]). To see that (6.2) is a $q$-analogue of (6.1), do the replacements $a \mapsto q^A - B$, $b \mapsto B$, $c \mapsto q^{-A-C}$, and then let $q \to 1$.

In [9], Carlitz gave a multidimensional extension of (6.1) being related to MacMahon's Master Theorem. A $q$-analogue of Carlitz's Rothe summation formula was derived in [30, Theorem 8.2]. Here we use special cases of our multidimensional matrix inversion in Theorem A.1 to derive several multiple extensions of (6.2) associated to the root system $A_r$.

*Remark 6.3.* Of course, there are different ways to write an identity with binomial coefficients like (6.1). After performing the substitutions $A \mapsto -A$, $B \mapsto -B$, $C \mapsto -C$ this identity may be written as

$$(A+C)_n = \sum_{k=0}^{n} \binom{n}{k} \frac{A}{A+Bk} (A+Bk)_k \, (C-Bk)_{n-k}. \tag{6.4}$$

We provide multidimensional extensions of this form of the Rothe summation (see Theorem 6.15).

*Remark 6.5.* If we iterate (6.2) $s-1$ times we obtain

$$\begin{aligned}(c;q)_N = \sum_{\substack{k_1,\dots,k_s \geq 0 \\ 0 \leq |\mathbf{k}| \leq N}} \Bigg( \begin{bmatrix} N \\ k_1,\dots,k_s \end{bmatrix}_q \prod_{i=1}^{s} \frac{1-(a_i+b_i)}{1-(a_i q^{-k_i}+b_i)} (a_i q^{-k_i}+b_i;q)_{k_i} \\ \times \left(c \textstyle\prod_{i=1}^{s}(a_i+b_i q^{k_i});q\right)_{N-|\mathbf{k}|} (-1)^{k_i} q^{\binom{k_i}{2}} \\ \times c^{|\mathbf{k}|} \prod_{i=1}^{s} \left(a_i + b_i q^{k_i}\right)^{\sum_{j=i+1}^{s} k_j} \Bigg)\end{aligned} \tag{6.6}$$

(see [30, Eq. (8.4)]), where

$$\begin{bmatrix} N \\ k_1,\dots,k_s \end{bmatrix}_q := \frac{(q;q)_N}{(q;q)_{k_1} \dots (q;q)_{k_s} \, (q;q)_{N-|\mathbf{k}|}}$$

is the *$q$-multinomial coefficient.* Identity (6.6) is a *Rothe-type generalization of the $q$-multinomial theorem* (for the $q$-multinomial theorem cf. [16, Exercise 1.3 (ii)]).

Other similar convolution formulas, in the $q=1$ case, are listed in [20]. For a combinatorial approach to convolution formulas containing many free parameters, see [44].

In the following, we give a couple of $A_r$ $q$-Rothe summations.

**Theorem 6.7** *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$\begin{aligned}\prod_{i=1}^{r}(cx_i;q)_{n_i} = \sum_{\substack{0 \leq k_i \leq n_i \\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \\ \times \frac{(1-(a+b))}{(1-(aq^{-|\mathbf{k}|}+b))} (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|} \, c^{|\mathbf{k}|} \\ \times (-1)^{|\mathbf{k}|} q^{\sum_{i=1}^{r}\binom{k_i}{2}} \prod_{i=1}^{r} x_i^{k_i} \left(cx_i q^{k_i}(aq^{-|\mathbf{k}|}+b);q\right)_{n_i-k_i} \Bigg).\end{aligned} \tag{6.8}$$

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.20)/(A.21). Then (2.2) holds for

$$a_{\mathbf{k}} = \left(\frac{q}{(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|}^{-1} \prod_{i=1}^{r} \left(\frac{qx_i}{c(aq^{|\mathbf{k}|}+b)};q\right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{|\mathbf{l}|+\sum_{i=1}^{r}\binom{l_i}{2}}\, c^{-|\mathbf{l}|} \prod_{i=1}^{r} x_i^{l_i} \left(\frac{c}{x_i} q^{|\mathbf{l}|-l_i};q\right)_{l_i}$$

by the $A_r$ $q$-Chu–Vandermonde summation (B.19) in Theorem B.18. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \ldots, r$. After performing the substitutions $b \mapsto bq^{|\mathbf{n}|}$, $c \mapsto cq^{-|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \ldots, r$, we eventually obtain (6.8). ■

**Theorem 6.9** *Let $a$, $b$, $c$, and $x_1, \ldots, x_r$ be indeterminate, and let $n_1, \ldots, n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^{r} \left(\frac{c}{x_i} q^{|\mathbf{n}|-n_i};q\right)_{n_i} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\ldots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \\ \times \frac{(1-(a+b))}{(1-(aq^{-|\mathbf{k}|}+b))} (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|}\, c^{|\mathbf{k}|}\, (-1)^{|\mathbf{k}|} \\ \times q^{e_2(\mathbf{k})+\binom{|\mathbf{k}|}{2}} \prod_{i=1}^{r} x_i^{-k_i} \left(\frac{c}{x_i} q^{|\mathbf{n}|-n_i}(a+bq^{|\mathbf{k}|});q\right)_{n_i-k_i} \Bigg), \tag{6.10}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \ldots, k_r\}$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.20)/(A.21). Then (2.2) holds for

$$a_{\mathbf{k}} = \left(\frac{q}{(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|}^{-1} \prod_{i=1}^{r} \left(\frac{q^{1+|\mathbf{k}|-k_i}}{cx_i(aq^{|\mathbf{k}|}+b)};q\right)_{k_i}$$

and

$$b_{\mathbf{l}} = q^{e_2(\mathbf{l})+\binom{|\mathbf{l}|+1}{2}}\, c^{-|\mathbf{l}|} \prod_{i=1}^{r} x_i^{-l_i}\, (cx_i;q)_{l_i}$$

by the $A_r$ $q$-Chu–Vandermonde summation (B.20) in Theorem B.18. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i = 1, \ldots, r$. After performing the substitutions $b \mapsto bq^{|\mathbf{n}|}$, $c \mapsto cq^{|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i = 1, \ldots, r$, we eventually obtain (6.10). ■

**Theorem 6.11** *Let $a$, $b$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$(c;q)_{|\mathbf{n}|} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}} \right.$$
$$\left. \times \frac{(1-(a+b))}{(1-(aq^{-|\mathbf{k}|}+b))} (aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|} \left(c(a+bq^{|\mathbf{k}|});q\right)_{|\mathbf{n}|-|\mathbf{k}|} c^{|\mathbf{k}|} \right). \tag{6.12}$$

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.20)/(A.21). Then (2.2) holds for

$$a_{\mathbf{k}} = \frac{\left(\frac{q}{c(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|}}{\left(\frac{q}{(aq^{|\mathbf{k}|}+b)};q\right)_{|\mathbf{k}|}}$$

and

$$b_{\mathbf{l}} = q^{\binom{|\mathbf{l}|+1}{2}} c^{-|\mathbf{l}|} (c;q)_{|\mathbf{l}|}$$

by the $A_r$ $q$-Chu–Vandermonde summation (B.21) in Theorem B.18. This implies the inverse relation (2.1), with the above values of $a_{\mathbf{k}}$ and $b_{\mathbf{l}}$. In the resulting identity, we reverse order of summations by performing the substitutions $k_i \mapsto n_i - k_i$, $i=1,\dots,r$. After performing the substitutions $b \mapsto bq^{|\mathbf{n}|}$, $x_i \mapsto q^{-n_i}/x_i$, $i=1,\dots,r$, we eventually obtain (6.12). ■

For illustration, we give the $A_r$ Rothe-type generalization of the $q$-multinomial theorem which follows from iterating (6.12).

**Theorem 6.13** *Let $a_1,\dots,a_s$, $b_1,\dots,b_s$, $c$, and $x_1,\dots,x_r$ be indeterminate, and let $N_1,\dots,N_r$ be nonnegative integers. Write $|\mathbf{k}_j|$ for $\sum_{i=1}^{r} k_{ij}$, for convenience. Then there holds*

$$(c;q)_{|\mathbf{N}|} = \sum_{\substack{k_{ij}\ge 0,\ i=1,\dots,r,\ j=1,\dots,s\\ 0\le\sum_{j=1}^{s}k_{ij}\le N_i,\ i=1,\dots,r}} \left( (-1)^{\sum_{j=1}^{s}|\mathbf{k}_j|} q^{\sum_{j=1}^{s}\binom{|\mathbf{k}_j|}{2}} \right.$$
$$\times \prod_{t,u=1}^{r} \left[ \frac{\left(\frac{x_t}{x_u}q;q\right)_{N_t}}{\left(\frac{x_t}{x_u}q^{1+\sum_{j=1}^{s}k_{tj}-\sum_{j=1}^{s}k_{uj}};q\right)_{N_t-\sum_{j=1}^{s}k_{tj}} \prod_{j=1}^{s}\left(\frac{x_t}{x_u}q^{1+\sum_{i=1}^{j-1}k_{ti}-\sum_{i=1}^{j-1}k_{ui}};q\right)_{k_{tj}}} \right]$$
$$\times \prod_{j=1}^{s} \frac{(1-(a_j+b_j))}{(1-(a_jq^{-|\mathbf{k}_j|}+b_j))} (a_jq^{-|\mathbf{k}_j|}+b_j;q)_{|\mathbf{k}_j|}$$
$$\left. \times \left(c\prod_{j=1}^{s}(a_j+b_jq^{|\mathbf{k}_j|});q\right)_{|\mathbf{N}|-\sum_{j=1}^{s}|\mathbf{k}_j|} c^{\sum_{j=1}^{s}|\mathbf{k}_j|} \prod_{j=1}^{s}\left(a_j+b_jq^{|\mathbf{k}_j|}\right)^{\sum_{i=j+1}^{s}|\mathbf{k}_i|} \right). \tag{6.14}$$

We finish this section with two $q \to 1$ limiting cases of our $A_r$ $q$-Rothe summations. These are higher-dimensional generalizations of the Rothe summation (6.4).

**Theorem 6.15** *Let $A$, $B$, $C$, and $X_1, \dots, X_r$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^{r} (A+C+X_i)_{n_i} \\ = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{(1+X_i-X_j)_{n_i}}{(1+X_i-X_j)_{k_i}\,(1+X_i+k_i-X_j-k_j)_{n_i-k_i}} \right] \\ \times \frac{A}{A+B|\mathbf{k}|}\,(A+B|\mathbf{k}|)_{|\mathbf{k}|} \prod_{i=1}^{r} (C-B|\mathbf{k}|-|\mathbf{k}|+X_i+k_i)_{n_i-k_i} \Bigg), \tag{6.16}$$

$$(A+C)_{|\mathbf{n}|} = \sum_{\substack{0 \le k_i \le n_i \\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{(1+X_i-X_j)_{n_i}}{(1+X_i-X_j)_{k_i}\,(1+X_i+k_i-X_j-k_j)_{n_i-k_i}} \right] \\ \times \frac{A}{A+B|\mathbf{k}|}\,(A+B|\mathbf{k}|)_{|\mathbf{k}|}\,(C-B|\mathbf{k}|)_{|\mathbf{n}|-|\mathbf{k}|} \Bigg). \tag{6.17}$$

**Proof:** In (6.8) and (6.12), respectively, do the replacements $a \mapsto q^{-A} + B$, $b \mapsto -B$, $c \mapsto q^{A+C}$, $x_i \mapsto q^{X_i}$, $i = 1, \dots, r$, and then let $q \to 1$. ■

*Remark 6.18.* Note, that if we specialize (6.10) as above, we obtain (6.16) with $A \mapsto -A$, $B \mapsto -1-B$, and $C \mapsto 1-|\mathbf{n}|-C$.

## 7. Some identities of a new type

In this section we derive some identities of an apparently new type. The types of factors appearing in the summand of the series are similar to those appearing in the Rothe identities. The identities of this section are different, and involve more factors, though. In our derivations we apply different summations than those we used for deriving the Abel- and Rothe-type identities in the previous sections.

We start by considering one-dimensional series. For the proof of absolute convergence of the series in the following two theorems, see Appendix C.

**Theorem 7.1** *Let $A$, $B$, $C$, and $D$ be indeterminate. Then there holds*

$$1 = \sum_{k=0}^{\infty} \left( \frac{A-C}{A+Bk-C} \right) \left( \frac{1-AC}{1-(A+Bk)C} \right) \left( \frac{1-(A+Bk)^2}{1-A(A+Bk)} \right) \\ \times \frac{(-1)^k}{(1)_k} \left( \frac{C}{D} - \frac{(A+Bk)}{D} \right)_k \left( \frac{C}{D} - \frac{1}{D(A+Bk)} \right)_k \\ \times \frac{\Gamma\Big(\frac{A}{B} - \frac{1}{B(A+Bk)} + 1\Big)\, \Gamma\Big(\frac{2A}{B} - \frac{2C}{D} + \frac{(B-D)(A+Bk)}{BD} + \frac{(B-D)}{BD(A+Bk)} + 1\Big)}{\Gamma\Big(\frac{A}{B} - \frac{C}{D} + \frac{(A+Bk)}{D} - \frac{1}{B(A+Bk)} + 1\Big)\, \Gamma\Big(\frac{A}{B} - \frac{C}{D} + \frac{(B-D)}{BD(A+Bk)} + 1\Big)}, \tag{7.2}$$

*provided* $\frac{B}{D} \not\leq 1$, *or,* $B = D$ *and* $\Re\left(\frac{C-A}{B}\right) < 2$.

**Proof:** In the $r = 1$ case of Theorem A.1 we set $d \mapsto 1$, $c_1(t) \mapsto A + Bt$, and $a_t \mapsto C + Dt$, and after some elementary manipulations we obtain that *the infinite matrices* $(f_{nk})_{n,k\in\mathbb{Z}}$ *and* $(g_{kl})_{k,l\in\mathbb{Z}}$ *are inverses of each other, where*

$$f_{nk} = \frac{\left(\frac{C}{D} - \frac{(A+Bk)}{D} + k\right)_{n-k} \left(\frac{C}{D} - \frac{1}{D(A+Bk)} + k\right)_{n-k}}{(1)_{n-k} \left(\frac{A}{B} - \frac{1}{B(A+Bk)} + k + 1\right)_{n-k}} \tag{7.3}$$

*and*

$$\begin{aligned} g_{kl} = {} & (-1)^{k-l} \left(\tfrac{A+Bl-C-Dl}{A+Bk-C-Dk}\right) \left(\tfrac{1-(A+Bl)(C+Dl)}{1-(A+Bk)(C+Dk)}\right) \\ & \times \frac{\left(\frac{C}{D} - \frac{(A+Bk)}{D} + l + 1\right)_{k-l} \left(\frac{C}{D} - \frac{1}{D(A+Bk)} + l + 1\right)_{k-l}}{(1)_{k-l} \left(\frac{A}{B} - \frac{1}{B(A+Bk)} + l\right)_{k-l}}. \end{aligned} \tag{7.4}$$

Then (2.3) holds for $a_n = 1$ and

$$b_k = \frac{\Gamma\left(\frac{A}{B} - \frac{1}{B(A+Bk)} + k + 1\right) \Gamma\left(\frac{2A}{B} - \frac{2C}{D} + \frac{(B-D)(A+Bk)}{BD} + \frac{(B-D)}{BD(A+Bk)} + 1\right)}{\Gamma\left(\frac{A}{B} - \frac{C}{D} + \frac{(A+Bk)}{D} - \frac{1}{B(A+Bk)} + 1\right) \Gamma\left(\frac{A}{B} - \frac{C}{D} + \frac{(B-D)}{BD(A+Bk)} + 1\right)}$$

by the classical *Gauß summation* (cf. [16, Eq. (1.2.11)]),

$$\sum_{k=0}^{\infty} \frac{(A)_k\,(B)_k}{(1)_k\,(C)_k} = \frac{\Gamma(C)\,\Gamma(C-A-B)}{\Gamma(C-A)\,\Gamma(C-B)} \tag{7.5}$$

(where $\Re(C) > \Re(A)+\Re(B)$). This implies the inverse relation (2.4), with the above values of $a_n$ and $b_k$. After performing the shift $k \mapsto k + l$, and the substitutions $A \mapsto A - Bl$, $C \mapsto C - Dl$, we get rid of $l$ and eventually obtain (7.2). ∎

The following theorem gives a "companion identity" of (7.2).

**Theorem 7.6** *Let* $A$, $B$, $C$, *and* $D$ *be indeterminate. Then there holds*

$$\begin{aligned} \frac{D}{B+D} = {} & \sum_{k=0}^{\infty} \left(\frac{1-AC}{1-(A+Bk)C}\right) \left(\frac{1-(A+Bk)^2}{1-A(A+Bk)}\right) \\ & \times \frac{(-1)^k}{(1)_k} \left(\frac{C}{D} - \frac{(A+Bk)}{D} + 1\right)_k \left(\frac{C}{D} - \frac{1}{D(A+Bk)}\right)_k \\ & \times \frac{\Gamma\left(\frac{A}{B} - \frac{1}{B(A+Bk)} + 1\right) \Gamma\left(\frac{2A}{B} - \frac{2C}{D} + \frac{(B-D)(A+Bk)}{BD} + \frac{(B-D)}{BD(A+Bk)}\right)}{\Gamma\left(\frac{A}{B} - \frac{C}{D} + \frac{(A+Bk)}{D} - \frac{1}{B(A+Bk)} + 1\right) \Gamma\left(\frac{A}{B} - \frac{C}{D} + \frac{(B-D)}{BD(A+Bk)}\right)}, \end{aligned} \tag{7.7}$$

*provided* $\frac{B}{D} \not\leq 1$, *or,* $B = D$ *and* $\Re\left(\frac{C-A}{B}\right) < 2$.

**Proof:** Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in equations (7.3)/(7.4). Then (2.3) holds for

$$a_n = (A - C + (B-D)n)$$

and

$$b_k = (A - C + (B-D)k) \\ \times \left(\frac{B+D}{D}\right) \frac{\Gamma\left(\frac{A}{B} - \frac{1}{B(A+Bk)} + k + 1\right)\Gamma\left(\frac{2A}{B} - \frac{2C}{D} + \frac{(B-D)(A+Bk)}{BD} + \frac{(B-D)}{BD(A+Bk)}\right)}{\Gamma\left(\frac{A}{B} - \frac{C}{D} + \frac{(A+Bk)}{D} - \frac{1}{B(A+Bk)} + 1\right)\Gamma\left(\frac{A}{B} - \frac{C}{D} + \frac{(B-D)}{BD(A+Bk)}\right)}.$$

Namely, we use

$$\frac{(A - C + (B-D)n)}{(A - C + (B-D)k)} = 1 + \frac{(B-D)\,(n-k)}{(A - C + (B-D)k)}$$

to split $\sum_{n\ge k} f_{nk}a_n$ into two sums, both of which can be evaluated by means of the classical Gauß summation (7.5). Addition of both evaluations and simplification yields $b_k$. This implies the inverse relation (2.4), with the above values of $a_n$ and $b_k$. After performing the shift $k \mapsto k + l$, and the substitutions $A \mapsto A - Bl$, $C \mapsto C - Dl$, we get rid of $l$ and eventually obtain (7.7). ∎

Next, we give two terminating summations.

**Theorem 7.8** *Let $A$, $B$, and $C$ be indeterminate, and let $n$ be a nonnegative integer. Then there holds*

$$\frac{(2C+1)_n}{(C+1)_n} = \sum_{k=0}^{n} \left(\frac{B + (A-C)A}{B + (A-C)(A+k)}\right)\left(\frac{B + (A+k)^2}{B + A(A+k)}\right) \\ \times \frac{(-n)_k\,(C)_k\,\left(A - C + \frac{B}{A+k}\right)_k}{(1)_k\,(-C-n)_k\,\left(A + C + \frac{B}{A+k} + 1\right)_k} \frac{\left(A + C + \frac{B}{A+k} + 1\right)_n}{\left(A + \frac{B}{A+k} + 1\right)_n}. \tag{7.9}$$

**Proof:** First we specialize the inverse pair of matrices (7.3)/(7.4) by setting $A \mapsto AB$, $C \mapsto (C + A)B$, $D \mapsto B$. Then we replace $B$ by $B^{-1/2}$ and transfer some factors from one matrix to the other to obtain that *the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$f_{nk} = \binom{n}{k}(-1)^k\left(\frac{B - (A+n)(A+C+n)}{B - (A+k)(A+C)}\right)\left(\frac{B - (A+k)^2}{B - A(A+k)}\right) \\ \times \frac{C}{(C-k)}\,(C-k)_n \frac{\left(A + C - \frac{B}{A+k}\right)_n}{\left(A - \frac{B}{A+k} + 1\right)_n} \tag{7.10}$$

*and*

$$g_{kl} = \binom{k}{l} \frac{(-1)^l \left(A - \frac{B}{A+k}\right)_l}{(C-k+1)_l \left(A + C - \frac{B}{A+k} + 1\right)_l}. \tag{7.11}$$

Then (2.2) holds for

$$a_k = \frac{(C+1)_k \left(A - C - \frac{B}{A+k}\right)_k}{(-C)_k \left(A + C - \frac{B}{A+k} + 1\right)_k} \qquad \text{and} \qquad b_l = (2C+1)_l$$

by the classical *Pfaff–Saalschütz summation* (cf. [16, Eq. (1.7.1)]),

$$\sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{(A)_l \, (B)_l}{(C)_l \, (A+B-C+1-k)_l} = \frac{(C-A)_k \, (C-B)_k}{(C)_k \, (C-A-B)_k}. \tag{7.12}$$

This implies the inverse relation (2.1), with the above values of $a_k$ and $b_l$. In the resulting identity, we reverse order of the summation by performing the substitution $k \mapsto n-k$. After performing the substitutions $A \mapsto -A-n$, $B \mapsto -B$, we eventually obtain (7.9). ■

The following theorem gives a companion identity of (7.9).

**Theorem 7.13** *Let $A$, $B$, and $C$ be indeterminate, and let $n$ be a nonnegative integer. Then there holds*

$$\begin{aligned} \frac{(2C)_n}{(C)_n} = \sum_{k=0}^{n} & \left( \frac{B + (A-C)A}{B + (A-C)(A+k)} \right) \left( \frac{B + (A+k)^2}{B + A(A+k)} \right) \\ & \times \left( \frac{A + C + \frac{B}{A+k} + 2n - k}{A + C + \frac{B}{A+k} + n} \right) \frac{\left(A + C + \frac{B}{A+k} + 1\right)_n}{\left(A + \frac{B}{A+k} + 1\right)_n} \\ & \times \frac{(-n)_k \, (C)_k \left(A - C + \frac{B}{A+k}\right)_k}{(1)_k \, (-C-n+1)_k \left(A + C + \frac{B}{A+k} + 1\right)_k}. \end{aligned} \tag{7.14}$$

**Proof:** Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in equations (7.10)/(7.11). Then (2.2) holds for

$$a_k = \frac{\left(A - C - \frac{B}{A+k} - k\right)}{\left(A - C - \frac{B}{A+k}\right)} \frac{(C)_k \left(A - C - \frac{B}{A+k}\right)_k}{(-C)_k \left(A + C - \frac{B}{A+k} + 1\right)_k} \qquad \text{and} \qquad b_l = (2C)_l$$

by the following *terminating 2-balanced ${}_3F_2$ summation,*

$$\begin{aligned} \sum_{l=0}^{k} \binom{k}{l} (-1)^l & \frac{(A)_l \, (B)_l}{(C)_l \, (A+B-C+2-k)_l} \\ & = \left(1 - \frac{Ak}{(C-B-1)(C-A+k-1)}\right) \frac{(C-A)_k \, (C-B-1)_k}{(C)_k \, (C-A-B-1)_k}. \end{aligned} \tag{7.15}$$

(The summation (7.15) follows easily from (7.12) by contiguous relations.) This implies the inverse relation (2.1), with the above values of $a_k$ and $b_l$. In the resulting identity, we reverse order of the summation by performing the substitution $k \mapsto n-k$. After performing the substitutions $A \mapsto -A-n$, $B \mapsto -B$, we eventually obtain (7.14). ∎

We proceed with some summations involving the base $q$.

**Theorem 7.16** *Let $a$, $b$, and $c$ be indeterminate. Then there holds*

$$\frac{(b^2q;q)_\infty}{(bq;q)_\infty} = \sum_{k=0}^{\infty} \left(\frac{c-(a+1)(a+b)}{c-(a+1)(a+bq^k)}\right)\left(\frac{c-(a+bq^k)^2}{c-(a+b)(a+bq^k)}\right) \times \frac{(b;q)_k \left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_k \left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k \left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty} b^k\,q^k, \tag{7.17}$$

*provided $|bq|<1$.*

**Proof:** In the $r=1$ case of Theorem A.1 we set $d \mapsto c$, $c_1(t) \mapsto a+bq^t$, and $a_t \mapsto a+q^t$, and after some elementary manipulations we obtain that *the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$f_{nk} = \frac{(1/b;q)_{n-k} \left(\frac{(a+bq^k)q^k}{c-a(a+bq^k)};q\right)_{n-k}}{(q;q)_{n-k} \left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_{n-k}} \tag{7.18}$$

*and*

$$g_{kl} = (-1)^{k-l}\,q^{\binom{k-l}{2}} \left(\frac{c-(a+bq^l)(a+q^l)}{c-(a+bq^k)(a+q^k)}\right) \times \frac{(q^{l-k+1}/b;q)_{k-l} \left(\frac{(a+bq^k)q^{l+1}}{c-a(a+bq^k)};q\right)_{k-l}}{(q;q)_{k-l} \left(\frac{(a+bq^k)bq^l}{c-a(a+bq^k)};q\right)_{k-l}}. \tag{7.19}$$

Then (2.3) holds for

$$a_n = q^n\,b^{2n} \qquad \text{and} \qquad b_k = q^k\,b^{2k}\,\frac{(bq;q)_\infty \left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(b^2q;q)_\infty \left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_\infty}$$

by the classical *$q$-Gauß summation* (cf. [16, Eq. (II.8)]),

$$\sum_{k=0}^{\infty} \frac{(a;q)_k\,(b;q)_k}{(q;q)_k\,(c;q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a;q)_\infty\,(c/b;q)_\infty}{(c;q)_\infty\,(c/ab;q)_\infty} \tag{7.20}$$

(where $|c|<|ab|$). This implies the inverse relation (2.4), with the above values of $a_n$ and $b_k$. After performing the shift $k \mapsto k+l$, and the substitutions $a \mapsto aq^l$, $c \mapsto cq^{2l}$, we get rid of $l$ and eventually obtain (7.17). ∎

**Theorem 7.21** *Let $a$, $b$, and $c$ be indeterminate. Then there holds*

$$(qb^2c/a;q)_\infty = \sum_{k=0}^{\infty} \left(\frac{c+1-(a+b)}{c+1-(a+bq^k)}\right)\left(\frac{bc+a+b}{bc+aq^{-k}+b}\right)\left(\frac{ac-(a+bq^k)^2}{ac-(a+b)(a+bq^k)}\right)$$
$$\times \frac{\left(\frac{1}{a+bq^k-c};q\right)_k \left(-\frac{1}{bc}(aq^{-k}+b);q\right)_k \left(\frac{b^2cq^{k+1}}{a(a+bq^k-c)};q\right)_\infty}{(q;q)_k \left(\frac{(a+bq^k)bq}{a(c-(a+bq^k))};q\right)_\infty}$$
$$\times \left(-\frac{bq}{a}(a+bq^k);q\right)_\infty (-1)^k q^{\binom{k}{2}} \left(\frac{qb^2c}{a}\right)^k, \tag{7.22}$$

*provided* $|bq|<1$.

**Proof:** In the $r=1$ case of Theorem A.1 we set $d \mapsto ac$, $c_1(t) \mapsto a+bq^t$, and $a_t \mapsto c+q^t$, and after some elementary manipulations we obtain that *the infinite matrices* $(f_{nk})_{n,k\in\mathbb{Z}}$ *and* $(g_{kl})_{k,l\in\mathbb{Z}}$ *are inverses of each other, where*

$$f_{nk} = \frac{\left(\frac{q^k}{a+bq^k-c};q\right)_{n-k} \left(-\frac{1}{bc}(a+bq^k);q\right)_{n-k}}{(q;q)_{n-k}\left(\frac{(a+bq^k)bq^{k+1}}{a(c-(a+bq^k))};q\right)_{n-k}} \tag{7.23}$$

*and*

$$g_{kl} = (-1)^{k-l} q^{\binom{k-l}{2}} \left(\frac{c+q^l-a-bq^l}{c+q^k-a-bq^k}\right)\left(\frac{bc+a+bq^l}{bc+a+bq^k}\right)$$
$$\times \frac{\left(\frac{q^{l+1}}{a+bq^k-c};q\right)_{k-l} \left(-\frac{1}{bc}(a+bq^k)q^{l-k+1};q\right)_{k-l}}{(q;q)_{k-l}\left(\frac{(a+bq^k)bq^l}{a(c-(a+bq^k))};q\right)_{k-l}}. \tag{7.24}$$

Then (2.3) holds for

$$a_n = \left(\frac{qb^2c}{a}\right)^n \qquad \text{and} \qquad b_k = \left(\frac{qb^2c}{a}\right)^k \frac{\left(\frac{b^2cq^{k+1}}{a(a+bq^k-c)};q\right)_\infty \left(-\frac{bq}{a}(a+bq^k);q\right)_\infty}{\left(\frac{b^2cq}{a};q\right)_\infty \left(\frac{(a+bq^k)bq^{k+1}}{a(c-(a+bq^k))};q\right)_\infty}$$

by the classical $q$-Gauß summation (7.20). This implies the inverse relation (2.4), with the above values of $a_n$ and $b_k$. After performing the shift $k\mapsto k+l$, and the substitutions $a\mapsto aq^l$, $c\mapsto cq^l$, we get rid of $l$ and eventually obtain (7.22). ∎

The following two theorems provide companion identities of (7.22).

**Theorem 7.25** *Let $a$, $b$, and $c$ be indeterminate. Then there holds*

$$\frac{(b^2c/a;q)_\infty}{1+b} = \sum_{k=0}^{\infty} \left(\frac{bc+a+b}{bc+aq^{-k}+b}\right)\left(\frac{ac-(a+bq^k)^2}{ac-(a+b)(a+bq^k)}\right)$$
$$\times \frac{\left(\frac{q}{a+bq^k-c};q\right)_k \left(-\frac{1}{bc}(aq^{-k}+b);q\right)_k \left(\frac{b^2cq^k}{a(a+bq^k-c)};q\right)_\infty}{(q;q)_k \left(\frac{(a+bq^k)bq}{a(c-(a+bq^k))};q\right)_\infty}$$
$$\times \left(-\frac{bq}{a}(a+bq^k);q\right)_\infty (-1)^k q^{\binom{k}{2}} \left(\frac{b^2c}{a}\right)^k, \tag{7.26}$$

*provided* $|b|<1$.

**Proof:** Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in equations (7.23)/(7.24). Then (2.3) holds for

$$a_n = \left(\frac{b^2c}{a}\right)^n (a-c+(b-1)q^n)$$

and

$$b_k = \left(\frac{b^2c}{a}\right)^k (a-c+(b-1)q^k)(1+b)\frac{\left(\frac{b^2cq^k}{a(a+bq^k-c)};q\right)_\infty \left(-\frac{bq}{a}(a+bq^k);q\right)_\infty}{\left(\frac{b^2c}{a};q\right)_\infty \left(\frac{(a+bq^k)bq^{k+1}}{a(c-(a+bq^k))};q\right)_\infty}.$$

Namely, we use

$$\frac{(a-c+(b-1)q^n)}{(a-c+(b-1)q^k)} = q^{n-k} + \frac{(a-c)(1-q^{n-k})}{(a-c+(b-1)q^k)}$$

to split $\sum_{n\ge k} f_{nk}a_n$ into two sums, both of which can be evaluated by means of the classical $q$-Gauß summation (7.20). Addition of both evaluations and simplification yields $b_k$. This implies the inverse relation (2.4), with the above values of $a_n$ and $b_k$. After performing the shift $k\mapsto k+l$, and the substitutions $a\mapsto aq^l$, $c\mapsto cq^l$, we get rid of $l$ and eventually obtain (7.26). ∎

**Theorem 7.27** *Let $a$, $b$, and $c$ be indeterminate. Then there holds*

$$(b^2c/a;q)_\infty = \sum_{k=0}^{\infty} \left(\frac{c+1-(a+b)}{c+1-(a+bq^k)}\right)\left(\frac{ac-(a+bq^k)^2}{ac-(a+b)(a+bq^k)}\right)\left(\frac{c-a}{c-(a+bq^k)}\right)$$
$$\times \frac{\left(\frac{1}{a+bq^k-c};q\right)_k \left(-\frac{q}{bc}(aq^{-k}+b);q\right)_k \left(\frac{b^2cq^{k+1}}{a(a+bq^k-c)};q\right)_\infty}{(q;q)_k \left(\frac{(a+bq^k)bq}{a(c-(a+bq^k))};q\right)_\infty}$$
$$\times \left(-\frac{b}{a}(a+bq^k);q\right)_\infty (-1)^k q^{\binom{k}{2}} \left(\frac{b^2c}{a}\right)^k, \tag{7.28}$$

*provided* $|b|<1$.

**Proof:** Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in equations (7.23)/(7.24). Then (2.3) holds for

$$a_n = \left(\frac{b^2c}{a}\right)^n (bc + a + bq^n)$$

and

$$b_k = \left(\frac{b^2c}{a}\right)^k (bc + a + bq^k) \frac{(c-a)}{(c-(a+bq^k))} \frac{\left(\frac{b^2cq^{k+1}}{a(a+bq^k-c)};q\right)_\infty \left(-\frac{b}{a}(a+bq^k);q\right)_\infty}{\left(\frac{b^2c}{a};q\right)_\infty \left(\frac{(a+bq^k)bq^{k+1}}{a(c-(a+bq^k))};q\right)_\infty}.$$

Namely, we use

$$\frac{(bc+a+bq^n)}{(bc+a+bq^k)} = q^{n-k} + \frac{(bc+a)(1-q^{n-k})}{(bc+a+bq^k)}$$

to split $\sum_{n\geq k} f_{nk} a_n$ into two sums, both of which can be evaluated by means of the classical $q$-Gauß summation (7.20). Addition of both evaluations and simplification yields $b_k$. This implies the inverse relation (2.4), with the above values of $a_n$ and $b_k$. After performing the shift $k \mapsto k+l$, and the substitutions $a \mapsto aq^l$, $c \mapsto cq^l$, we get rid of $l$ and eventually obtain (7.28). ■

Next, we provide some terminating basic summations.

**Theorem 7.29** *Let $a$, $b$, and $c$ be indeterminate, and let $n$ be a nonnegative integer. Then there holds*

$$\begin{aligned}(qa/b^2c;q)_n = \sum_{k=0}^{n} &\left(\frac{c+1-(a+b)}{c+1-(a+bq^{-k})}\right)\left(\frac{bc+a+b}{bc+a+bq^{-k}}\right)\\ \times &\left(\frac{ac-(a+bq^{-k})^2}{ac-(a+b)(a+bq^{-k})}\right)\frac{\left(\frac{aq}{b^2c}(a+bq^{-k}-c);q\right)_n}{\left(aq\frac{c-(a+bq^{-k})}{b(a+bq^{-k})};q\right)_n}\left(-\frac{aq}{b(a+bq^{-k})};q\right)_n\\ &\times \frac{(q^{-n};q)_k\,(a+bq^{-k}-c;q)_k\,\left(-\frac{1}{bc}(a+bq^{-k});q\right)_k}{(q;q)_k\,\left(-\frac{b}{a}(a+bq^{-k})q^{-n};q\right)_k\,\left(\frac{aq}{b^2c}(a+bq^{-k}-c);q\right)_k}\,q^k.\end{aligned} \tag{7.30}$$

**Proof:** From the inverse pair (7.23)/(7.24) we easily obtain, by transferring some factors, that *the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$\begin{aligned}f_{nk} = (-1)^k q^{\binom{k}{2}} &\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{c+q^n-a-bq^n}{c+q^n-a-bq^k}\right)\left(\frac{ac-(a+bq^k)^2}{ac-(a+bq^k)(a+bq^n)}\right)\\ \times &\left(\frac{bc+a+bq^n}{bc+aq^{n-k}+bq^n}\right)\left(-\frac{1}{bc}(a+bq^k)q^{1-k};q\right)_n \frac{\left(\frac{q}{a+bq^k-c};q\right)_n}{\left(\frac{b(a+bq^k)}{a(c-(a+bq^k))};q\right)_n}\end{aligned} \tag{7.31}$$

*and*

$$g_{kl} = (-1)^l q^{\binom{l}{2}-kl+l} \begin{bmatrix} k \\ l \end{bmatrix}_q \frac{\left(\frac{b(a+bq^k)}{a(c-(a+bq^k))};q\right)_l}{\left(\frac{q}{a+bq^k-c};q\right)_l \left(-\frac{1}{bc}(a+bq^k)q^{1-k};q\right)_l}. \tag{7.32}$$

Then (2.2) holds for

$$a_k = \frac{\left(-\frac{aq}{b(a+bq^k)};q\right)_k \left(\frac{b^2c}{a(a+bq^k-c)};q\right)_k}{\left(\frac{q}{a+bq^k-c};q\right)_k \left(-\frac{bc}{a+bq^k};q\right)_k} \qquad \text{and} \qquad b_l = \left(\frac{aq}{b^2c};q\right)_l$$

by the classical *q-Pfaff–Saalschütz summation* (cf. [16, Eq. (II.12)]),

$$\sum_{l=0}^{k}(-1)^l q^{\binom{l}{2}} \begin{bmatrix} k \\ l \end{bmatrix}_q \frac{(a;q)_l\,(b;q)_l}{(c;q)_l\,(abq^{1-k}/c;q)_l}\, q^{(1-k)l} = \frac{(c/a;q)_k\,(c/b;q)_k}{(c;q)_k\,(c/ab;q)_k}. \tag{7.33}$$

This implies the inverse relation (2.1), with the above values of $a_k$ and $b_l$. In the resulting identity, we reverse order of the summations by performing the substitution $k \mapsto n-k$. After performing the substitutions $a \mapsto aq^n$, $c \mapsto cq^n$, we eventually obtain (7.30). ■

**Theorem 7.34** *Let $a$, $b$, and $c$ be indeterminate, and let $n$ be a nonnegative integer. Then there holds*

$$\frac{(c^2q;q)_n}{(cq;q)_n} = \sum_{k=0}^{n}\left(\frac{b+(a-c)(a-1)}{b+(a-c)(a-q^{-k})}\right)\left(\frac{b+(a-q^{-k})^2}{b+(a-1)(a-q^{-k})}\right)$$
$$\times \frac{(q^{-n};q)_k\,(c;q)_k\,\left(\frac{b+a(a-q^{-k})}{c(a-q^{-k})};q\right)_k}{(q;q)_k\,(q^{-n}/c;q)_k\,\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_k} \frac{\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n}{\left(q\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n} q^k. \tag{7.35}$$

**Proof:** From the inverse pair (7.18)/(7.19) we easily obtain, by performing the substitutions $a \mapsto -a/c$, $b \mapsto 1/c$, $c \mapsto -b/c^2$, and by transferring some factors, that *the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$f_{nk} = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{b+(a-cq^n)(a-q^n)}{b+(a-c)(a-q^k)}\right)\left(\frac{b+(a-q^k)^2}{b+(a-q^n)(a-q^k)}\right)$$
$$\times (1/c;q)_k\,(c;q)_{n-k} \frac{\left(c\frac{(a-q^k)}{b+a(a-q^k)};q\right)_n}{\left(\frac{(a-q^k)}{b+a(a-q^k)};q\right)_n} \tag{7.36}$$

*and*

$$g_{kl} = (-1)^l q^{\binom{l}{2}-kl+l} \begin{bmatrix} k \\ l \end{bmatrix}_q \frac{\left(\frac{(a-q^k)}{b+a(a-q^k)};q\right)_l}{(cq^{1-k};q)_l\,\left(cq\frac{(a-q^k)}{b+a(a-q^k)};q\right)_l}. \tag{7.37}$$

Then (2.2) holds for

$$a_k = \frac{(cq;q)_k\,\left(\frac{(a-q^k)}{c(b+a(a-q^k))};q\right)_k}{(1/c;q)_k\,\left(cq\frac{(a-q^k)}{b+a(a-q^k)};q\right)_k} \qquad \text{and} \qquad b_l = (c^2q;q)_l$$

by the classical $q$-Pfaff–Saalschütz summation (7.33). This implies the inverse relation (2.1), with the above values of $a_k$ and $b_l$. In the resulting identity, we reverse order of the summation by performing the substitution $k \mapsto n-k$. After performing the substitutions $a \mapsto aq^n$, $b \mapsto bq^{2n}$, we eventually obtain (7.35). ∎

The following theorem gives a companion identity of (7.35).

**Theorem 7.38** *Let $a$, $b$, and $c$ be indeterminate, and let $n$ be a nonnegative integer. Then there holds*

$$\begin{aligned}\frac{(c^2;q)_n}{(c;q)_n} &= \sum_{k=0}^{n} \left(\frac{b+(a-c)(a-1)}{b+(a-c)(a-q^{-k})}\right)\left(\frac{b+(a-q^{-k})^2}{b+(a-1)(a-q^{-k})}\right)\\ &\quad\times\left(\frac{bc+(a-q^{-k})(ac-q^{-n}+c(q^{-k}-q^{-n}))}{bc+(a-q^{-k})(ac-q^{-n})}\right)\\ &\quad\times\frac{(q^{-n};q)_k\,(c;q)_k\,\left(\frac{b+a(a-q^{-k})}{c(a-q^{-k})};q\right)_k}{(q;q)_k\,(q^{1-n}/c;q)_k\,\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_k}\,\frac{\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n}{\left(q\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n}\,q^{2k}.\end{aligned}\tag{7.39}$$

**Proof:** Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in equations (7.36)/(7.37). Then (2.2) holds for

$$a_k = \frac{\left(1-\frac{(a-q^k)(1+c-cq^k)}{c(b+a(a-q^k))}\right)}{\left(1-\frac{(a-q^k)}{c(b+a(a-q^k))}\right)}\,\frac{(c;q)_k\,\left(\frac{(a-q^k)}{c(b+a(a-q^k))};q\right)_k}{(1/c;q)_k\,\left(cq\frac{(a-q^k)}{b+a(a-q^k)};q\right)_k} \qquad \text{and} \qquad b_l=(c^2;q)_l$$

by the following *terminating* 2*-balanced* ${}_3\phi_2$ *summation*,

$$\begin{aligned}&\sum_{l=0}^{k}(-1)^l\,q^{\binom{l}{2}}\begin{bmatrix}k\\ l\end{bmatrix}_q\frac{(a;q)_l\,(b;q)_l}{(c;q)_l\,(abq^{2-k}/c;q)_l}\,q^{(1-k)l}\\ &\qquad=\left(1-\frac{(1-a)(1-q^{-k})}{(1-c/bq)(1-aq^{1-k}/c)}\right)\frac{(c/a;q)_k\,(c/bq;q)_k}{(c;q)_k\,(c/abq;q)_k}.\end{aligned}\tag{7.40}$$

(The summation (7.40) follows easily from (7.33) by contiguous relations.) This implies the inverse relation (2.1), with the above values of $a_k$ and $b_l$. In the resulting identity, we reverse order of the summation by performing the substitution $k \mapsto n-k$. After performing the substitutions $a \mapsto aq^n$, $b \mapsto bq^{2n}$, we eventually obtain (7.39). ∎

Finally, we give some multiple series expansions.

*Remark 7.41.* We have associated the multidimensional summations in the previous sections with the root system $A_r$ due to the factor

$$\prod_{1\le i<j\le r}(y_i-y_j)\tag{7.42}$$

occurring in the summand of the series where $y_i = x_i q^{k_i}$, or $y_i = X_i + k_i$, for $i = 1, \dots, r$. (Concerning our terminating multidimensional summations this factor is hidden in the "$A_r$ $q$-binomial coefficient" (B.4).) The multiple series in this section belong to a different type though. We may associate them to the root system $D_r$. This is because we can make out the factor

$$\prod_{1\le i<j\le r} [(y_i - y_j)(1 - y_i y_j)] \tag{7.43}$$

in the respective summands of the series where $y_i = (bx_i q^{k_i} - 1)$ in Theorems 7.44 and 7.46, and where $y_i = (A_i + Bk_i)$ in Theorem 7.48, for $i = 1, \dots, r$. A reason for naming these series as $A_r$ or $D_r$ series is that (7.42) and (7.43) are basically the product sides of the Weyl denominator formula for the respective root systems, see [4], [43]. For multiple basic hypergeometric series associated to $D_r$, see [4], [6], [42].

The series appearing in the following three theorems all converge absolutely in value, provided specific conditions hold. We investigate their convergence in Appendix C.

**Theorem 7.44** *Let $a$, $b$, and $x_1, \dots, x_r$ be indeterminate. Then there holds*

$$\begin{aligned}
(aq;q)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( &\prod_{1\le i<j\le r} \left( \frac{1 - \frac{x_i}{x_j} q^{k_i - k_j}}{1 - \frac{x_i}{x_j}} \right) \\
&\times \prod_{1\le i<j\le r} \left( \frac{1 - (bx_j q^{k_j} - \frac{x_j}{x_i} q^{k_j - k_i})}{1 - (bx_j - \frac{x_j}{x_i})} \right) \\
\times \prod_{i=1}^{r} \left( \frac{bx_i - 1}{bx_i q^{k_i} - 1} \right) &\prod_{i=1}^{r} \left( \frac{1 - a(bx_i q^{k_i} - 1)\prod_{j=1}^{r}(bx_j q^{k_j} - 1)}{1 + a\prod_{j=1}^{r}(bx_j q^{k_j} - 1)} \right) \\
\times \prod_{i,j=1}^{r} &\left( \frac{1 - (bx_i - \frac{x_i}{x_j})}{1 - (bx_i - \frac{x_i}{x_j} q^{-k_j})} \right) \prod_{i,j=1}^{r} \frac{\left(bx_i - \frac{x_i}{x_j} q^{-k_j}; q\right)_{k_i}}{\left(\frac{x_i}{x_j} q; q\right)_{k_i}} \\
\times \prod_{i=1}^{r} &\frac{\left( \frac{bx_i q}{1 + a\prod_{j=1}^{r}(bx_j q^{k_j} - 1)}; q \right)_{k_i} \left( \frac{abx_i q^{1+k_i}}{1 + a\prod_{j=1}^{r}(bx_j q^{k_j} - 1)}; q \right)_\infty}{\left( \frac{abx_i \prod_{j=1}^{r}(bx_j q^{k_j} - 1)}{1 + a\prod_{j=1}^{r}(bx_j q^{k_j} - 1)}; q \right)_\infty} \\
&\times \left(aq\prod_{i=1}^{r}(bx_i q^{k_i} - 1); q\right)_\infty (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2} + |\mathbf{k}|} a^{|\mathbf{k}|} \Bigg),
\end{aligned} \tag{7.45}$$

*provided* $|aq| < 1$.

**Proof:** Let the multidimensional inverse matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ be defined as in (A.27)/(A.28). Then (2.3) holds for

$$a_{\mathbf{n}} = a^{|\mathbf{n}|} q^{|\mathbf{n}|} \prod_{1\le i<j\le r} \left( 1 - \frac{x_i}{x_j} q^{n_i - n_j} \right)$$

and

$$b_{\mathbf{k}} = a^{|\mathbf{k}|} q^{|\mathbf{k}|} \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right) \\ \times \frac{\left(aq\prod_{i=1}^r (bx_iq^{k_i}-1);q\right)_\infty}{(aq;q)_\infty} \prod_{i=1}^r \frac{\left(\frac{abx_iq^{1+k_i}}{1+a\prod_{j=1}^r(bx_jq^{k_j}-1)};q\right)_\infty}{\left(\frac{abx_iq^{1+k_i}\prod_{j=1}^r(bx_jq^{k_j}-1)}{1+a\prod_{j=1}^r(bx_jq^{k_j}-1)};q\right)_\infty}$$

by the $A_r$ $q$-Gauß summation (B.24) in Theorem B.23. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (7.45). ∎

**Theorem 7.46** *Let $z$, $b$, and $x_1, \ldots, x_r$ be indeterminate. Then there holds*

$$(z;q)_\infty = \sum_{k_1,\ldots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right) \right. \\ \times \prod_{1\le i<j\le r} \left(\frac{1-(bx_jq^{k_j}-\frac{x_j}{x_i}q^{k_j-k_i})}{1-(bx_j-\frac{x_j}{x_i})}\right) \prod_{i=1}^r \left(\frac{bx_i-1}{bx_iq^{k_i}-1}\right) \\ \times \prod_{i,j=1}^r \left(\frac{1-(bx_i-\frac{x_i}{x_j})}{1-(bx_i-\frac{x_i}{x_j}q^{-k_j})}\right) \prod_{i,j=1}^r \frac{\left(bx_i-\frac{x_i}{x_j}q^{-k_j};q\right)_{k_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}} \\ \left. \times \left(z\prod_{i=1}^r (bx_iq^{k_i}-1);q\right)_\infty (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}} z^{|\mathbf{k}|} \right), \tag{7.47}$$

*provided $|z| < 1$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.30)/(A.31). Then (2.3) holds for

$$a_{\mathbf{n}} = z^{|\mathbf{n}|} \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{n_i-n_j}\right)$$

and

$$b_{\mathbf{k}} = \frac{\left(z\prod_{i=1}^r (bx_iq^{k_i}-1);q\right)_\infty}{(z;q)_\infty} z^{|\mathbf{k}|} \prod_{1\le i<j\le r} \left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)$$

by the $A_r$ nonterminating $q$-binomial theorem (B.30) in Theorem B.29. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts $k_i \mapsto k_i + l_i$, $i = 1, \ldots, r$, and the substitutions $x_i \mapsto x_i q^{-l_i}$, $i = 1, \ldots, r$, we get rid of the $l_i$'s and eventually obtain (7.47). ∎

**Theorem 7.48** *Let $A_1 \ldots, A_r$, $B$, and $C$ be indeterminate. Then there holds*

$$
\begin{aligned}
1 = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( & \prod_{1\le i<j\le r} \left(\frac{A_i+Bk_i-A_j-Bk_j}{A_i-A_j}\right) \\
& \times \prod_{1\le i<j\le r} \left(\frac{1-(A_i+Bk_i)(A_j+Bk_j)}{1-A_iA_j}\right) \\
& \times \prod_{i=1}^{r}\left(\frac{A_i}{A_i+Bk_i}\right) \prod_{i=1}^{r} \left(\frac{1-C(A_i+Bk_i)\prod_{j=1}^r(A_j+Bk_j)}{1-CA_i\prod_{j=1}^r(A_j+Bk_j)}\right) \\
& \times \prod_{i,j=1}^{r}\left(\frac{1-A_iA_j}{1-A_i(A_j+Bk_j)}\right) \prod_{i,j=1}^{r} \frac{\left(\frac{A_i}{B}-\frac{1}{B(A_j+Bk_j)}\right)_{k_i}}{\left(\frac{A_i}{B}-\frac{A_j}{B}+1\right)_{k_i}} (-1)^{|\mathbf{k}|} \\
\times \prod_{i=1}^{r} & \frac{\left(\frac{A_i}{B}-\frac{C}{B}\prod_{j=1}^r(A_j+Bk_j)+1\right)_{k_i} \Gamma\left(\frac{A_i}{B}-\frac{1}{BC\prod_{j=1}^r(A_j+Bk_j)}+1\right)}{\Gamma\left(\frac{1}{B(A_i+Bk_i)}-\frac{1}{BC\prod_{j=1}^r(A_j+Bk_j)}+1\right)} \\
& \times \frac{\Gamma\left(\begin{array}{c}\frac{C}{B}\prod_{i=1}^r(A_i+Bk_i)-\frac{1}{BC\prod_{i=1}^r(A_i+Bk_i)} \\ +\frac{1}{B}\sum_{i=1}^r\frac{1}{(A_i+Bk_i)}-\sum_{i=1}^r(\frac{A_i}{B}+k_i)+1\end{array}\right)}{\Gamma\left(\frac{C}{B}\prod_{i=1}^r(A_i+Bk_i)-\frac{1}{BC\prod_{i=1}^r(A_i+Bk_i)}+1\right)}\Bigg),
\end{aligned}
\tag{7.49}
$$

*provided $C\prod_{j\ne i}(A_j+Bk_j) \not\le 1$ for $i=1,\dots,r$ and $k_j=0,1,2,\dots$ ($j=1,\dots,r$, $j\ne i$), or, if $C\prod_{j\ne i}(A_j+Bk_j)=1$ for an $i=1,\dots,r$ and a $k_j=0,1,2,\dots$ ($j=1,\dots,r$, $j\ne i$), then $\sum_{1\le j\le r,\, j\ne i}\Re(\frac{A_j}{B})<2$.*

**Proof:** Let the multidimensional inverse matrices $f_{\mathbf{nk}}$ and $g_{\mathbf{kl}}$ be defined as in (A.33)/(A.34). Then (2.3) holds for

$$
a_{\mathbf{n}} = \prod_{1\le i<j\le r}(A_i+Bn_i-A_j-Bn_j)
$$

and

$$
\begin{aligned}
b_{\mathbf{k}} = \prod_{1\le i<j\le r}(A_i+Bk_i-A_j-Bk_j) & \prod_{i=1}^{r}\frac{\Gamma\left(\frac{A_i}{B}+k_i-\frac{1}{BC\prod_{j=1}^r(A_j+Bk_j)}+1\right)}{\Gamma\left(\frac{1}{B(A_i+Bk_i)}-\frac{1}{BC\prod_{j=1}^r(A_j+Bk_j)}+1\right)} \\
& \times \frac{\Gamma\left(\begin{array}{c}\frac{C}{B}\prod_{i=1}^r(A_i+Bk_i)-\frac{1}{BC\prod_{i=1}^r(A_i+Bk_i)} \\ +\frac{1}{B}\sum_{i=1}^r\frac{1}{(A_i+Bk_i)}-\sum_{i=1}^r(\frac{A_i}{B}+k_i)+1\end{array}\right)}{\Gamma\left(\frac{C}{B}\prod_{i=1}^r(A_i+Bk_i)-\frac{1}{BC\prod_{i=1}^r(A_i+Bk_i)}+1\right)}
\end{aligned}
$$

by the $A_r$ Gauß summation (B.27) in Theorem B.26. This implies the inverse relation (2.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After performing the shifts

$k_i \mapsto k_i + l_i$, $i = 1, \dots, r$, and the substitutions $A_i \mapsto A_i - Bl_i$, $i = 1, \dots, r$, we get rid of the $l_i$'s and eventually obtain (7.49). ∎

## Appendix A

## Multidimensional matrix inversions

In this appendix we state our multidimensional matrix inversions from [42] and specialize them as they are needed for deriving the identities in this article.

The following multidimensional matrix inverse appeared as Theorem 3.1 in [42]. There it was utilized in the derivation of several new summation theorems for basic hypergeometric series associated to the root systems of type $A_r$ and $D_r$.

**Theorem A.1** *Let $(a_t)_{t\in\mathbb{Z}}$, $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1, \dots, r$, be arbitrary sequences, $d$ arbitrary, such that none of the denominators in* (A.2) *or* (A.3) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1} (a_t - d/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - d/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \tag{A.2}$$

*and*

$$\begin{aligned} g_{\mathbf{kl}} = & \prod_{1\le i<j\le r} \frac{(c_i(l_i) - c_j(l_j))}{(c_i(k_i) - c_j(k_j))} \\ & \times \frac{(d - a_{|\mathbf{l}|}\prod_{j=1}^r c_j(l_j))}{(d - a_{|\mathbf{k}|}\prod_{j=1}^r c_j(k_j))} \prod_{i=1}^r \frac{(a_{|\mathbf{l}|} - c_i(l_i))}{(a_{|\mathbf{k}|} - c_i(k_i))} \\ & \times \frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|} (a_t - d/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - d/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))}. \end{aligned} \tag{A.3}$$

The following multidimensional matrix inversion is used in Section 3 for deriving nonterminating $q$-Abel identities.

**Proposition A.4** *Let $a$, $b$, and $x_1, \dots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.5) *or* (A.6) *vanish. Then the multidimensional matrices the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|} q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}} \prod_{i=1}^r \left(a + bx_i q^{k_i}\right)^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i,j=1}^r \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}^{-1} \tag{A.5}$$

*and*

$$g_{\mathbf{k}\mathbf{l}} = \prod_{i=1}^{r}(a+bx_iq^{l_i})\left(a+bx_iq^{k_i}\right)^{|\mathbf{k}|-|\mathbf{l}|-1} \prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q^{1+l_i-l_j};q\right)_{k_i-l_i}^{-1}. \tag{A.6}$$

**Proof:** In Theorem A.1 we set $a_t \mapsto -a$ and $c_i(t_i) \mapsto bx_iq^{t_i}$ for $i = 1, \ldots, r$. After some elementary manipulations (like transferring some factors from one matrix to the other), which include Lemma B.1, we let $d \to 0$ and obtain the inverse pair (A.5)/(A.6). ∎

In Proposition A.4, if we interchange $a$ and $b$ and transfer some factors from one matrix to the other we obtain the equivalent Proposition A.7. In this form the inverse matrices are used in Section 4 for deriving terminating $q$-Abel summations.

**Proposition A.7** *Let $a$, $b$, and $x_1, \ldots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.8) *or* (A.9) *vanish. Then the multidimensional matrices* $(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$\begin{aligned} f_{\mathbf{n}\mathbf{k}} = (-1)^{|\mathbf{k}|}\, q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}} \prod_{i=1}^{r}(ax_iq^{n_i}+b)\left(ax_iq^{k_i}+b\right)^{|\mathbf{n}|-1} \\ \times \prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}}\right] \end{aligned} \tag{A.8}$$

*and*

$$g_{\mathbf{k}\mathbf{l}} = (-1)^{|\mathbf{l}|}\prod_{i=1}^{r}\left(ax_iq^{k_i}+b\right)^{-|\mathbf{l}|} \prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q;q\right)_{k_i}}{\left(\frac{x_i}{x_j}q;q\right)_{l_i}\left(\frac{x_i}{x_j}q^{1+l_i-l_j};q\right)_{k_i-l_i}}\right]. \tag{A.9}$$

By another specialization of Theorem A.1, we deduce the following multidimensional matrix inversion which is also used in Section 3 for deriving nonterminating $q$-Abel identities.

**Proposition A.10** *Let $a$, $b$, and $x_1, \ldots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.11) *or* (A.12) *vanish. Then the multidimensional matrices* $(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{n}\mathbf{k}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|}\, q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}}\left(a+bq^{|\mathbf{k}|}\right)^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}^{-1} \tag{A.11}$$

*and*

$$g_{\mathbf{k}\mathbf{l}} = (a+bq^{|\mathbf{l}|})\left(a+bq^{|\mathbf{k}|}\right)^{|\mathbf{k}|-|\mathbf{l}|-1} \prod_{i,j=1}^{r}\left(\frac{x_i}{x_j}q^{1+l_i-l_j};q\right)_{k_i-l_i}^{-1}. \tag{A.12}$$

**Proof:** In Theorem A.1 we set $a_t \mapsto -d/a \prod_{j=1}^r x_j$ and $c_i(t_i) \mapsto b^{1/r} x_i q^{t_i}$ for $i = 1, \ldots, r$. After some elementary manipulations, which include Lemma B.1, we let $d \to 0$ and obtain the inverse pair (A.11)/(A.12). ■

In Proposition A.10, if we interchange $a$ and $b$ and transfer some factors from one matrix to the other we obtain the equivalent Proposition A.13. In this form the inverse matrices are used in Section 4 for deriving terminating $q$-Abel summations.

**Proposition A.13** *Let $a$, $b$, and $x_1, \ldots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.14) *or* (A.15) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}} \left(aq^{|\mathbf{n}|} + b\right) \left(aq^{|\mathbf{k}|} + b\right)^{|\mathbf{n}|-1} \times \prod_{i,j=1}^r \left[ \frac{\left(\frac{x_i}{x_j} q; q\right)_{n_i}}{\left(\frac{x_i}{x_j} q; q\right)_{k_i} \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}} \right] \tag{A.14}$$

*and*

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{l}|} \left(aq^{|\mathbf{k}|} + b\right)^{-|\mathbf{l}|} \prod_{i,j=1}^r \left[ \frac{\left(\frac{x_i}{x_j} q; q\right)_{k_i}}{\left(\frac{x_i}{x_j} q; q\right)_{l_i} \left(\frac{x_i}{x_j} q^{1+l_i-l_j}; q\right)_{k_i-l_i}} \right]. \tag{A.15}$$

By the following specialization of Theorem A.1, we deduce a multidimensional matrix inversion which is used in Section 5 for deriving nonterminating $q$-Rothe identities.

**Proposition A.16** *Let $a$, $b$, and $x_1, \ldots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.17) *or* (A.18) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|} q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}} \left(a + bq^{|\mathbf{k}|}\right)^{|\mathbf{n}|-|\mathbf{k}|} \left(\frac{q^{|\mathbf{k}|}}{(a + bq^{|\mathbf{k}|})}; q\right)_{|\mathbf{n}|-|\mathbf{k}|} \times \prod_{i,j=1}^r \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}^{-1} \tag{A.17}$$

*and*

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|} q^{\binom{|\mathbf{k}|+1}{2} - \binom{|\mathbf{l}|+1}{2}} \frac{(a + bq^{|\mathbf{l}|} - q^{|\mathbf{l}|})}{(a + bq^{|\mathbf{k}|} - q^{|\mathbf{k}|})} \left(aq^{-|\mathbf{k}|} + b; q\right)_{|\mathbf{k}|-|\mathbf{l}|} \times \prod_{i,j=1}^r \left(\frac{x_i}{x_j} q^{1+l_i-l_j}; q\right)_{k_i-l_i}^{-1}. \tag{A.18}$$

**Proof:** In Theorem A.1 we set $a_t \mapsto d/(q^t - a)\prod_{j=1}^r x_j$ and $c_i(t_i) \mapsto b^{1/r} x_i q^{t_i}$ for $i = 1, \ldots, r$. After some elementary manipulations, which include Lemma B.1, we let $d \to 0$ and obtain the inverse pair (A.17)/(A.18). ∎

In Proposition A.16, if we interchange $a$ and $b$ and transfer some factors from one matrix to the other we obtain the equivalent Proposition A.19. In this form the inverse matrices are used in Section 6 for deriving terminating $q$-Rothe summations.

**Proposition A.19** *Let $a$, $b$, and $x_1, \ldots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.20) *or* (A.21) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{n}|-|\mathbf{k}|}{2}} \frac{(b + aq^{|\mathbf{n}|} - q^{|\mathbf{n}|})}{(b + aq^{|\mathbf{k}|} - q^{|\mathbf{n}|})} \left(\frac{q}{(aq^{|\mathbf{k}|} + b)}; q\right)_{|\mathbf{n}|} \left(aq^{|\mathbf{k}|} + b\right)^{|\mathbf{n}|} \times \prod_{i,j=1}^r \left[\frac{\left(\frac{x_i}{x_j} q; q\right)_{n_i}}{\left(\frac{x_i}{x_j} q; q\right)_{k_i} \left(\frac{x_i}{x_j} q^{1+k_i-k_j}; q\right)_{n_i-k_i}}\right] \tag{A.20}$$

*and*

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{l}|} \frac{\left(aq^{|\mathbf{k}|} + b\right)^{-|\mathbf{l}|}}{\left(\frac{q}{(aq^{|\mathbf{k}|}+b)}; q\right)_{|\mathbf{l}|}} \prod_{i,j=1}^r \left[\frac{\left(\frac{x_i}{x_j} q; q\right)_{k_i}}{\left(\frac{x_i}{x_j} q; q\right)_{l_i} \left(\frac{x_i}{x_j} q^{1+l_i-l_j}; q\right)_{k_i-l_i}}\right]. \tag{A.21}$$

*Remark A.22.* In this Appendix, we have deduced the preceding Propositions from Theorem A.1. However, it should be mentioned that they also could have been derived from Bhatnagar and Milne's multidimensional matrix inversion [5, Theorem 3.48] (which is contained in Theorem A.1 as a special case).

The following multidimensional matrix inverse appeared as Theorem 4.1 in [42].

**Theorem A.23** *Let $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1, \ldots, r$, be arbitrary sequences, $d$ arbitrary, such that none of the denominators in* (A.24) *or* (A.25) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \prod_{i=1}^r \frac{\prod_{t_i=k_i}^{n_i-1} (1 - dc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - d/\prod_{j=1}^r c_j(k_j))} \prod_{i,j=1}^r \frac{\prod_{t_i=k_i}^{n_i-1} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \tag{A.24}$$

*and*

$$
\begin{aligned}
g_{\mathbf{kl}} = \prod_{1\le i<j\le r} &\left(\frac{(c_i(l_i)-c_j(l_j))}{(c_i(k_i)-c_j(k_j))}\frac{(1-c_i(l_i)c_j(l_j))}{(1-c_i(k_i)c_j(k_j))}\right)\\
&\times\prod_{i=1}^r\frac{(1-c_i(l_i)^2)}{(1-c_i(k_i)^2)}\prod_{i=1}^r\frac{c_i(l_i)}{c_i(k_i)}\\
\times\prod_{i=1}^r&\frac{\prod\limits_{t_i=l_i+1}^{k_i}(1-dc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-d/\prod_{j=1}^r c_j(k_j))}\prod_{i,j=1}^r\frac{\prod\limits_{t_i=l_i+1}^{k_i}(1-c_i(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}.
\end{aligned}
\tag{A.25}
$$

No applications of the above multidimensional matrix inversion were given in [42]. Here, this is made up for, as from the following specializations of Theorem A.23 we derive certain new multiple series expansions in Section 7.

**Proposition A.26** *Let $a$, $b$, and $x_1,\dots,x_r$ be indeterminate, and suppose that none of the denominators in* (A.27) *or* (A.28) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$
\begin{aligned}
f_{\mathbf{nk}} = q^{\sum_{i=1}^r (i-1)(n_i-k_i)}&\prod_{i=1}^r\frac{\left(\frac{bx_iq^{k_i}}{1+a\prod_{j=1}^r(bx_jq^{k_j}-1)};q\right)_{n_i-k_i}}{\left(\frac{abx_iq^{1+k_i}\prod_{j=1}^r(bx_jq^{k_j}-1)}{1+a\prod_{j=1}^r(bx_jq^{k_j}-1)};q\right)_{n_i-k_i}}\\
&\times\prod_{i,j=1}^r\frac{\left(bx_iq^{k_i}-\frac{x_i}{x_j}q^{k_i-k_j};q\right)_{n_i-k_i}}{\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}}
\end{aligned}
\tag{A.27}
$$

*and*

$$
\begin{aligned}
g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|}\,q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}}&\prod_{1\le i\le j\le r}\left(\frac{1-(bx_iq^{l_i}-\frac{x_i}{x_j}q^{l_i-l_j})}{1-(bx_iq^{k_i}-\frac{x_i}{x_j}q^{k_i-k_j})}\right)\\
\times\prod_{i=1}^r\frac{(bx_iq^{l_i}-1)}{(bx_iq^{k_i}-1)}&\prod_{i=1}^r\frac{\left(\frac{bx_iq^{1+l_i}}{1+a\prod_{j=1}^r(bx_jq^{k_j}-1)};q\right)_{k_i-l_i}}{\left(\frac{abx_iq^{l_i}\prod_{j=1}^r(bx_jq^{k_j}-1)}{1+a\prod_{j=1}^r(bx_jq^{k_j}-1)};q\right)_{k_i-l_i}}\\
&\times\prod_{i,j=1}^r\frac{\left(bx_iq^{1+l_i}-\frac{x_i}{x_j}q^{1+l_i-k_j};q\right)_{k_i-l_i}}{\left(\frac{x_i}{x_j}q^{1+l_i-l_j};q\right)_{k_i-l_i}}.
\end{aligned}
\tag{A.28}
$$

**Proof:** In Theorem A.23 we set $d\mapsto 1/a$ and $c_i(t_i)\mapsto bx_iq^{t_i}-1$ for $i=1,\dots,r$. After some elementary manipulations, which include Lemma B.1, we obtain the inverse pair (A.27)/(A.28). ∎

**Proposition A.29** *Let $b$, and $x_1, \ldots, x_r$ be indeterminate, and suppose that none of the denominators in* (A.30) *or* (A.31) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = q^{\sum_{i=1}^r (i-1)(n_i-k_i)} \prod_{i,j=1}^r \frac{\left(bx_iq^{k_i} - \frac{x_i}{x_j}q^{k_i-k_j};q\right)_{n_i-k_i}}{\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \tag{A.30}$$

*and*

$$\begin{aligned} g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|} q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}} &\prod_{1\le i\le j\le r} \left( \frac{1-(bx_iq^{l_i} - \frac{x_i}{x_j}q^{l_i-l_j})}{1-(bx_iq^{k_i} - \frac{x_i}{x_j}q^{k_i-k_j})} \right) \\ &\times \prod_{i,j=1}^r \frac{\left(bx_iq^{1+l_i} - \frac{x_i}{x_j}q^{1+l_i-k_j};q\right)_{k_i-l_i}}{\left(\frac{x_i}{x_j}q^{1+l_i-l_j};q\right)_{k_i-l_i}}. \end{aligned} \tag{A.31}$$

**Proof:** In Proposition A.26 we set $a \mapsto 0$ and transfer some factors from one matrix to the other to obtain the inverse pair (A.30)/(A.31). ■

**Proposition A.32** *Let $A_1, \ldots, A_r$, $B$, and $C$ be indeterminate, and suppose that none of the denominators in* (A.33) *or* (A.34) *vanish. Then the multidimensional matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$\begin{aligned} f_{\mathbf{nk}} = \prod_{i=1}^r &\frac{\left(\frac{A_i}{B} + k_i - \frac{C}{B}\prod_{j=1}^r (A_j + Bk_j)\right)_{n_i-k_i}}{\left(1 + \frac{A_i}{B} + k_i - \frac{1}{BC\prod_{j=1}^r (A_j+Bk_j)}\right)_{n_i-k_i}} \\ &\times \prod_{i,j=1}^r \frac{\left(\frac{A_i}{B} + k_i - \frac{1}{B(A_j+Bk_j)}\right)_{n_i-k_i}}{\left(1 + \frac{A_i}{B} + k_i - \frac{A_j}{B} - k_j\right)_{n_i-k_i}} \end{aligned} \tag{A.33}$$

*and*

$$\begin{aligned} g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|} &\prod_{1\le i\le j\le r} \left( \frac{1-(A_i+Bl_i)(A_j+Bl_j)}{1-(A_i+Bk_i)(A_j+Bk_j)} \right) \\ \times \prod_{i=1}^r \frac{(A_i+Bl_i)}{(A_i+Bk_i)} &\prod_{i=1}^r \frac{\left(1 + \frac{A_i}{B} + l_i - \frac{C}{B}\prod_{j=1}^r (A_j + Bk_j)\right)_{k_i-l_i}}{\left(\frac{A_i}{B} + l_i - \frac{1}{BC\prod_{j=1}^r (A_j+Bk_j)}\right)_{k_i-l_i}} \\ &\times \prod_{i,j=1}^r \frac{\left(1 + \frac{A_i}{B} + l_i - \frac{1}{B(A_j+Bk_j)}\right)_{k_i-l_i}}{\left(1 + \frac{A_i}{B} + l_i - \frac{A_j}{B} - l_j\right)_{k_i-l_i}}. \end{aligned} \tag{A.34}$$

**Proof:** In Theorem A.23 we set $d \mapsto 1/C$ and $c_i(t_i) \mapsto A_i + Bt_i$ for $i = 1, \ldots, r$. After some elementary manipulations, which include the $q \to 1$ case of Lemma B.1, we obtain the inverse pair (A.33)/(A.34). ■

## Appendix B

## Background information – $A_r$ basic hypergeometric summation theorems

Here we state a simplification lemma, a partial fraction decomposition, and the $A_r$ basic hypergeometric summations (mainly taken from Milne [37]) we need in the proofs of our multiple summation theorems.

**Lemma B.1**

$$\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}q^{l_i-l_j}}\right)\prod_{i,j=1}^{r}\frac{\left(\frac{x_i}{x_j}q^{l_i-k_j};q\right)_{k_i-l_i}}{\left(\frac{x_i}{x_j}q^{1+l_i-l_j};q\right)_{k_i-l_i}}
= (-1)^{|\mathbf{k}|-|\mathbf{l}|}\, q^{-\binom{|\mathbf{k}|-|\mathbf{l}|}{2}}\, q^{-\sum_{i=1}^{r} i(k_i-l_i)}.$$

*Remark B.2.* Lemma B.1 is equivalent to Lemma 4.3 of [37], which is proved by some elementary manipulations.

*Remark B.3.* When reversing order of the summations in the proofs of Sections 4 and 6, we permanently make use of the fact that the "$A_r$ $q$-binomial coefficient"

$$\prod_{i,j=1}^{r}\left[\frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}}\right] \tag{B.4}$$

remains unchanged after performing the substitutions $k_i\mapsto n_i-k_i$ and $x_i\mapsto q^{-n_i}/x_i$, for $i=1,\dots,r$. This can be seen using Lemma B.1.

Throughout this article, we use the following ($q$-analogue of the) *partial fraction decomposition* for the derivation of some multidimensional "companion identities":

$$\prod_{i=1}^{r}\frac{(1-tx_iy_i)}{(1-tx_i)} = y_1y_2\dots y_r + \sum_{j=1}^{r}\frac{\prod_{i=1}^{r}(1-y_ix_i/x_j)}{(1-tx_j)\prod_{\substack{i=1\\ i\neq j}}^{r}(1-x_i/x_j)} \tag{B.5}$$

(see [35, Appendix]). Partial fraction decompositions have been an essential tool in the study of multiple (basic) hypergeometric series associated to root systems, see e.g. [5], [35].

In the remainder of this Appendix we list the $A_r$ basic hypergeometric summation theorems we need for proving the theorems of this article.

**Theorem B.6 (An $\mathbf{A_r}$ ${}_0\phi_0$-summation)** *Let $a$ and $x_1,\dots,x_r$ be indeterminate. Then there holds*

$$(a;q)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j}q;q \right)_{k_i}^{-1} \right. \\ \left. \times (-1)^{r|\mathbf{k}|} a^{|\mathbf{k}|} q^{r\sum_{i=1}^r \binom{k_i}{2} + \sum_{i=1}^r (i-1)k_i} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \right). \tag{B.7}$$

*Remark B.8.* Theorem B.6 can be obtained from Theorem B.29 by substituting $z \mapsto z/\prod_{i=1}^r a_i$, then letting $a_1 \to \infty, \dots, a_r \to \infty$, and relabelling $z \mapsto a$.

**Theorem B.9 ($\mathbf{A_r}$ terminating q-binomial theorems)** *Let $x_1, \dots, x_r$, and $z$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$(z;q)_{|\mathbf{n}|} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}} z^{|\mathbf{k}|} \right), \tag{B.10}$$

$$\prod_{i=1}^{r} (zx_i;q)_{n_i} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \right. \\ \left. \times (-1)^{|\mathbf{k}|} q^{\sum_{i=1}^r \binom{k_i}{2}} z^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{k_i} \right), \tag{B.11}$$

$$\prod_{i=1}^{r} \left( \frac{z}{x_i} q^{|\mathbf{n}|-n_i};q \right)_{n_i} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \left( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \right. \\ \left. \times (-1)^{|\mathbf{k}|} q^{e_2(\mathbf{k})+\binom{|\mathbf{k}|}{2}} z^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{-k_i} \right), \tag{B.12}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \dots, k_r\}$.*

*Remark B.13.* The summations (B.10), (B.11), and (B.12) are Theorems 5.44, 5.46, and 5.48 of [37], respectively (slightly rewritten using Lemma B.1).

**Theorem B.14 ($\mathbf{A_r}$ $_1\phi_1$-summations)** *Let $a$, $c$, and $x_1, \dots, x_r$ be indeterminate. Then there holds*

$$\frac{(c/a;q)_\infty}{(c;q)_\infty} = \sum_{k_1,\dots,k_r=0}^{\infty} \left( \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j}q;q \right)_{k_i}^{-1} \right. \\ \left. \times \frac{(a;q)_{|\mathbf{k}|}}{(c;q)_{|\mathbf{k}|}} (-1)^{r|\mathbf{k}|} q^{r\sum_{i=1}^r \binom{k_i}{2} + \sum_{i=1}^r (i-1)k_i} \left(\frac{c}{a}\right)^{|\mathbf{k}|} \prod_{i=1}^{r} x_i^{rk_i-|\mathbf{k}|} \right), \tag{B.15}$$

$$\frac{(c/a;q)_\infty}{(c;q)_\infty} = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg( \prod_{1\le i<j\le r} \left( \frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}} \right) \prod_{i,j=1}^{r} \left( \frac{x_i}{x_j}q;q \right)_{k_i}^{-1} \\ \times \frac{(a;q)_{|\mathbf{k}|}}{(c;q)_{|\mathbf{k}|}} (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}+\sum_{i=1}^r (i-1)k_i} \left(\frac{c}{a}\right)^{|\mathbf{k}|} \Bigg). \tag{B.16}$$

*Remark B.17.* The summation (B.15) can be obtained from Theorem 7.6 of [37] by letting $a_1 \to \infty, \dots, a_r \to \infty$, and relabelling $b \mapsto a$. The summation (B.16) can be obtained from Theorem 7.9 of [37] by letting $b \to \infty$.

**Theorem B.18 ($\mathbf{A_r}$ q-Chu–Vandermonde summations)** *Let $x_1, \dots, x_r$, $a$, and $c$ be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$\frac{\prod_{i=1}^r (cx_i/a;q)_{n_i}}{(c;q)_{|\mathbf{n}|}} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \\ \times \frac{\prod_{i=1}^r \left(\frac{a}{x_i}q^{|\mathbf{k}|-k_i};q\right)_{k_i}}{(c;q)_{|\mathbf{k}|}} (-1)^{|\mathbf{k}|} q^{\sum_{i=1}^r \binom{k_i}{2}} \left(\frac{c}{a}\right)^{|\mathbf{k}|} \prod_{i=1}^r x_i^{k_i} \Bigg), \tag{B.19}$$

$$\frac{\prod_{i=1}^r \left(\frac{c}{ax_i}q^{|\mathbf{n}|-n_i};q\right)_{n_i}}{(c;q)_{|\mathbf{n}|}} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \\ \times \frac{\prod_{i=1}^r (ax_i;q)_{k_i}}{(c;q)_{|\mathbf{k}|}} (-1)^{|\mathbf{k}|} q^{e_2(\mathbf{k})+\binom{|\mathbf{k}|}{2}} \left(\frac{c}{a}\right)^{|\mathbf{k}|} \prod_{i=1}^r x_i^{-k_i} \Bigg), \tag{B.20}$$

*where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1, \dots, k_r\}$,*

$$\frac{(c/a;q)_{|\mathbf{n}|}}{(c;q)_{|\mathbf{n}|}} = \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg( \prod_{i,j=1}^{r} \left[ \frac{\left(\frac{x_i}{x_j}q;q\right)_{n_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i} \left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_{n_i-k_i}} \right] \\ \times \frac{(a;q)_{|\mathbf{k}|}}{(c;q)_{|\mathbf{k}|}} (-1)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2}} \left(\frac{c}{a}\right)^{|\mathbf{k}|} \Bigg). \tag{B.21}$$

*Remark B.22.* The summations (B.19) and (B.20) are Theorems 5.28 and 5.32 of [37], respectively (slightly rewritten using Lemma B.1). The summation (B.21) can be obtained from Theorem 7.6 of [37] by letting $a_i \to q^{-n_i}$, $i = 1, \dots, r$, and relabelling $b \mapsto a$.

**Theorem B.23 (An $\mathbf{A_r}$ q-Gauß summation)** *Let $x_1, \ldots, x_r, a_1, \ldots, a_r, b$, and $c$ be indeterminate. Then there holds*

$$\frac{(c/b;q)_\infty}{(c/b\prod_{i=1}^r a_i;q)_\infty}\prod_{i=1}^r\frac{(cx_i/a_i;q)_\infty}{(cx_i;q)_\infty}$$
$$=\sum_{k_1,\ldots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)\prod_{i,j=1}^r\frac{\left(\frac{x_i}{x_j}a_j;q\right)_{k_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}}\right.$$
$$\left.\times\prod_{i=1}^r\frac{(bx_i;q)_{k_i}}{(cx_i;q)_{k_i}}\cdot\left(\frac{c}{b\prod_{i=1}^r a_i}\right)^{|\mathbf{k}|}q^{\sum_{i=1}^r(i-1)k_i}\right), \tag{B.24}$$

*provided $|c|<|b\prod_{i=1}^r a_i|$.*

*Remark B.25.* The summation (B.24) is Theorem 5.1 of [37].

**Theorem B.26 (An $\mathbf{A_r}$ Gauß summation)** *Let $X_1, \ldots, X_r, A_1, \ldots, A_r, B$, and $C$ be indeterminate. Then there holds*

$$\frac{\Gamma(C-B-\sum_{i=1}^r A_i)}{\Gamma(C-B)}\prod_{i=1}^r\frac{\Gamma(C+X_i)}{\Gamma(C+X_i-A_i)}$$
$$=\sum_{k_1,\ldots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{X_i+k_i-X_j-k_j}{X_i-X_j}\right)\right.$$
$$\left.\prod_{i,j=1}^r\frac{(A_j+X_i-X_j)_{k_i}}{(1+X_i-X_j)_{k_i}}\prod_{i=1}^r\frac{(B+X_i)_{k_i}}{(C+X_i)_{k_i}}\right), \tag{B.27}$$

*provided $\Re(C)>\Re(B)+\sum_{i=1}^r\Re(A_i)$.*

*Remark B.28.* This $q\to 1$ case of Theorem B.23 is equivalent to Theorem 1 of [24].

**Theorem B.29 (An $\mathbf{A_r}$ nonterminating q-binomial theorem)** *Let $a_1, \ldots, a_r$, $z$, and $x_1,\ldots,x_r$ be indeterminate. Then there holds*

$$\frac{(z\prod_{i=1}^r a_i;q)_\infty}{(z;q)_\infty}=\sum_{k_1,\ldots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{1-\frac{x_i}{x_j}q^{k_i-k_j}}{1-\frac{x_i}{x_j}}\right)q^{\sum_{i=1}^r(i-1)k_i}\right.$$
$$\left.\prod_{i,j=1}^r\frac{\left(\frac{x_i}{x_j}a_j;q\right)_{k_i}}{\left(\frac{x_i}{x_j}q;q\right)_{k_i}}\cdot z^{|\mathbf{k}|}\right), \tag{B.30}$$

*provided $|z|<1$.*

*Remark B.31.* The summation (B.30) is Theorem 1.47 of [34]. It can be obtained from Theorem B.23 by taking $c=bz\prod_{i=1}^r a_i$ and then letting $b\to 0$.

## Appendix C
### Convergence of the (multiple) ($q$-)series

In this appendix we discuss and prove the absolute convergence of the nonterminating summations in this article. Similar convergence proofs for multiple (basic) hypergeometric series are given in [12, Appendix].

First, we consider the convergence of our nonterminating multiple $q$-series (recall that we always assume $0<|q|<1$). To the $q$-rising factorials of the summand of the series we apply, if necessary (i.e., if there is an occurrence of a negative power of $q$ in the parameter), the relation

$$(a;q)_k=(q^{1-k}/a;q)_k\,(-a)^k\,q^{\binom{k}{2}} \tag{C.1}$$

(cf. [16, Eq. (I.3)]), so that we are only left with positive powers of $q$ in the $q$-factorials. Then we rewrite all $q$-factorials using (2.9). Since $\lim_{k\to+\infty}(aq^k;q)_\infty=1$, factors of the form $(\,\cdot\,;q)_\infty$ will then not influence the convergence of the series.

As an explicit example we consider (5.8) where we apply (C.1) to the factor $(aq^{-|\mathbf{k}|}+b;q)_{|\mathbf{k}|}$. After performing some manipulations we see that the series in (5.8) converges provided

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left|(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|}z^{|\mathbf{k}|}q^{\sum_{i=1}^r(i-1)k_i}\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)\right|<\infty. \tag{C.2}$$

We use the Vandermonde determinant expansion

$$q^{\sum_{i=1}^r(i-1)k_i}\prod_{1\le i<j\le r}\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)=\prod_{i=1}^r x_i^{1-i}\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\prod_{i=1}^r x_{\sigma(i)}^{i-1}q^{(\sigma^{-1}(i)-1)k_i},$$

where $\mathcal{S}_r$ denotes the symmetric group of order $r$. Then we interchange summation in (C.2) and obtain $r!$ multiple sums each corresponding to a permutation $\sigma\in\mathcal{S}_r$. Thus we see that the series in (5.8) converges provided

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left|(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|}z^{|\mathbf{k}|}\prod_{i=1}^r q^{(\sigma^{-1}(i)-1)k_i}\right|<\infty, \tag{C.3}$$

for any $\sigma\in\mathcal{S}_r$. The series on the left hand side of (C.3) is dominated by

$$\prod_{i=1}^r\sum_{k_i=0}^{\infty}\left|(|a|+|b||q|^{k_i})^{k_i}z^{k_i}q^{(\sigma^{-1}(i)-1)k_i}\right|.$$

Therefore we deduce (by d'Alembert's ratio test, or, equivalently, comparison with the geometrical series) that the series in (5.8) converges whenever $|azq^{j-1}|<1$, for $j=1,\dots,r$, or, equivalently, whenever $|az|<1$.

In some of our series also the factor

$$q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r \binom{k_i}{2}} \prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}$$

appears in the summand. Consider, for example, the series in (5.6). Here we use the identity

$$-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r \binom{k_i}{2} = -\frac{(r-1)}{2}|\mathbf{k}|+\frac{1}{2}\sum_{1\le i<j\le r}(k_i-k_j)^2$$

to replace $q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r \binom{k_i}{2}}$ by $q^{-\frac{(r-1)}{2}|\mathbf{k}|}$ for comparison with a dominating multiple series. In a way similar to as before, we see that the series in (5.6) converges provided

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left|(a+bq^{|\mathbf{k}|})^{|\mathbf{k}|}z^{|\mathbf{k}|}q^{-\frac{(r-1)}{2}|\mathbf{k}|}\prod_{i=1}^r q^{(\sigma^{-1}(i)-1)k_i}x_i^{rk_i-|\mathbf{k}|}\right|<\infty, \tag{C.4}$$

for any $\sigma\in\mathcal{S}_r$. Now, the series on the left hand side of (C.4) is dominated by

$$\prod_{i=1}^r\sum_{k_i=0}^{\infty}\left|(|a|+|b||q|^{k_i})^{k_i}z^{k_i}q^{\left(\sigma^{-1}(i)-1-\frac{(r-1)}{2}\right)k_i}x_i^{rk_i}(\textstyle\prod_{j=1}^r x_j)^{-k_i}\right|,$$

from which we eventually deduce that the series in (5.6) converges whenever $|az|<\left|q^{\frac{r-1}{2}}x_i^{-r}\prod_{j=1}^r x_j\right|$, for $i=1,\dots,r$.

Next, we check convergence of the multiple $q$-series in (7.45) and (7.47). Due to symmetry of the summands (observe that some factors are quotients of the form (7.43)) it suffices to consider convergence in the summation region $k_1\ge k_2\ge\dots\ge k_r$. Using some elementary identities from [16, Appendix I] we may rewrite

$$\prod_{1\le i<j\le r}\left[\left(1-\frac{x_i}{x_j}q^{k_i-k_j}\right)\left(1-\left(bx_jq^{k_j}-\frac{x_j}{x_i}q^{k_j-k_i}\right)\right)\right]\prod_{i,j=1}^r\left(1-\left(bx_i-\frac{x_i}{x_j}q^{-k_j}\right)\right)^{-1}$$

as

$$q^{-|\mathbf{k}|+2\sum\limits_{i=1}^r ik_i}\prod_{i=1}^r x_i^{r-1}\prod_{i,j=1}^r\frac{\left(bx_iq^{1+k_i}-\frac{x_i}{x_j}q^{1+k_i};q\right)_\infty}{\left(bx_iq^{k_i}-\frac{x_i}{x_j}q^{k_i};q\right)_\infty}$$
$$\times\prod_{1\le i<j\le r}\frac{\left(\frac{x_i}{x_j}q^{k_i-k_j};q\right)_\infty\left(bx_iq^{k_i}-\frac{x_i}{x_j}q^{k_i-k_j};q\right)_\infty}{\left(\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_\infty\left(bx_iq^{1+k_i}-\frac{x_i}{x_j}q^{1+k_i-k_j};q\right)_\infty},$$

and also

$$\prod_{i,j=1}^r\left(bx_i-\frac{x_i}{x_j}q^{-k_j};q\right)_{k_i}$$

as

$$q^{-\binom{|\mathbf{k}|}{2}-\sum_{i=1}^{r} ik_i} \prod_{i,j=1}^{r} \left(\frac{q}{bx_iq^{k_j}-\frac{x_i}{x_j}};q\right)_{k_j}$$
$$\times \prod_{1\le i<j\le r} \frac{\left(bx_iq^{k_j}-\frac{x_i}{x_j};q\right)_{k_i-k_j}}{\left(\frac{q}{bx_jq^{k_i}-\frac{x_j}{x_i}};q\right)_{k_i-k_j}} \left(\frac{x_j}{x_i}-bx_jq^{k_i}\right)^{k_j} \left(\frac{x_i}{x_j}-bx_iq^{k_j}\right)^{k_j}.$$

Now, if we rewrite all $q$-factorials in the summand of (7.45) in terms of (2.9) we see that the series converges in the summation region $k_1\ge\ldots\ge k_r$ provided

$$\sum_{k_1\ge\ldots\ge k_r\ge 0} \left| a^{|\mathbf{k}|} q^{\sum_{i=1}^r ik_i} \prod_{1\le i<j\le r} \left(\frac{x_j}{x_i}-bx_jq^{k_i}\right)^{k_j} \left(\frac{x_i}{x_j}-bx_iq^{k_j}\right)^{k_j} \right| < \infty. \tag{C.5}$$

The series on the left hand side of (C.5) is dominated by

$$\sum_{k_1\ge\ldots\ge k_r\ge 0} \left| a^{|\mathbf{k}|} q^{\sum_{j=1}^r jk_j} \prod_{j=1}^{r} \prod_{1\le i\le j} \left(1+|bx_i||q|^{k_j}\right)^{(j-1)k_j} \right|,$$

from which we eventually deduce that the series in (7.45) converges whenever $|aq|<1$. (Similarly, the series in (7.47) converges absolutely provided $|z|<1$.)

Next, we consider the convergence of our nonterminating (one-dimensional and multiple) "ordinary" series. First we apply to some of the rising factorials in the summands of our series the relation

$$(a)_k = (1-k-a)_k\,(-1)^k. \tag{C.6}$$

(The purpose of this manipulation is to enable the application of (C.8) below.) Then we rewrite all rising factorials in the summands of our series as quotients of gamma functions using (2.6). Next we apply *Stirling's formula* (cf. [45, Sec. 3.12, Eq. (5)])

$$\Gamma(z) = z^{z+\frac{1}{2}}\,e^{-z}\,\sqrt{2\pi}\left(1+O\left(\frac{1}{z}\right)\right), \tag{C.7}$$

valid for $|z|\to\infty$, where $z$ is in the complex plane but not on the negative real axis. A consequence of (C.7) is

$$\left|\frac{\Gamma(a+ck)}{\Gamma(b+ck)}\right| = O\left(k^{\Re(a)-\Re(b)}\right), \tag{C.8}$$

for $k\to+\infty$, provided $c$ is not a nonpositive real number, which we conveniently make use of, if possible.

As an explicit example we consider (7.2), with a one-dimensional series on the right hand side. Here we apply (C.6) to the factor $(\frac{C}{D}-\frac{(A+Bk)}{D})_k$ before we apply

(2.6) to all rising factorials in the summand. After some manipulations we obtain that the series in (7.2) converges if

$$\sum_{k=0}^{\infty}\left|\frac{\Gamma\left(\frac{A-C}{D}-1+\frac{B}{D}k\right)\,\Gamma\left(\frac{C}{D}-\frac{1}{D(A+Bk)}+k\right)}{\Gamma\left(\frac{A-C}{D}+\left(\frac{B}{D}-1\right)k\right)\,\Gamma(1+k)}\right.$$
$$\left.\times\frac{\Gamma\left(\frac{A}{B}+\frac{A}{D}-\frac{2C}{D}+\frac{(B-D)}{BD(A+Bk)}+1+\left(\frac{B}{D}-1\right)k\right)}{\Gamma\left(\frac{A}{B}+\frac{A}{D}-\frac{C}{D}-\frac{1}{B(A+Bk)}+1+\frac{B}{D}k\right)}\right|<\infty.$$

Using (C.8) and the absolute convergence of *Riemann's zeta function* (cf. [45, Sec. 3.14, Eq. (1)]) $\zeta(s)=\sum_{k\geq 0}k^{-s}$ for $\Re(s)>1$ we see that the series in (7.2) converges provided $\frac{B}{D}\not\leq 1$ (by comparison with $\zeta(2)$), or, $B=D$ and $\Re(\frac{C-A}{B})<2$ (by comparison with $\zeta(3+\frac{A-C}{B})$). The convergence of the series in (7.6) can be shown similar.

Finally, we consider multiple ordinary series. The convergence of the series in (3.14), (3.15), (5.15), and (5.16), for $r\geq 2$, can easily be checked by application of the multiple power series ratio test [25], [28], for which we omit the details.

Before investigating the convergence of the series in (7.49) we give another illustration of the method by checking the convergence of the multiple Gauß series in Theorem B.26. First we convert all rising factorials into quotients of gamma functions by (2.6). Then we use (C.8) and see that that the series in (B.27) converges absolutely provided

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left|\prod_{1\leq i<j\leq r}(X_i+k_i-X_j-k_j)\prod_{i,j=1}^{r}k_i^{A_j-1}\prod_{i=1}^{r}k_i^{B-C}\right|<\infty. \tag{C.9}$$

We use the Vandermonde determinant expansion

$$\prod_{1\leq i<j\leq r}(X_i+k_i-X_j-k_j)=\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\prod_{i=1}^{r}(x_i+k_i)^{r-\sigma(i)},$$

where, again, $\mathcal{S}_r$ denotes the symmetric group of order $r$. Then we interchange summation in (C.9) and obtain $r!$ multiple sums each corresponding to a permuation $\sigma\in\mathcal{S}_r$. Then we see that the series in (B.27) converges provided

$$\prod_{i=1}^{r}\sum_{k_i=0}^{\infty}\left|(x_i+k_i)^{r-\sigma(i)}\,k_i^{B-C-r+\sum_{j=1}^{r}A_j}\right|<\infty,$$

for any $\sigma\in\mathcal{S}_r$, or equivalently, if

$$\prod_{i=1}^{r}\sum_{k_i=0}^{\infty}k_i^{-\sigma(i)+B-C+\sum_{j=1}^{r}A_j}<\infty,$$

from which we deduce that the series in (B.27) converges whenever $-j+\Re(B)-\Re(C)+\sum_{i=1}^r\Re(A_i)<-1$, for $j=1,\dots,r$, or, equivalently, whenever $\Re(C)>\Re(B)+\sum_{i=1}^r\Re(A_i)$.

Finally, we consider the series in (7.49). The whole machinery can be applied to prove the absolute convergence of the series in question. We apply (C.6) to the factors $\prod_{1\le i\le r}\left(\frac{A_i}{B}-\frac{C}{B}\prod_{j=1}^r(A_j+Bk_j)+1\right)_{k_i}$ before we apply (2.6) to all rising factorials in the summand. We use both asymptotics (C.7) and (C.8) (where applicable). We also use the (orthogonal Vandermonde) determinant expansion

$$\prod_{1\le i<j\le r}\left[(A_i+Bk_i-A_j-Bk_j)(1-(A_i+Bk_i)(A_j+Bk_j))\right]$$
$$=\frac{1}{2}\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\prod_{i=1}^r\left((A_i+Bk_i)^{r-\sigma(i)}+(A_i+Bk_i)^{r+\sigma(i)-2}\right),$$

which we may simply estimate as $O\big(\prod_{i=1}^r k_i^{2r-2}\big)$. After a further amount of calculations we can compare the series with a product of $r$ zeta functions. If $C\prod_{j\ne i}(A_j+Bk_j)\not\le 1$ for $i=1,\dots,r$ and $k_j=0,1,2,\dots$ $(j=1,\dots,r,\ j\ne i)$, we use the convergence of $\zeta(2)$. Else, if $C\prod_{j\ne i}(A_j+Bk_j)=1$ for an $i=1,\dots,r$ and a $k_j=0,1,2,\dots$ $(j=1,\dots,r,\ j\ne i)$, then we compare the series with $\zeta\Big(3+\frac{A_i}{B}-\sum_{j=1}^r\frac{A_j}{B}\Big)$, in which latter case we deduce that the series in (7.49) converges whenever $\sum_{1\le j\le r,\,j\ne i}\Re(\frac{A_j}{B})<2$.

## References


1. G. E. Andrews, "Connection coefficient problems and partitions", D. Ray-Chaudhuri, ed., *Proc. Symp. Pure Math.*, vol. 34, Amer. Math. Soc., Providence, R. I., 1979, 1–24.
2. W. N. Bailey, "Some identities in combinatory analysis", *Proc. London Math. Soc.* (2) **49** (1947), 421–435.
3. G. Bhatnagar, *Inverse relations, generalized bibasic series and their $U(n)$ extensions*, Ph. D. thesis, The Ohio State University, 1995.
4. G. Bhatnagar, "$D_n$ basic hypergeometric series", to appear in *The Ramanujan J.*
5. G. Bhatnagar and S. C. Milne, "Generalized bibasic hypergeometric series and their $U(n)$ extensions", *Adv. in Math.* **131** (1997), 188–252.
6. G. Bhatnagar and M. Schlosser, "$C_n$ and $D_n$ very-well-poised ${}_{10}\phi_9$ transformations", *Constr. Approx.* **14** (1998), 531–567.
7. D. M. Bressoud, "A matrix inverse", *Proc. Amer. Math. Soc.* **88** (1983), 446–448.
8. L. Carlitz, "Some inverse relations", *Duke Math. J.* **40** (1973), 893–901.
9. L. Carlitz, "Some expansions and convolution formulas related to MacMahon's Master Theorem", *SIAM J. Appl. Math.* **8** (1977), 320–336.
10. W. Chu, "Some multifold reciprocal transformations with applications to series expansions", *Europ. J. Comb.* **12** (1991), 1–8.
11. W. Chu, "Inversion techniques and combinatorial identities – Jackson's $q$-analogue of the Dougall-Dixon theorem and the dual formulae", *Compositio Math.* **95** (1995), 43–68.
12. J. F. van Diejen, "On certain multiple Bailey, Rogers and Dougall type summation formulas", *Publ. RIMS, Kyoto Univ.* **33** (1997), 483–508.
13. L. Euler, *De serie Lambertiana plurimisque eius insignibus proprietatibus*, Acta Acad. Sci. Petro, 1779; II, 1783, pp. 29–51; reprinted in *Opera Omnia* Ser. I, vol. 6, Teubner, Leibzig, 1921, pp. 350–369.

14. G. Gasper, "Summation, transformation and expansion formulas for bibasic series", *Trans. Amer. Soc.* **312** (1989), 257-278.
15. G. Gasper and M. Rahman, "An indefinite bibasic summation formula and some quadratic, cubic and quartic summation and transformation formulae", *Canad. J. Math.* **42** (1990), 1–27.
16. G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990.
17. I. M. Gessel and D. Stanton, "Application of $q$-Lagrange inversion to basic hypergeometric series", *Trans. Amer. Math. Soc.* **277** (1983), 173–203.
18. I. M. Gessel and D. Stanton, "Another family of $q$-Lagrange inversion formulas", *Rocky Mountain J. Math.* **16** (1986), 373–384.
19. H. W. Gould, "Final analysis of Vandermonde's convolution", *Amer. Math. Monthly* **64** (1957), 409–415.
20. H. W. Gould, *Combinatorial Identities*, Morgantown, 1972.
21. H. W. Gould and L. C. Hsu, "Some new inverse series relations", *Duke Math. J.* **40** (1973), 885–891.
22. R. A. Gustafson, "A Whipple's transformation for hypergeometric series in $U(n)$ and multivariable orthogonal polynomials", *SIAM J. Math. Anal.* **18**, (1987), 495–530.
23. R. A. Gustafson, "The Macdonald identities for affine root systems of classical type and hypergeometric series very well-poised on semi-simple Lie algebras", *in* Ramanujan International Symposium on Analysis (Dec. 26th to 28th, 1987, Pune, India), N. K. Thakare (ed.) (1989) 187–224.
24. W. J. Holman III, "Summation theorems for hypergeometric series in $U(n)$", *SIAM J. Math. Anal.* **II** (1980), 523–532.
25. J. Horn, "Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen", *Math. Ann.* **34** (1889), 544–600.
26. F. H. Jackson, "A $q$-generalization of Abel's series", *Rend. Circ. Math. Palermo* **29** (1910), 340–346.
27. W. P. Johnson, "$q$-Extensions of identities of Abel–Rothe type", *Discrete Math.* **159** (1996), 161–177.
28. P. W. Karlsson and H. M. Srivastava, *Multiple Gaussian hypergeometric series*, Ellis Horwood Series: Mathematics and Its Applications, Halsted Press, J. Wiley & Sons, New York, 1985.
29. C. Krattenthaler, "A new matrix inverse", *Proc. Amer. Math. Soc.* **124** (1996), 47–59.
30. C. Krattenthaler and M. Schlosser, "A new multidimensional matrix inverse with applications to multiple $q$-series", to appear in *Discrete Math.*, spec. vol. in honor of H. W. Gould.
31. G. M. Lilly and S. C. Milne, "The $C_l$ Bailey Transform and Bailey Lemma", *Constr. Approx.* **9** (1993), 473–500.
32. S. C. Milne, "A $q$-analog of hypergeometric series very-well-poised in $SU(n)$ and invariant $G$-functions", *Adv. in Math.* **58** (1985), 1–60.
33. S. C. Milne, "A $q$-analog of the ${}_5F_4(1)$ summation theorem for hypergeometric series well-poised in $SU(n)$", *Adv. in Math.* **57** (1985), 14–33.
34. S. C. Milne, "An elementary proof of the Macdonald identities for $A_\ell^{(1)}$", *Adv. in Math.* **57** (1985), 34–70.
35. S. C. Milne, "A $q$-analog of the Gauss summation theorem for hypergeometric series in $U(n)$", *Adv. in Math.* **72** (1988), 59–131.
36. S. C. Milne, "A $q$-analog of the balanced ${}_3F_2$ summation theorem for hypergeometric series in $U(n)$", *Adv. in Math.* **99** (1993), 162–237.
37. S. C. Milne, "Balanced ${}_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series", *Adv. in Math.* **131** (1997), 93–187.
38. S. C. Milne and G. M. Lilly, "Consequences of the $A_l$ and $C_l$ Bailey Transform and Bailey Lemma", *Discrete Math.* **139** (1995), 319–346.
39. M. Rahman, "Some cubic summation formulas for basic hypergeometric series", *Utilitas Math.* **36** (1989), 161–172.
40. M. Rahman, "Some quadratic and cubic summation formulas for basic hypergeometric series", *Can. J. Math.* **45** (1993), 394–411.
41. J. Riordan, *Combinatorial identities*, J. Wiley, New York, 1968.

42. M. Schlosser, "Multidimensional matrix inversions and $A_r$ and $D_r$ basic hypergeometric series", *The Ramanujan J.* **1** (1997), 243–274.
43. D. Stanton, "An elementary approach to the Macdonald identities", *in* $q$-Series and partitions (D. Stanton, ed.), The IMA volumes in mathematics and its applications, vol. 18, Springer–Verlag, 1989, 139–150.
44. V. Strehl, "Identities of Rothe–Abel–Schläfli–Hurwitz-type", *Discrete Math.* **99** (1992), 321–340.
45. Z. X. Wang and D. R. Guo, *Special functions*, World Scientific, Singapore, 1989.