A MULTIDIMENSIONAL GENERALIZATION OF SHUKLA'S ${}_8\psi_8$ SUMMATION

MICHAEL SCHLOSSER

Abstract. We give an r-dimensional generalization of H. S. Shukla's very-well-poised ${}_8\psi_8$ summation formula. We work in the setting of multiple basic hypergeometric series very-well-poised over the root system A_{r-1}, or equivalently, the unitary group $U(r)$. Our proof, which is already new in the one-dimensional case, utilizes an A_{r-1} nonterminating very-well-poised ${}_6\phi_5$ summation by S. C. Milne, a partial fraction decomposition, and analytic continuation.

1. Introduction

In this article, we provide a multidimensional generalization of H. S. Shukla's [23, Eq. (4.1)] very-well-poised ${}_8\psi_8$ summation. In the classical (one-dimensional) case, it reads as follows (cf. H. Exton [7, Eq. (3.8.1.2) or (A.28)]):

$$
\begin{aligned}
(1.1)\quad {}_8\psi_8&\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, aq^2/c, e, f, g \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, c/q, aq/e, aq/f, aq/g \end{matrix}; q, \frac{a^2}{befg}\right] \\
&= \left(1 - \frac{(1-be/a)(1-bf/a)(1-bg/a)}{(1-bq/c)(1-bc/aq)(1-befg/a^2)}\right) \frac{(1-c/bq)(1-bc/aq)}{(1-c/aq)(1-c/q)} \\
&\qquad \times \frac{(q, aq, q/a, aq/be, aq/bf, aq/bg, aq/ef, aq/eg, aq/fg; q)_\infty}{(aq/b, aq/e, aq/f, aq/g, q/b, q/e, q/f, q/g, a^2q/befg; q)_\infty},
\end{aligned}
$$

provided $|q| < 1$ and $|a^2/befg| < 1$.

Here, and throughout the article, we are using the following notations. For a complex number q with $|q| < 1$, the *q-shifted factorial* is defined by

$$
(1.2)\qquad (a; q)_k := \prod_{j=0}^{\infty} \frac{(1-aq^j)}{(1-aq^{j+k})}, \quad \text{and} \quad (a_1, \dots, a_m; q)_k := (a_1; q)_k \dots (a_m; q)_k,
$$

where k is an integer or infinity. Further, we use

$$
(1.3)\qquad {}_t\phi_{t-1}\left[\begin{matrix} a_1, a_2, \dots, a_t \\ b_1, b_2, \dots, b_{t-1} \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_t; q)_k}{(q, b_1, \dots, b_{t-1}; q)_k} z^k,
$$

Date: first version: March 5, 2001; revised version: June 13, 2001.

2000 *Mathematics Subject Classification.* Primary 33D15; Secondary 33D67.

Key words and phrases. basic hypergeometric series, q-series, bilateral series, ${}_8\psi_8$ summation, ${}_6\phi_5$ summation, A_r series, $U(r)$ series.

and

$$(1.4)\qquad {}_t\psi_t\left[\begin{matrix}a_1,a_2,\dots,a_t\\ b_1,b_2,\dots,b_t\end{matrix};q,z\right]:=\sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\dots,a_t;q)_k}{(b_1,b_2,\dots,b_t;q)_k}z^k,$$

to denote the *basic hypergeometric* ${}_t\phi_{t-1}$ *series*, and *bilateral basic hypergeometric* ${}_t\psi_t$ *series*, respectively. For a survey of classical results in the theory of basic hypergeometric series, see G. Gasper and M. Rahman [8]. For applications of basic hypergeometric series to various areas, including number theory, combinatorics, and physics, see G. E. Andrews [1, 2].

The ${}_8\psi_8$ summation in (1.1) generalizes W. N. Bailey's [4, Eq. (4.7)] very-well-poised ${}_6\psi_6$ summation, to which it reduces for $c\to 0$.

By an elementary computation it follows that H. S. Shukla's ${}_8\psi_8$ summation can also be written in the following eqivalent form:

$$(1.5)\quad {}_8\psi_8\left[\begin{matrix}q\sqrt{a},-q\sqrt{a},b,c,aq^2/c,e,f,g\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,c/q,aq/e,aq/f,aq/g\end{matrix};q,\frac{a^2}{befg}\right]$$
$$=\left(1-\frac{(1-a/bg)(1-cf/aq)(1-ce/aq)}{(1-c/gq)(1-ef/a)(1-c/bq)}\right)\frac{(1-c/bq)(1-c/gq)}{(1-c/aq)(1-c/q)}$$
$$\times\frac{(q,aq,q/a,aq/be,aq/bf,aq/bg,a/ef,aq/eg,aq/fg;q)_\infty}{(aq/b,aq/e,aq/f,aq/g,q/b,q/e,q/f,q/g,a^2/befg;q)_\infty},$$

provided $|q|<1$ and $|a^2/befg|<1$. In Section 3, we provide a multidimensional generalization of (1.5), see Theorem 3.4.

H. S. Shukla [23] derived (1.1) by specializing a transformation of a very-well-poised ${}_8\psi_8$ into a sum of three balanced ${}_4\phi_3$ series due to M. Jackson [13, Eq. (3.1)]. Unlike H. S. Shukla [23], we give a proof of the ${}_8\psi_8$ summation formula (1.5) using a weaker result, namely L. J. Rogers' [19] ${}_6\phi_5$ summation, which is a special case of (1.5). As further ingredients in our derivation of (1.5) we utilize a simple decomposition identity, see Eq. (2.4), and an application of M. E. H. Ismail's [12] analytic continuation argument. We display our proof of (1.5) in Section 2.

Surprisingly, the whole analysis carries over to the multivariate case, in the setting of *multiple basic hypergeometric series very-well-poised over the root-system* A_{r-1}. In fact, the main achievement of this article is an A_{r-1} generalization of H. S. Shukla's ${}_8\psi_8$ summation (1.5), see Theorem 3.4.

Our article is organized as follows. In Section 2, after briefly explaining some basic concepts which we need from the theory of basic hypergeometric series [8], we give a new proof of H. S. Shukla's ${}_8\psi_8$ summation. The one-dimensional analysis in Section 2 turns out to be very much motivated by the multivariate case. In Section 3, after some preparations, we state and prove our A_{r-1} generalization of H. S. Shukla's very-well-poised ${}_8\psi_8$ summation theorem. Our A_{r-1} ${}_8\psi_8$ summation in Theorem 3.4 includes R. A. Gustafson's [10] A_{r-1} ${}_6\psi_6$ summation as a special case. Our proof utilizes an A_{r-1} nonterminating very-well-poised $_6\phi_5$ summation by S. C. Milne [15], a partial fraction decomposition (see Lemma 3.2), and analytic continuation.

2. Some basic concepts and proof of the $_8\psi_8$ summation in one-dimension

2.1. **Some basic concepts.** We first recall some basic concepts from the theory of basic hypergeometric series (cf. G. Gasper and M. Rahman [8]).

By definition (1.2) it is clear that $(q;q)_k^{-1} = 0$, for $k = -1, -2, \ldots$. More generally, $(q^{1+m};q)_k^{-1} = 0$ for $k = -m-1, -m-2, \ldots$. Thus, using the definitions (1.3) and (1.4), a bilateral $_t\psi_t$ series becomes a unilateral $_t\phi_{t-1}$ series if one of the lower parameters, say b_t, is q (or more generally, q^j where j is a positive integer). In this case, the $_t\psi_t$ series terminates naturally from below. Similarly, a $_t\phi_{t-1}$ series terminates naturally from above if one of the upper parameters, say a_t, equals q^{-n}, $n = 0, 1, 2, \ldots$.

The ratio test gives simple criteria of when the series in (1.3) and (1.4) converge, if they do not terminate. Remember that we assume $|q| < 1$. The $_t\phi_{t-1}$ series in (1.3) converges absolutely in the radius $|z| < 1$, while the $_t\psi_t$ series in (1.4) converges absolutely in the annulus $|b_1 \ldots b_t/a_1 \ldots a_t| < |z| < 1$.

The classical theory of basic hypergeometric series consists of several summation and transformation formulae involving $_t\phi_{t-1}$ or $_t\psi_t$ series. Some classical summation theorems for these series require that the parameters satisfy the condition of being very-well-poised. A $_t\phi_{t-1}$ basic hypergeometric series is called *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_t b_{t-1}$. It is called *very-well-poised* if it is well-poised and if $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$. Note that the factor

$$\frac{(q\sqrt{a_1}, -q\sqrt{a_1};q)_k}{(\sqrt{a_1}, -\sqrt{a_1};q)_k} = \frac{1 - a_1 q^{2k}}{1 - a_1} \tag{2.1}$$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series, and we call (2.1) the *very-well-poised term* of the series. Similarly, a bilateral $_t\psi_t$ basic hypergeometric series is well-poised if $a_1 b_1 = a_2 b_2 \cdots = a_t b_t$ and very-well-poised if, in addition, $a_1 = -a_2 = q b_1 = -q b_2$.

In our subsequent computations (in this section and in Section 3), we make heavy use of some elementary identities involving q-shifted factorials, listed in G. Gasper and M. Rahman [8, Appendix I].

2.2. **Proof of the $_8\psi_8$ summation.** The main ingredient in our derivation of H. S. Shukla's $_8\psi_8$ summation (1.5) is L. J. Rogers' [19, p. 29, second eq.] nonterminating very-well-poised $_6\phi_5$ summation:

$$ {}_6\phi_5\left[\begin{matrix} a,\ q\sqrt{a}, -q\sqrt{a}, b, c, d \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix}; q, \frac{aq}{bcd}\right] = \frac{(aq, aq/bc, aq/bd, aq/cd;q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd;q)_\infty}, \tag{2.2}$$

provided $|q|<1$ and $|aq/bcd|<1$. Note that (2.2) is equivalent to the special case $b\to a$, $c\to 0$ of (1.5).

We derive the ${}_8\psi_8$ summation (1.5) in two steps. In the first step, we establish the $b\to a$ case of (1.5) by using L. J. Rogers' ${}_6\phi_5$ summation (2.2) twice, i.e., we first establish the unilateral summation

$$
{}_8\phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},c,aq^2/c,e,f,g\\ \sqrt{a},-\sqrt{a},aq/c,c/q,aq/e,aq/f,aq/g\end{matrix};q,\frac{a}{efg}\right]
= \left(1-\frac{(1-1/g)(1-cf/aq)(1-ce/aq)}{(1-c/gq)(1-ef/a)(1-c/aq)}\right)\frac{(1-c/gq)}{(1-c/q)}
\times\frac{(aq,a/ef,aq/eg,aq/fg;q)_\infty}{(aq/e,aq/f,aq/g,a/efg;q)_\infty}, \tag{2.3}
$$

where $|q|<1$ and $|a/efg|<1$. In the second step, we extend (2.3) to (1.5) by analytic continuation, or equivalently, by an application of M. E. H. Ismail's [12] argument.

The details of the first step are as follows. Since

$$
\frac{(1-cq^{k-1})(1-aq^{k+1}/c)}{(1-c/q)(1-aq/c)} = q^k + \frac{(1-aq^k)(1-q^k)}{(1-c/q)(1-aq/c)}, \tag{2.4}
$$

we have

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},c,aq^2/c,e,f,g\\ \sqrt{a},-\sqrt{a},aq/c,c/q,aq/e,aq/f,aq/g\end{matrix};q,\frac{a}{efg}\right]\\
&= \sum_{k=0}^\infty \frac{(1-aq^{2k})}{(1-a)}\frac{(a,e,f,g;q)_k}{(q,aq/e,aq/f,aq/g;q)_k}\left(\frac{a}{efg}\right)^k\frac{(1-cq^{k-1})(1-aq^{k+1}/c)}{(1-c/q)(1-aq/c)}\\
&= \sum_{k=0}^\infty \frac{(1-aq^{2k})}{(1-a)}\frac{(a,e,f,g;q)_k}{(q,aq/e,aq/f,aq/g;q)_k}\left(\frac{aq}{efg}\right)^k\\
&\quad+\sum_{k=0}^\infty \frac{(1-aq^{2k})}{(1-a)}\frac{(a,e,f,g;q)_k}{(q,aq/e,aq/f,aq/g;q)_k}\left(\frac{a}{efg}\right)^k\frac{(1-aq^k)(1-q^k)}{(1-c/q)(1-aq/c)}.
\end{aligned}
$$

Now in the second sum, because of the factor $(1-q^k)$ in the numerator of the summand, we shift the index $k\mapsto k+1$. We then obtain

$$
\begin{aligned}
&\sum_{k=0}^\infty \frac{(1-aq^{2k})}{(1-a)}\frac{(a,e,f,g;q)_k}{(q,aq/e,aq/f,aq/g;q)_k}\left(\frac{aq}{efg}\right)^k\\
&\quad+\frac{a(1-aq)(1-aq^2)(1-e)(1-f)(1-g)}{efg(1-c/q)(1-aq/c)(1-aq/e)(1-aq/f)(1-aq/g)}
\end{aligned}
$$

$$\times \sum_{k=0}^{\infty} \frac{(1-aq^{2+2k})}{(1-aq^2)} \frac{(aq^2,eq,fq,gq;q)_k}{(q,aq^2/e,aq^2/f,aq^2/g;q)_k} \left(\frac{a}{efg}\right)^k.$$

Next, we simplify both sums by the ${}_6\phi_5$ summation in (2.2) and obtain

$$\begin{aligned}
(2.5)\quad & \frac{(aq,aq/ef,aq/eg,aq/fg;q)_\infty}{(aq/e,aq/f,aq/g,aq/efg;q)_\infty} \\
&\quad + \frac{a(1-aq)(1-aq^2)(1-e)(1-f)(1-g)}{efg(1-c/q)(1-aq/c)(1-aq/e)(1-aq/f)(1-aq/g)} \\
&\qquad \times \frac{(aq^3,aq/ef,aq/eg,aq/fg;q)_\infty}{(aq^2/e,aq^2/f,aq^2/g,a/efg;q)_\infty} \\
&= \left(1-\frac{(1-e)(1-f)(1-g)}{(1-c/q)(1-aq/c)(1-efg/a)}\right) \frac{(aq,aq/ef,aq/eg,aq/fg;q)_\infty}{(aq/e,aq/f,aq/g,aq/efg;q)_\infty}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(1-\frac{(1-e)(1-f)(1-g)}{(1-c/q)(1-aq/c)(1-efg/a)}\right) \frac{(1-a/efg)}{(1-a/ef)} \\
&\qquad = \left(1-\frac{(1-1/g)(1-cf/aq)(1-ce/aq)}{(1-c/gq)(1-ef/a)(1-c/aq)}\right) \frac{(1-c/gq)}{(1-c/q)}
\end{aligned}$$

(as can be readily checked by using a symbolic computer algebra program such as *Maple* or *Mathematica*), the last expression in (2.5) is equivalent to

$$\left(1-\frac{(1-1/g)(1-cf/aq)(1-ce/aq)}{(1-c/gq)(1-ef/a)(1-c/aq)}\right) \frac{(1-c/gq)}{(1-c/q)} \frac{(aq,a/ef,aq/eg,aq/fg;q)_\infty}{(aq/e,aq/f,aq/g,a/efg;q)_\infty},$$

which is the right side of (2.3).

Having established (2.3), we are now ready to proceed with the second step, where we extend the unilateral summation (2.3) to the bilateral (1.5) by analytic continuation, by a method commonly referred to as "Ismail's argument" (see M. E. H. Ismail [12], and R. Askey and M. E. H. Ismail [3]). This works as follows: Both sides of the identity in (1.5) are analytic in b^{-1} in a domain around the origin. Now, the identity is true for $b = aq^{-m}$, for all $m = 0,1,2,\ldots$, by the ${}_8\phi_7$ summation in (2.3) (see below for the details). Since $\lim_{m\to\infty} q^m/a = 0$ is an interior point in the domain of analyticity of b^{-1}, by the identity theorem of analytic functions, we establish the identity (1.5) for b^{-1} throughout the whole domain. Finally, by analytic continuation we establish the identity (1.5) to be valid for $|b^{-1}| < |efg/a^2|$, the region of convergence of the series.

We still need to show that the identity (1.5) is true when $b = aq^{-m}$. In this case, the left side of (1.5) is

$$\sum_{k=-m}^{\infty} \frac{(1-aq^{2k})}{(1-a)} \frac{(aq^{-m}, c, aq^2/c, e, f, g; q)_k}{(q^{1+m}, aq/c, c/q, aq/e, aq/f, aq/g; q)_k} \left(\frac{aq^m}{efg}\right)^k. \tag{2.6}$$

We shift the summation index in (2.6) by $k \mapsto k-m$ and obtain

$$\frac{(1-aq^{-2m})}{(1-a)} \frac{(aq^{-m}, c, aq^2/c, e, f, g; q)_{-m}}{(q^{1+m}, aq/c, c/q, aq/e, aq/f, aq/g; q)_{-m}} \left(\frac{aq^m}{efg}\right)^{-m}$$
$$\times \sum_{k=0}^{\infty} \frac{(1-aq^{-2m+2k})(aq^{-2m}, cq^{-m}, aq^{2-m}/c, eq^{-m}, fq^{-m}, gq^{-m}; q)_k}{(1-a^{-2m})(q, aq^{1-m}/c, cq^{-m-1}, aq^{1-m}/e, aq^{1-m}/f, aq^{1-m}/g; q)_k} \left(\frac{aq^m}{efg}\right)^k.$$

Now we apply the $a \mapsto aq^{-2m}$, $c \mapsto cq^{-m}$, $e \mapsto eq^{-m}$, $f \mapsto fq^{-m}$, and $g \mapsto gq^{-m}$ case of the summation formula in (2.3) to simplify this expression to

$$\frac{(1-aq^{-2m})}{(1-a)} \frac{(aq^{-m}, c, aq^2/c, e, f, g; q)_{-m}}{(q^{1+m}, aq/c, c/q, aq/e, aq/f, aq/g; q)_{-m}} \left(\frac{aq^m}{efg}\right)^{-m}$$
$$\times \left(1 - \frac{(1-q^m/g)(1-cf/aq)(1-ce/aq)}{(1-c/gq)(1-ef/a)(1-cq^m/aq)}\right) \frac{(1-c/gq)}{(1-cq^{-m-1})}$$
$$\times \frac{(aq^{1-2m}, a/ef, aq/eg, aq/fg; q)_\infty}{(aq^{1-m}/e, aq^{1-m}/f, aq^{1-m}/g, aq^m/efg; q)_\infty}.$$

Now, this can easily be further transformed into

$$\left(1 - \frac{(1-q^m/g)(1-cf/aq)(1-ce/aq)}{(1-c/gq)(1-ef/a)(1-cq^m/aq)}\right) \frac{(1-cq^m/aq)(1-c/gq)}{(1-c/aq)(1-c/q)}$$
$$\times \frac{(q, aq, q/a, q^{1+m}/e, q^{1+m}/f, q^{1+m}/g, a/ef, aq/eg, aq/fg; q)_\infty}{(q^{1+m}, aq/e, aq/f, aq/g, q^{1+m}/a, q/e, q/f, q/g, aq^m/efg; q)_\infty},$$

which is exactly the $b = aq^{-m}$ case of the right side of (1.5). $\square$

A natural question that arises is: Why did we prove the ${}_8\psi_8$ summation (1.5) in two steps, by first establishing a unilateral summation and then applying Ismail's argument [12] to it? A conceptual expanation is that the presence of the contiguous factor

$$\frac{(1-cq^{k-1})(1-aq^{k+1}/c)}{(1-c/q)(1-aq/c)}$$

in the summand of the ${}_8\psi_8$ series led us to seek a decomposition, after which the ${}_8\psi_8$ series could be summed by two applications of the ${}_6\psi_6$ summation. Using the decomposition in (2.4), the first series can be indeed summed by the ${}_6\psi_6$ summation. However, for the second series, because of the appearance of the factor $(1-q^k)$ on the right side of (2.4), we are forced to consider the unilateral specialization of the series to be able to apply the ${}_6\psi_6$ (or shifted ${}_6\phi_5$) summation. Nevertheless, in the end Ismail's argument allows us to extend the unilateral result (2.3) to the bilateral identity (1.5).

A very similar analysis holds in the multidimensional setting, regarding multiple basic hypergeometric series very-well-poised over the root-system A_{r-1}.

3. An A_{r-1} very-well-poised $_8\psi_8$ summation formula

3.1. **Preliminaries on A_{r-1} basic hypergeometric series.** We consider multiple series of the form

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} S(\mathbf{k}), \tag{3.1}$$

where $\mathbf{k}=(k_1,\dots,k_r)$, which reduce to classical (bilateral) basic hypergeometric series when $r=1$. We call such a multiple basic hypergeometric series *well-poised* if it reduces to a well-poised series when $r=1$. *Very-well-poised* multiple basic hypergeometric series are defined analogously. In case these series do not terminate from below, we also call such series *multilateral* basic hypergeometric series.

In our particular cases, we also have

$$\prod_{1\le i<j<r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right) \tag{3.2}$$

as a factor of $S(\mathbf{k})$. A typical example is the right side of (3.3). Since we may associate (3.2) with the product side of the Weyl denominator formula for the root system A_{r-1} (see e.g. D. Stanton [24]), we call our series A_{r-1} basic hypergeometric series, in accordance with I. M. Gessel and C. Krattenthaler [9, Eq. (7.1)]. Very often these series are also called $U(r)$ basic hypergeometric series, where $U(r)$ is the unitary group. For some selected results in the theory of A_{r-1} basic hypergeometric series, see the references [5, 6, 9, 10, 11, 14, 15, 16, 17, 18, 20, 21, 22].

For convenience, we frequently use the notation $|\mathbf{k}|:=k_1+\dots+k_r$. Furthermore, we often use capital letters to abbreviate the (r-fold) products of certain variables. Specifically, in this article we use $B:=b_1\cdots b_r$, $E:=e_1\cdots e_r$, and $Y:=y_1\cdots y_r$, respectively.

Since multidimensional $_6\phi_5$ series play a significant role in our derivation of our A_{r-1} $_8\psi_8$ summation, we find it useful to make the following definition. Let a, $b_1,\dots,b_r$, c, d, $z_1,\dots,z_r$, and w be indeterminate. We define for $r\ge 1$,

$$\begin{aligned}
&{}_6\Phi_5^{(r)}\left[a;b_1,\dots,b_r;c,d;z_1,\dots,z_r\,\middle|\,q,w\right]\\
&:=\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^{r}\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)\prod_{i,j=1}^{r}\frac{(b_jz_i/z_j;q)_{k_i}}{(qz_i/z_j;q)_{k_i}}\right.
\end{aligned} \tag{3.3}$$

$$\times \prod_{i=1}^r \frac{(az_i;q)_{|\mathbf{k}|}\,(cz_i;q)_{k_i}}{(az_iq/b_i;q)_{|\mathbf{k}|}\,(az_iq/d;q)_{k_i}} \cdot \frac{(d;q)_{|\mathbf{k}|}}{(aq/c;q)_{|\mathbf{k}|}}\, w^{|\mathbf{k}|}\Bigg).$$

The above ${}_6\Phi_5^{(r)}$ series is an r-dimensional ${}_6\phi_5$ series (which reduces to a classical very-well-posied ${}_6\phi_5$ when $r=1$).

In our proof of Theorem 3.4, or more precisely, of the intermediate Proposition 3.5, we utilize S. C. Milne's [15, Theorem 1.44] A_{r-1} extension of L. J. Rogers' ${}_6\phi_5$ summation theorem.

Theorem 3.1 ((Milne) An A_{r-1} nonterminating very-well-poised ${}_6\phi_5$ summation)**.** *Let a, $b_1,\dots,b_r$, c, d, and $z_1,\dots,z_r$, be indeterminate, let $B := b_1\cdots b_r$, $r\ge 1$, and suppose that none of the denominators in* (3.4) *vanishes. Then*

$$(3.4)\quad {}_6\Phi_5^{(r)}\left[a;b_1,\dots,b_r;c,d;z_1,\dots,z_r\,\middle|\,q,\frac{aq}{Bcd}\right] = \frac{(aq/Bc,aq/cd;q)_\infty}{(aq/Bcd,aq/c;q)_\infty}\prod_{i=1}^r \frac{(az_iq,az_iq/b_id;q)_\infty}{(az_iq/d,az_iq/b_i;q)_\infty},$$

provided $|q|<1$ and $|aq/Bcd|<1$.

The $r=1$ case of (3.4) clearly reduces to (2.2).

Further, we make use of the following (q-analogue of the) *partial fraction decomposition*

$$(3.5)\qquad \prod_{i=1}^r \frac{(1-tz_iy_i)}{(1-tz_i)} = y_1y_2\dots y_r + \sum_{l=1}^r \frac{\prod_{i=1}^r(1-y_iz_i/z_l)}{(1-tz_l)\prod_{i=1,\, i\neq l}^r(1-z_i/z_l)}$$

(see [16, Appendix]). In particular, for a multivariate extension of (2.4), we utilize the following extension of (3.5):

Lemma 3.2. *Let $Y := y_1y_2\dots y_r$. Then*

$$\frac{(1-uY)}{(1-u)}\prod_{i=1}^r \frac{(1-tz_iy_i)}{(1-tz_i)} = Y + \sum_{l=1}^r \frac{(1-uYtz_l)\prod_{i=1}^r(1-y_iz_i/z_l)}{(1-u)(1-tz_l)\prod_{i=1,\, i\neq l}^r(1-z_i/z_l)}.$$

Proof. Since

$$1-Y = \sum_{l=1}^r \frac{\prod_{i=1}^r(1-y_iz_i/z_l)}{\prod_{i=1,\, i\neq l}^r(1-z_i/z_l)},$$

by the $t=0$ case of (3.5), we have

$$\frac{(1-uY)}{(1-u)}\prod_{i=1}^r \frac{(1-tz_iy_i)}{(1-tz_i)}$$

$$= \left(1 + \frac{(1-Y)}{(1-u)} u\right) \left(Y + \sum_{l=1}^{r} \frac{\prod_{i=1}^{r} (1 - y_i z_i / z_l)}{(1 - t z_l) \prod_{\substack{i=1 \\ i \neq l}}^{r} (1 - z_i / z_l)} \right)$$

$$= Y + \sum_{l=1}^{r} \left(\frac{(1 - u + (1 - t z_l) u Y + (1 - Y) u)}{(1-u)} \cdot \frac{\prod_{i=1}^{r} (1 - y_i z_i / z_l)}{(1 - t z_l) \prod_{\substack{i=1 \\ i \neq l}}^{r} (1 - z_i / z_l)} \right)$$

$$= Y + \sum_{l=1}^{r} \frac{(1 - u Y t z_l) \prod_{i=1}^{r} (1 - y_i z_i / z_l)}{(1-u)(1 - t z_l) \prod_{\substack{i=1 \\ i \neq l}}^{r} (1 - z_i / z_l)}.$$

□

Remark 3.3. In the multivariate analysis of our proof of Theorem 3.4, the partial fraction decomposition of Lemma 3.2 plays a crucial role. Applications of partial fraction decompositions have often proved to be useful in the derivation of results for A_{r-1} series, see e.g. [5, 11, 14, 16, 20, 21].

After these preparations, we are ready to state and prove our multiple extension of (1.5).

3.2. **The main result.** Our r-dimensional generalization of H. S. Shukla's [23, Eq. (4.1)] very-well-poised ${}_8\psi_8$ summation is as follows:

Theorem 3.4 (An A_{r-1} very-well-poised ${}_8\psi_8$ summation)**.** *Let a, $b_1, \dots, b_r$, c, $e_1, \dots, e_r$, f, g, and $z_1, \dots, z_r$ be indeterminate, let $B := b_1 \cdots b_r$, $E := e_1 \cdots e_r$, $r \geq 1$, and suppose that none of the denominators in* (3.6) *vanishes. Then*

$$\sum_{k_1, \dots, k_r = -\infty}^{\infty} \left(\prod_{1 \leq i < j \leq r} \left(\frac{z_i q^{k_i} - z_j q^{k_j}}{z_i - z_j} \right) \prod_{i=1}^{r} \left(\frac{1 - a z_i q^{k_i + |\mathbf{k}|}}{1 - a z_i} \right) \right.$$
$$\times \prod_{i,j=1}^{r} \frac{(e_j z_i / z_j; q)_{k_i}}{(a z_i q / b_j z_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(b_i z_i; q)_{|\mathbf{k}|} (f z_i; q)_{k_i}}{(a z_i q / e_i; q)_{|\mathbf{k}|} (a z_i q / g; q)_{k_i}} \cdot \frac{(g; q)_{|\mathbf{k}|}}{(aq/f; q)_{|\mathbf{k}|}}$$
$$\left. \times \frac{(1 - c q^{|\mathbf{k}| - 1})}{(1 - c/q)} \prod_{i=1}^{r} \frac{(1 - a z_i q^{k_i + 1} / c)}{(1 - a z_i q / c)} \cdot \left(\frac{a^{r+1}}{B E f g} \right)^{|\mathbf{k}|} \right) \tag{3.6}$$
$$= \left(1 - \frac{(1 - a^r / B g)(1 - c f / a q)}{(1 - c / g q)(1 - E f / a)} \prod_{i=1}^{r} \frac{(1 - c e_i / a z_i q)}{(1 - c / b_i z_i q)} \right) \prod_{i=1}^{r} \frac{(1 - c / b_i z_i q)}{(1 - c / a z_i q)}$$
$$\times \frac{(1 - c / g q)}{(1 - c/q)} \frac{(a / E f, a q / f g, a^r q / B g; q)_\infty}{(a^{r+1} / B E f g, a q / f, q / g; q)_\infty} \prod_{i,j=1}^{r} \frac{(q z_i / z_j, a z_i q / b_j e_i z_j; q)_\infty}{(a z_i q / b_j z_j, z_i q / e_i z_j; q)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(a z_i q, q / a z_i, a z_i q / e_i g, a q / b_i f z_i; q)_\infty}{(a z_i q / e_i, a z_i q / g, q / b_i z_i, q / f z_i; q)_\infty},$$

provided $|q|<1$ *and* $|a^{r+1}/BEfg|<1$.

Theorem 3.4 generalizes R. A. Gustafson's [10, Theorem 1.15] A_{r-1} ${}_6\psi_6$ summation, to which it reduces for $c\to 0$.

Following closely the univariate analysis of Section 2, we prove Theorem 3.4 in two steps. First, we prove the $b_i=a$, $i=1,\dots,r$, special case of Theorem 3.4, which is Proposition 3.5 below. Then we extend Proposition 3.5 to Theorem 3.4 by an r-fold application of M. E. H. Ismail's [12] analytic continuation argument.

Proposition 3.5 (An A_{r-1} nonterminating very-well-poised ${}_8\phi_7$ summation)**.** *Let* a, c, $e_1,\dots,e_r$, f, g, *and* $z_1,\dots,z_r$ *be indeterminate, let* $E:=e_1\cdots e_r$, $r\ge 1$, *and suppose that none of the denominators in* (3.7) *vanishes. Then*

$$\begin{aligned}(3.7)\qquad &\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^{r}\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)\\ &\quad\times\prod_{i,j=1}^{r}\frac{(e_jz_i/z_j;q)_{k_i}}{(qz_i/z_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(az_i;q)_{|\mathbf{k}|}\,(fz_i;q)_{k_i}}{(az_iq/e_i;q)_{|\mathbf{k}|}\,(az_iq/g;q)_{k_i}}\cdot\frac{(g;q)_{|\mathbf{k}|}}{(aq/f;q)_{|\mathbf{k}|}}\\ &\quad\times\frac{(1-cq^{|\mathbf{k}|-1})}{(1-c/q)}\prod_{i=1}^{r}\frac{(1-az_iq^{k_i+1}/c)}{(1-az_iq/c)}\cdot\left(\frac{a}{Efg}\right)^{|\mathbf{k}|}\Bigg)\\ &=\left(1-\frac{(1-1/g)(1-cf/aq)}{(1-c/gq)(1-Ef/a)}\prod_{i=1}^{r}\frac{(1-ce_i/az_iq)}{(1-c/az_iq)}\right)\frac{(1-c/gq)}{(1-c/q)}\\ &\quad\times\frac{(a/Ef,aq/fg;q)_\infty}{(a/Efg,aq/f;q)_\infty}\prod_{i=1}^{r}\frac{(az_iq,az_iq/e_ig;q)_\infty}{(az_iq/e_i,az_iq/g;q)_\infty},\end{aligned}$$

provided $|q|<1$ *and* $|a/Efg|<1$.

Proposition 3.5 generalizes S. C. Milne's A_{r-1} ${}_6\phi_5$ summation in Theorem 3.1, to which it reduces for $c=0$.

Proof of Proposition 3.5. Since

$$\begin{aligned}&\frac{(1-cq^{|\mathbf{k}|-1})}{(1-c/q)}\prod_{i=1}^{r}\frac{(1-az_iq^{k_i+1}/c)}{(1-az_iq/c)}\\ &\qquad=q^{|\mathbf{k}|}+\sum_{l=1}^{r}\frac{(1-az_lq^{|\mathbf{k}|})\prod_{i=1}^{r}(1-q^{k_i}z_i/z_l)}{(1-c/q)(1-az_lq/c)\prod_{i=1,\,i\neq l}^{r}(1-z_i/z_l)},\end{aligned}$$

by the $t\mapsto aq/c$, $u\mapsto c/q$, $y_i\mapsto q^{k_i}$, $i=1,\dots,r$, case of Lemma 3.2, we have

$$\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^{r}\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)$$

$$\times \prod_{i,j=1}^{r} \frac{(e_j z_i/z_j;q)_{k_i}}{(qz_i/z_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(az_i;q)_{|\mathbf{k}|}\,(fz_i;q)_{k_i}}{(az_iq/e_i;q)_{|\mathbf{k}|}\,(az_iq/g;q)_{k_i}} \cdot \frac{(g;q)_{|\mathbf{k}|}}{(aq/f;q)_{|\mathbf{k}|}}$$

$$\times \left(\frac{a}{Efg}\right)^{|\mathbf{k}|} \frac{(1-cq^{|\mathbf{k}|-1})}{(1-c/q)} \prod_{i=1}^{r} \frac{(1-az_iq^{k_i+1}/c)}{(1-az_iq/c)}\Bigg)$$

$$= \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right) \prod_{i=1}^{r} \left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right) \prod_{i,j=1}^{r} \frac{(e_jz_i/z_j;q)_{k_i}}{(qz_i/z_j;q)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(az_i;q)_{|\mathbf{k}|}\,(fz_i;q)_{k_i}}{(az_iq/e_i;q)_{|\mathbf{k}|}\,(az_iq/g;q)_{k_i}} \cdot \frac{(g;q)_{|\mathbf{k}|}}{(aq/f;q)_{|\mathbf{k}|}} \left(\frac{aq}{Efg}\right)^{|\mathbf{k}|}\Bigg)$$

$$+ \sum_{l=1}^{r} \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right) \prod_{i=1}^{r} \left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)$$

$$\times \prod_{i,j=1}^{r} \frac{(e_j z_i/z_j;q)_{k_i}}{(qz_i/z_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(az_i;q)_{|\mathbf{k}|}\,(fz_i;q)_{k_i}}{(az_iq/e_i;q)_{|\mathbf{k}|}\,(az_iq/g;q)_{k_i}} \cdot \frac{(g;q)_{|\mathbf{k}|}}{(aq/f;q)_{|\mathbf{k}|}}$$

$$\times \left(\frac{a}{Efg}\right)^{|\mathbf{k}|} \frac{(1-az_lq^{|\mathbf{k}|})\prod_{i=1}^{r}(1-q^{k_i}z_i/z_l)}{(1-c/q)(1-az_lq/c)\prod_{\substack{i=1\\ i\neq l}}^{r}(1-z_i/z_l)}\Bigg).$$

We have arrived at a sum of $1+r$ infinite multisums. In the $(1+l)$th sum, for $l=1,\dots,r$, due to the factor $\prod_{i=1}^{r}(1-q^{k_i}z_i/z_l)$ in the numerator of the summand, we shift the index $k_l \mapsto k_l+1$. We then obtain

$$ {}_6\Phi_5^{(r)}\left[a;e_1,\dots,e_r;f,g;z_1,\dots,z_r \,\middle|\, q,\frac{aq}{Efg}\right]$$

$$+\sum_{l=1}^{r} \frac{a(1-fz_l)(1-g)(1-az_lq^2)}{Efg(1-c/q)(1-az_lq/c)(1-az_lq/g)(1-aq/f)}$$

$$\times \frac{\prod_{i=1}^{r}(1-az_iq)\prod_{i=1}^{r}(1-e_iz_l/z_i)}{\prod_{i=1}^{r}(1-az_iq/e_i)\prod_{\substack{i=1\\ i\neq l}}^{r}(1-z_l/z_i)}$$

$$\times {}_6\Phi_5^{(r)}\Big[aq;e_1,\dots,e_{l-1},e_lq,e_{l+1},\dots,e_r;f,gq;$$

$$z_1,\dots,z_{l-1},z_lq,z_{l+1},\dots,z_r \,\Big|\, q,\frac{a}{Efg}\Big].$$

Next, we simplify all the $1+r$ ${}_6\Phi_5^{(r)}$ series according to Theorem 3.1, and obtain

$$\frac{(aq/Ef,aq/fg;q)_\infty}{(aq/Efg,aq/f;q)_\infty} \prod_{i=1}^{r} \frac{(az_iq,az_iq/e_ig;q)_\infty}{(az_iq/g,az_iq/e_i;q)_\infty} \tag{3.8}$$

$$+\sum_{l=1}^{r}\frac{a(1-fz_l)(1-g)(1-az_lq^2)}{Efg(1-c/q)(1-az_lq/c)(1-az_lq/g)(1-aq/f)}$$
$$\times\frac{\prod_{i=1}^{r}(1-az_iq)\prod_{i=1}^{r}(1-e_iz_l/z_i)}{\prod_{i=1}^{r}(1-az_iq/e_i)\prod_{\substack{i=1\\i\neq l}}^{r}(1-z_l/z_i)}$$
$$\times\frac{(aq/Ef,aq/fg;q)_\infty}{(a/Efg,aq^2/f;q)_\infty}\frac{(1-az_lq/g)}{(1-az_lq^2)}\prod_{i=1}^{r}\frac{(az_iq^2,az_iq/e_ig;q)_\infty}{(az_iq/g,az_iq^2/e_i;q)_\infty}$$
$$=\left(1-\sum_{l=1}^{r}\frac{(1-fz_l)(1-g)\prod_{i=1}^{r}(1-e_iz_l/z_i)}{(1-c/q)(1-az_lq/c)(1-Efg/a)\prod_{\substack{i=1\\i\neq l}}^{r}(1-z_l/z_i)}\right)$$
$$\times\frac{(aq/Ef,aq/fg;q)_\infty}{(aq/Efg,aq/f;q)_\infty}\prod_{i=1}^{r}\frac{(az_iq,az_iq/e_ig;q)_\infty}{(az_iq/g,az_iq/e_i;q)_\infty}.$$

Now we apply the $t\mapsto c/aq$, $u\mapsto aq/cEf$, $y_i\mapsto e_i$, and $z_i\mapsto 1/z_i$, $i=1,\dots,r$, case of Lemma 3.2, which can be rewritten as

$$\sum_{l=1}^{r}\frac{(1-fz_l)\prod_{i=1}^{r}(1-e_iz_l/z_i)}{(1-cEf/aq)(1-az_lq/c)\prod_{\substack{i=1\\i\neq l}}^{r}(1-z_l/z_i)}$$
$$=1-\frac{(1-cf/aq)}{(1-cEf/aq)}\prod_{i=1}^{r}\frac{(1-ce_i/az_iq)}{(1-c/az_iq)},$$

to simplify the expression obtained in (3.8) to

$$\left(1-\frac{(1-g)(1-cEf/aq)}{(1-c/q)(1-Efg/a)}\left(1-\frac{(1-cf/aq)}{(1-cEf/aq)}\prod_{i=1}^{r}\frac{(1-ce_i/az_iq)}{(1-c/az_iq)}\right)\right)$$
$$\times\frac{(aq/Ef,aq/fg;q)_\infty}{(aq/Efg,aq/f;q)_\infty}\prod_{i=1}^{r}\frac{(az_iq,az_iq/e_ig;q)_\infty}{(az_iq/g,az_iq/e_i;q)_\infty}. \tag{3.9}$$

Finally, using

$$1-\frac{(1-g)(1-cEf/aq)}{(1-c/q)(1-Efg/a)}=\frac{(1-c/gq)(1-a/Ef)}{(1-c/q)(1-a/Efg)},$$

we can easily transform the expression in (3.9) into

$$\left(1-\frac{(1-1/g)(1-cf/aq)}{(1-c/gq)(1-Ef/a)}\prod_{i=1}^{r}\frac{(1-ce_i/az_iq)}{(1-c/az_iq)}\right)\frac{(1-c/gq)}{(1-c/q)}$$
$$\times\frac{(a/Ef,aq/fg;q)_\infty}{(a/Efg,aq/f;q)_\infty}\prod_{i=1}^{r}\frac{(az_iq,az_iq/e_ig;q)_\infty}{(az_iq/e_i,az_iq/g;q)_\infty},$$

which is the right side of (3.7), as desired. □

Similar to the one-dimensional case, where we deduced the bilateral summation (1.5) from the unilateral summation (2.3) by using M. E. H. Ismail's [12] argument, we can now readily deduce Theorem 3.4 from Proposition 3.5.

Proof of Theorem 3.4. To establish (3.6), we apply Ismail's argument successively to the parameters $b_1^{-1},\dots,b_r^{-1}$ using Proposition 3.5. Both sides of the multiple series identity in (3.6) are analytic in each of the parameters $b_1^{-1},\dots,b_r^{-1}$ in a domain around the origin. Now, the identity is true for $b_1=aq^{-m_1}, b_2=aq^{-m_2},\dots,$ and $b_r=aq^{-m_r}$, by the A_{r-1} summation in Proposition 3.5 (see below for the details). This holds for all $m_1,\dots,m_r\ge 0$. Since $\lim_{m_1\to\infty}q^{m_1}/a=0$ is an interior point in the domain of analyticity of b_1^{-1}, by the identity theorem of analytic functions, we obtain an identity for b_1^{-1}. By iterating this argument for $b_2^{-1},\dots,b_r^{-1}$, and analytic continuation, we establish (3.6) for general $b_1^{-1},\dots,b_r^{-1}$ where $|B^{-1}|<|Efg/a^{r+1}|$.

The details are displayed as follows. Setting $b_i=aq^{-m_i}$, for $i=1,\dots,r$, the left side of (3.6) becomes

$$\sum_{\substack{-m_i\le k_i\le\infty\\ i=1,\dots,r}}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^r\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)$$
$$\times\prod_{i,j=1}^r\frac{(e_jz_i/z_j;q)_{k_i}}{(q^{1+m_j}z_i/z_j;q)_{k_i}}\prod_{i=1}^r\frac{(az_iq^{-m_i};q)_{|\mathbf{k}|}\,(fz_i;q)_{k_i}}{(az_iq/e_i;q)_{|\mathbf{k}|}\,(az_iq/g;q)_{k_i}}$$
$$\times\frac{(g;q)_{|\mathbf{k}|}}{(aq/f;q)_{|\mathbf{k}|}}\frac{(1-cq^{|\mathbf{k}|-1})}{(1-c/q)}\prod_{i=1}^r\frac{(1-az_iq^{k_i+1}/c)}{(1-az_iq/c)}\cdot\left(\frac{aq^{|\mathbf{m}|}}{Efg}\right)^{|\mathbf{k}|}\Bigg).\tag{3.10}$$

We shift the summation indices in (3.10) by $k_i\mapsto k_i-m_i$, for $i=1,\dots,r$, and obtain

$$\prod_{1\le i<j\le r}\left(\frac{z_iq^{-m_i}-z_jq^{-m_j}}{z_i-z_j}\right)\prod_{i=1}^r\left(\frac{1-az_iq^{-m_i-|\mathbf{m}|}}{1-az_i}\right)$$
$$\times\prod_{i,j=1}^r\frac{(e_jz_i/z_j;q)_{-m_i}}{(q^{1+m_j}z_i/z_j;q)_{-m_i}}\prod_{i=1}^r\frac{(az_iq^{-m_i};q)_{-|\mathbf{m}|}\,(fz_i;q)_{-m_i}}{(az_iq/e_i;q)_{-|\mathbf{m}|}\,(az_iq/g;q)_{-m_i}}$$
$$\times\frac{(g;q)_{-|\mathbf{m}|}}{(aq/f;q)_{-|\mathbf{m}|}}\frac{(1-cq^{-|\mathbf{m}|-1})}{(1-c/q)}\prod_{i=1}^r\frac{(1-az_iq^{1-m_i}/c)}{(1-az_iq/c)}\cdot\left(\frac{aq^{|\mathbf{m}|}}{Efg}\right)^{-|\mathbf{m}|}$$
$$\times\sum_{\substack{-m_i\le k_i\le\infty\\ i=1,\dots,r}}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{-m_i+k_i}-z_jq^{-m_j+k_j}}{z_iq^{-m_i}-z_jq^{-m_j}}\right)\prod_{i=1}^r\left(\frac{1-az_iq^{-m_i-|\mathbf{m}|+k_i+|\mathbf{k}|}}{1-az_iq^{-m_i-|\mathbf{m}|}}\right)\tag{3.11}$$

$$\times \prod_{i,j=1}^{r} \frac{(e_j q^{-m_i} z_i/z_j;q)_{k_i}}{(q^{1+m_j-m_i} z_i/z_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(az_i q^{-m_i-|\mathbf{m}|};q)_{|\mathbf{k}|}\,(fz_i q^{-m_i};q)_{k_i}}{(az_i q^{1-|\mathbf{m}|}/e_i;q)_{|\mathbf{k}|}\,(az_i q^{1-m_i}/g;q)_{k_i}}$$

$$\times \frac{(gq^{-|\mathbf{m}|};q)_{|\mathbf{k}|}}{(aq^{1-|\mathbf{m}|}/f;q)_{|\mathbf{k}|}} \frac{(1-cq^{|\mathbf{k}|-|\mathbf{m}|-1})}{(1-cq^{-|\mathbf{m}|-1})} \prod_{i=1}^{r} \frac{(1-az_i q^{k_i+1-m_i}/c)}{(1-az_i q^{1-m_i}/c)} \cdot \left(\frac{aq^{|\mathbf{m}|}}{Efg}\right)^{|\mathbf{k}|} \Bigg).$$

Next, using the identities

$$\prod_{i,j=1}^{r} (e_j z_i/z_j;q)_{-m_i} = (-1)^{r|\mathbf{m}|} E^{-|\mathbf{m}|} q^{r\sum_{i=1}^{r}\binom{m_i+1}{2}} \prod_{i=1}^{r} z_i^{|\mathbf{m}|-rm_i} \prod_{i,j=1}^{r} (z_i q/e_i z_j;q)_{m_j}^{-1},$$

and

$$\prod_{i,j=1}^{r} (q^{1+m_j} z_i/z_j;q)_{-m_i}^{-1} = \prod_{i,j=1}^{r} \frac{(qz_i/z_j;q)_{m_j}}{(qz_i/z_j;q)_{m_j-m_i}},$$

together with the $n \mapsto r$, $x_i \mapsto z_i$, and $y_i \mapsto -m_i$, $i=1,\dots,r$, case of [17, Lemma 3.12], specifically

$$\prod_{i,j=1}^{r} (qz_i/z_j;q)_{m_j-m_i} = (-1)^{(r-1)|\mathbf{m}|} q^{-\binom{|\mathbf{m}|+1}{2}+r\sum_{i=1}^{r}\binom{m_i+1}{2}} \times \prod_{i=1}^{r} z_i^{|\mathbf{m}|-rm_i} \prod_{1\le i<j\le r} \left(\frac{z_i q^{-m_i} - z_j q^{-m_j}}{z_i - z_j}\right), \tag{3.12}$$

and further the $a \mapsto aq^{-|\mathbf{m}|}$, $c \mapsto cq^{-|\mathbf{m}|}$, $e_i \mapsto e_i q^{-m_i}$, $g \mapsto gq^{-|\mathbf{m}|}$, and $z_i \mapsto z_i q^{-m_i}$, $i=1,\dots,r$, case of the multidimensional summation formula in (3.7), we simplify the expression in (3.11) to

$$(-1)^{|\mathbf{m}|} q^{-\binom{|\mathbf{m}|}{2}} \left(\frac{fg}{a}\right)^{|\mathbf{m}|} \prod_{i=1}^{r} \left(\frac{1-az_i q^{-m_i-|\mathbf{m}|}}{1-az_i}\right)$$

$$\times \prod_{i,j=1}^{r} \frac{(qz_i/z_j;q)_{m_j}}{(z_i q/e_i z_j;q)_{m_j}} \prod_{i=1}^{r} \frac{(az_i q^{-m_i};q)_{-|\mathbf{m}|}\,(fz_i;q)_{-m_i}}{(az_i q/e_i;q)_{-|\mathbf{m}|}\,(az_i q/g;q)_{-m_i}}$$

$$\times \frac{(g;q)_{-|\mathbf{m}|}}{(aq/f;q)_{-|\mathbf{m}|}} \frac{(1-cq^{-|\mathbf{m}|-1})}{(1-c/q)} \prod_{i=1}^{r} \frac{(1-az_i q^{1-m_i}/c)}{(1-az_i q/c)}$$

$$\times \left(1 - \frac{(1-q^{|\mathbf{m}|}/g)(1-cf/aq)}{(1-c/gq)(1-Ef/a)} \prod_{i=1}^{r} \frac{(1-ce_i/az_i q)}{(1-cq^{m_i}/az_i q)}\right) \frac{(1-c/gq)}{(1-cq^{-|\mathbf{m}|-1})}$$

$$\times \frac{(a/Ef,aq/fg;q)_\infty}{(aq^{|\mathbf{m}|}/Efg,aq^{1-|\mathbf{m}|}/f;q)_\infty} \prod_{i=1}^{r} \frac{(az_i q^{1-m_i-|\mathbf{m}|},az_i q/e_i g;q)_\infty}{(az_i q^{1-|\mathbf{m}|}/e_i,az_i q^{1-m_i}/g;q)_\infty}.$$

Now, this can easily be further transformed into

$$\left(1-\frac{(1-q^{|\mathbf{m}|}/g)(1-cf/aq)}{(1-c/gq)(1-Ef/a)}\prod_{i=1}^{r}\frac{(1-ce_i/az_iq)}{(1-cq^{m_i}/az_iq)}\right)\prod_{i=1}^{r}\frac{(1-cq^{m_i}/az_iq)}{(1-c/az_iq)}$$
$$\times\frac{(1-c/gq)}{(1-c/q)}\frac{(a/Ef,aq/fg,q^{1+|\mathbf{m}|}/g;q)_\infty}{(aq^{|\mathbf{m}|}/Efg,aq/f,q/g;q)_\infty}\prod_{i,j=1}^{r}\frac{(qz_i/z_j,z_iq^{1+m_j}/e_iz_j;q)_\infty}{(q^{1+m_j}z_i/z_j,z_iq/e_iz_j;q)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(az_iq,q/az_i,az_iq/e_ig,q^{1+m_i}/fz_i;q)_\infty}{(az_iq/e_i,az_iq/g,q^{1+m_i}/az_i,q/fz_i;q)_\infty},$$

which is exactly the $b_i = aq^{-m_i}$, $i = 1, \dots, r$, case of the right side of (3.6). □

References

[1] G. E. Andrews, "Applications of basic hypergeometric functions", *SIAM Rev.* **16** (1974), 441–484.

[2] G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Regional Conference Lectures Series **66** (Amer. Math. Soc., Providence, RI, 1986).

[3] R. Askey and M. E. H. Ismail, "The very well poised $_6\psi_6$", *Proc. Amer. Math. Soc.* **77** (1979), 218–222.

[4] W. N. Bailey, "Series of hypergeometric type which are infinite in both directions", *Quart. J. Math.* (Oxford) **7** (1936), 105–115.

[5] G. Bhatnagar and S. C. Milne, "Generalized bibasic hypergeometric series and their $U(n)$ extensions", *Adv. Math.* **131** (1997), 188–252.

[6] G. Bhatnagar and M. Schlosser, "C_n and D_n very-well-poised $_{10}\phi_9$ transformations", *Constr. Approx.* **14** (1998), 531–567.

[7] H. Exton, *q-Hypergeometric functions and applications*, Ellis Horwood Series in Mathematics and its Applications, Chichester; Halsted Press, John Wiley & Sons, New York, 1983.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990.

[9] I. M. Gessel and C. Krattenthaler, "Cylindric Partitions", *Trans. Amer. Math. Soc.* **349** (1997), 429–479.

[10] R. A. Gustafson, "Multilateral summation theorems for ordinary and basic hypergeometric series in $U(n)$", *SIAM J. Math. Anal.* **18** (1987), 1576–1596.

[11] R. A. Gustafson, "The Macdonald identities for affine root systems of classical type and hypergeometric series very well-poised on semi-simple Lie algebras", *in* Ramanujan International Symposium on Analysis (Dec. 26th to 28th, 1987, Pune, India), N. K. Thakare (ed.) (1989), 187–224.

[12] M. E. H. Ismail, "A simple proof of Ramanujan's $_1\psi_1$ sum", *Proc. Amer. Math. Soc.* **63** (1977), 185–186.

[13] M. Jackson, "On well-poised bilateral hypergeometric series of the type $_8\Psi_8$", *Quart. J. Math.* (Oxford) (2) **1** (1950), 63–68.

[14] S. C. Milne, "A q-analog of hypergeometric series very-well-poised in $SU(n)$ and invariant G-functions", *Adv. Math.* **58** (1985), 1–60.

[15] S. C. Milne, "Basic hypergeometric series very well-poised in $U(n)$", *J. Math. Anal. Appl.* **122** (1987), 223–256.

[16] S. C. Milne, "A q-analog of the Gauss summation theorem for hypergeometric series in $U(n)$", *Adv. Math.* **72** (1988), 59–131.
[17] S. C. Milne, "Balanced ${}_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series", *Adv. Math.* **131** (1997), 93–187.
[18] S. C. Milne and M. Schlosser, "A new A_n extension of Ramanujan's ${}_1\psi_1$ summation with applications to multilateral A_n series", to appear in *Rocky Mount. J. Math.*, spec. vol. "Special Functions 2000" (Tempe, AZ, May 29 – June 9, 2000).
[19] L. J. Rogers, "Third memoir on the expansion of certain infinite products", *Proc. London Math. Soc.* **26** (1894), 15–32.
[20] M. Schlosser, "Multidimensional matrix inversions and A_r and D_r basic hypergeometric series", *The Ramanujan J.* **1** (1997), 243–274.
[21] M. Schlosser, "Some new applications of matrix inversions in A_r", *The Ramanujan J.* **3** (1999), 405–461.
[22] M. Schlosser, "Multilateral transformations of q-series with quotients of parameters that are nonnegative integral powers of q", to appear in *Contemp. Math.*, spec. vol. "q-Series with Applications to Combinatorics, Number Theory, and Physics" (Urbana-Champaign, October 26–28, 2000).
[23] H. S. Shukla, "A note on the sums of certain bilateral hypergeometric series", *Proc. Cambridge Phil. Soc.* **55** (1959), 262–266.
[24] D. Stanton, "An elementary approach to the Macdonald identities", *in* q-Series and partitions (D. Stanton, ed.), The IMA volumes in mathematics and its applications, vol. 18, Springer–Verlag, 1989, 139–150.

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210, USA

Current address: Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: `schlosse@ap.univie.ac at`

URL: `http://www.mat.univie.ac.at/People/mschloss`