WRONSKIANS OF THETA FUNCTIONS AND SERIES FOR $1/\pi$

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Dedicated to the memory of Professor J.M. Borwein

Abstract. In this article, we define functions analogous to Ramanujan's function $f(n)$ defined in his famous paper "Modular equations and approximations to π". We then use these new functions to study Ramanujan's series for $1/\pi$ associated with the classical, cubic and quartic bases.

1. Introduction

Let $q = e^{\pi i \tau}$ with $\operatorname{Im}\tau > 0$ and let

$$\vartheta_2(q) = \sum_{k=-\infty}^{\infty} q^{(k+1/2)^2}, \quad \vartheta_3(q) = \sum_{k=-\infty}^{\infty} q^{k^2}, \quad \text{and} \quad \vartheta_4(q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}.$$

Further, let

$$P(q) = 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}$$

and

$$\alpha(q) = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}. \tag{1.1}$$

In his paper "Modular equations and approximations to π", S. Ramanujan gave a table [19, Table III] expressing the function

$$f(\ell) := \frac{\ell P(q^{2\ell}) - P(q^2)}{\vartheta_3^2(q)\vartheta_3^2(q^\ell)} \tag{1.2}$$

in terms of $\alpha(q)$ and $\alpha(q^\ell)$ for $\ell = 2, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31$ and 35. (To be exact, Ramanujan actually defined f as $f(l) = \ell P(q^{2\ell}) - P(q^2)$, i.e., without the denominator in (1.2). We have modified Ramanujan's function for simplicity of the entries in the table.) Examples of such relations are

$$f(3) = 1 + \sqrt{\alpha(q)\alpha(q^3)} + \sqrt{(1-\alpha(q))(1-\alpha(q^3))},$$

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and

$$f(7) = 3\left(1 + \sqrt{\alpha(q)\alpha(q^7)} + \sqrt{(1-\alpha(q))(1-\alpha(q^7))}\right).$$

Unfortunately, Ramanujan did not provide any proofs of these identities. Ramanujan's table for $f(\ell)$ was reproduced by J.M. Borwein and P.B. Borwein in their book "Pi and the AGM" [5, p. 159, Table 5.1]. The Borweins remarked that "*The verification that $f(\ell)$ has the given form is tedious but straightforward for small ℓ. For larger ℓ, we rely on Ramanujan.*" This remark added more mysteries to Ramanujan's table of identities for $f(\ell)$.

In the paragraph after Table III of [19], Ramanujan outlined the relation of these identities with his series for $1/\pi$ [19, Section 13]. A more detailed explanation of Ramanujan's method of deriving series for $1/\pi$ using $f(\ell)$ was first made available by the Borweins in their book [5, Chapter 5].

Let

$$\eta(\tau) = q^{1/12}\prod_{k=1}^{\infty}\left(1-q^{2k}\right)$$

be the Dedekind η-function. It is immediate that

$$P(q^2) = 12q\frac{d\eta(\tau)}{dq}\cdot\frac{1}{\eta(\tau)},$$

and we can rewrite $f(\ell)$ in terms of the Wronskian of $\eta(\tau)$ and $\eta(\ell\tau)$ as follows:

$$f(\ell) = \frac{12}{\eta(\tau)\eta(\ell\tau)\vartheta_3^2(q)\vartheta_3^2(q^\ell)}\det\begin{pmatrix}\eta(\tau) & \eta(\ell\tau)\\ q\dfrac{d\eta(\tau)}{dq} & q\dfrac{d\eta(\ell\tau)}{dq}\end{pmatrix}. \tag{1.3}$$

In this article, we define analogues of $f(\ell)$ by replacing the Wronskian involving $\eta(\tau)$ in (1.3) by Wronskians of various theta functions. For example, associated with the classical Jacobi theta functions, we define the function

$$D_\ell(q) = \frac{1}{\vartheta_3^3(q)\vartheta_3^3(q^\ell)}\det\begin{pmatrix}\vartheta_3(q) & \vartheta_3(q^\ell)\\ q\dfrac{d\vartheta_3(q)}{dq} & q\dfrac{d\vartheta_3(q^\ell)}{dq}\end{pmatrix}.$$

The relation of $D_\ell(q)$ with the series for $1/\pi$ is illustrated in the following theorem:

Theorem 1.1. *Let $N > 2$ be an integer and let*

$$\alpha_N = \alpha\left(e^{-\pi\sqrt{N}}\right), \tag{1.4}$$

where $\alpha(q)$ is given by (1.1). Then

$$\frac{1}{\sqrt{N}\pi} = \sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(k\left(1-2\alpha_N\right) - \frac{2}{\sqrt{N}}D_N\left(e^{-\pi/\sqrt{N}}\right)\right)\left(4\alpha_N(1-\alpha_N)\right)^k. \tag{1.5}$$

Comparing (1.5) with the following simplified version of the Borweins' series (see (5.5.13) of [5])

$$\frac{1}{\sqrt{N}\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(k\left(1-2\alpha_N\right)+\frac{1-2\alpha_N}{3}-\frac{\sigma(N)}{6\sqrt{N}}\right)\left(4\alpha_N(1-\alpha_N)\right)^k,$$

where

$$\sigma(N)=f(N)\Big|_{q=e^{-\pi/\sqrt{N}}},$$

we conclude that (1.5) is perhaps the simplest form of Ramanujan's series for $1/\pi$ associated with the "classical base". Using (1.5), we can derive a series for $1/\pi$ whenever α_N and $-D_N\left(e^{-\pi/\sqrt{N}}\right)$ are known. For example, we will show using modular equations satisfied by $\alpha(q),\alpha(q^{13})$ and $D_{13}(q)$ that

$$\alpha_{13}=\frac12-3\sqrt{-18+5\sqrt{13}},$$

and

$$-D_{13}\left(e^{-\pi/\sqrt{13}}\right)=\frac{(-7+3\sqrt{13})\sqrt{-18+5\sqrt{13}}}{4}.$$

The series corresponding to $N=13$ is then given by

$$\frac{1}{(6\sqrt{13}\sqrt{-18+5\sqrt{13}})\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(k+\frac14-\frac{7}{156}\sqrt{13}\right)\left(649-180\sqrt{13}\right)^k. \qquad (1.6)$$

The identity (1.6) was implicitly given by the Borweins [5, p. 172, Table 5.2a][1] but since Ramanujan did not provide an expression for $f(13)$, the Borweins probably arrived at the series without using any specific identity associated with $f(13)$.

The article is organized as follows: In Section 2, we use the general series found by H.H. Chan, S.H. Chan and Z.-G. Liu [9, Theorem 2.1] to prove Theorem 1.1. We then state a result that is an extension of [9, Theorem 2.1] and use it to derive the following analogue of (1.5):

Theorem 1.2. *Let $N>1$ be a positive integer and α_N be given as in* (1.4)*. Then*

$$\frac{1}{\sqrt{N}\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(\frac{1+\alpha_N}{1-\alpha_N}k+\widehat{a}_N\right)\left(-4\frac{\alpha_N}{(1-\alpha_N)^2}\right)^k, \qquad (1.7)$$

where

$$\widehat{D}_\ell(q)=\frac{1}{\vartheta_4^3(q)\vartheta_4^3(q^\ell)}\det\begin{pmatrix}\vartheta_4(q) & \vartheta_4(q^\ell)\\ q\dfrac{d\vartheta_4(q)}{dq} & q\dfrac{d\vartheta_4(q^\ell)}{dq}\end{pmatrix} \qquad (1.8)$$

[1]Tables 5.2a and 5.2b on page 172 of [5] list certain quantities which are used in formulas for $1/\pi$ given by the Borweins in their book as (5.5.13) and (5.5.14), respectively. We should warn the reader that our notation is different from that used by the Borweins. In particular, the Borweins' $\lambda^\star(r)$ and $\lambda^{\star\prime}(r)$ translate in our notation to $\alpha(r)$ and $\sqrt{1-\alpha(r)}$, respectively, while the Borweins' $\alpha(r)$ can be expressed as $\left(\pi^{-1}-4\sqrt{r}(q\frac{d}{dq}(\log(\vartheta_4(q)))\right)/\vartheta_3^4(q)$ evaluated at $q=e^{-\pi\sqrt{r}}$ (see [5, (5.1.10)]).

and

$$\widehat{a}_N = -\frac{2}{\sqrt{N}}\sqrt{\frac{\alpha_N}{1-\alpha_N}}\widehat{D}_N\left(e^{-\pi/\sqrt{N}}\right) + \frac{1}{2(1-\alpha_N)}.$$

Note that for odd prime ℓ,

$$\widehat{D}_\ell(q) = D_\ell(-q). \tag{1.9}$$

Using (1.7), we derive some explicit examples, some of which are due to the Borweins. The series which we will prove with complete details is

$$\frac{1}{\sqrt{6}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(\sqrt{3}\left(2-\sqrt{2}\right)k + \frac{2}{3}\sqrt{3} - \frac{5}{12}\sqrt{6}\right)(-1)^k\left(17-12\sqrt{2}\right)^k. \tag{1.10}$$

Series (1.10) follows from the values

$$\alpha_6 = 35 + 24\sqrt{2} - 20\sqrt{3} - 14\sqrt{6},$$

and

$$\widehat{D}_6\left(e^{-\pi/\sqrt{6}}\right) = \sqrt{\frac{111}{16} + 5\sqrt{2} + \frac{33}{8}\sqrt{3} + \frac{45}{16}\sqrt{6}}.$$

We observe that the terms in the sum on the right-hand side of (1.10) have alternating signs. Although series with alternating signs in the "quartic base" are present in Ramanujan's work [19, (35)–(38)], no series with alternating signs in the "classical base" was recorded by Ramanujan. It is likely that the study of series such as (1.10) began with the Borweins.

In Section 3, we study the function $D_\ell(-q^2)$ and express $D_\ell(-q^2)$ in terms of Hauptmoduls when $\ell = 3, 5, 7, 11$ and 23.

In Section 4, we use the identities established in Section 3, modular equations satisfied by $\alpha(q)$ and $\alpha(q^\ell)$, Theorem 1.1 and Theorem 1.2 to derive several explicit series for $1/\pi$. We also provide a table of identities associated with $D_\ell(q)$ that is an analogue of Ramanujan's table for $f(\ell)$. This table of formulas allows us to derive series for $1/\pi$ associated with primes other than $3, 5, 7, 11$ and 23. In particular, we give an expression for $D_{13}(q)$, for which its counterpart $f(13)$ is missing in Ramanujan's table. The discovery of an expression for $D_{13}(q)$ in terms of $\alpha(q)$ and $\alpha(q^{13})$ leads to a proof of (1.6).

In Section 5, we turn our attention to the Borweins' cubic theta functions (see [6], [7]) and define the following cubic analogue of $D_\ell(q)$:

$$C_\ell(q) = \frac{1}{a^2(q)a^2(q^\ell)} \det\begin{pmatrix} a(q) & a(q^\ell) \\ q\dfrac{da(q)}{dq} & q\dfrac{da(q^\ell)}{dq}\end{pmatrix},$$

where

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Using $C_\ell(q)$, we present Theorem 5.1 and Theorem 5.2, which are cubic analogues of Theorem 1.1 and Theorem 1.2, respectively. Ramanujan did not offer any series for

$1/\pi$ arising from the class of series given in Theorem 5.2. The first few examples of such series are given by H.H. Chan, W.-C. Liaw and V. Tan [14].

We also derive representations of $C_\ell(q)$ in terms of Hauptmoduls for $\ell = 2, 5$ and 11 and provide a table of identities representing $C_\ell(q)$ in terms of the cubic singular modulus. This table is an analogue of Ramanujan's table for $f(\ell)$. Using the representations of $C_\ell(q)$ in terms of Hauptmoduls and cubic singular modulus, we derive several series for $1/\pi$ associated with the cubic base.

In Section 6, we state the following quartic analogue of Theorem 1.1:

$$D_\ell^\perp(q) = \frac{1}{\sqrt{A^3(q)A^3(q^\ell)}} \det\begin{pmatrix} A(q) & A(q^\ell) \\ q\dfrac{dA(q)}{dq} & q\dfrac{dA(q^\ell)}{dq}\end{pmatrix},$$

where

$$A(q^2) = \frac{\eta^8(\tau) + 32\eta^8(4\tau)}{\eta^4(2\tau)}.$$

Instead of providing a table for $D_\ell^\perp(q)$ analogous to Ramanujan's table for $f(\ell)$ for the purpose of deriving Ramanujan's series for $1/\pi$ in the quartic base, we establish a relation between $D_\ell^\perp(q)$ and $D_\ell(q)$ and show that Ramanujan's series for $1/\pi$ in the quartic base can be derived from the table of identities for $D_\ell(q)$. In particular, we provide a proof of Ramanujan's series

$$\frac{1}{\pi} = 2\sqrt{2}\sum_{k=0}^\infty \frac{\left(\frac12\right)_k\left(\frac14\right)_k\left(\frac34\right)_k}{(1)_k^3}(1103 + 26390k)\left(\frac{1}{99^2}\right)^{2n+1}.$$

This series is perhaps Ramanujan's most famous series for $1/\pi$ as it was the series used by B. Gosper in 1985 to compute π to 17526200 digits (cf. [1, p. 387 and p. 685]).

2. New representations of the Ramanujan–Borweins series for $1/\pi$ for the "classical base"

Let $\mathbf{Q}$ denote the field of rational numbers. We begin this section with a general series for $1/\pi$ given by H.H. Chan, S.H. Chan and Z.-G. Liu [9, Theorem 2.1].

Theorem 2.1. *Suppose $Z(q)$, $X(q)$ and $U(q)$ are functions satisfying*

$$rZ(e^{-2\pi\sqrt{r/s}}) = Z(e^{-2\pi/\sqrt{rs}}),$$

$$q\frac{dX(q)}{dq} = U(q)X(q)Z(q) \tag{2.1}$$

and

$$Z(q) = \sum_{k=0}^\infty A_k X^k(q), \quad A_k \in \mathbf{Q}.$$

Suppose

$$M_N(q) = \frac{Z(q)}{Z(q^N)},$$

for a positive integer $N>1$. Let

$$a_N = \frac{U(q)X(q)}{2N}\frac{dM_N(q)}{dX(q)}\bigg|_{q=e^{-2\pi/\sqrt{Ns}}},$$

$$b_N = U(e^{-2\pi\sqrt{N/s}}),$$

and

$$X_N = X(e^{-2\pi\sqrt{N/s}}).$$

If the series

$$\sum_{k=0}^{\infty}(b_N k + a_N)A_k X_N^k$$

converges, then

$$\sqrt{\frac{s}{N}}\frac{1}{2\pi} = \sum_{k=0}^{\infty}(b_N k + a_N)A_k X_N^k.$$

We will now establish Theorem 1.1 using Theorem 2.1.

Proof of Theorem 1.1. We begin by applying Theorem 2.1 with $Z(q)=\vartheta_3^4(q)$ and

$$X(q) = 4\alpha(q)\left(1-\alpha(q)\right).$$

This implies that $X_N = 4\alpha_N(1-\alpha_N)$. It is known that [9, (3.5)]

$$Z(q) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}X^k(q), \tag{2.2}$$

and this implies that

$$A_k = \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}.$$

The function $\vartheta_3(q)$ satisfies the transformation formula (see for example [2, p. 43, Entry 27(ii)])

$$\vartheta_3^2\left(e^{-\pi/\sqrt{N}}\right) = N^{1/2}\vartheta_3^2\left(e^{-\pi\sqrt{N}}\right), \tag{2.3}$$

and this implies that

$$Z\left(e^{-\pi/\sqrt{N}}\right) = NZ\left(e^{-\pi\sqrt{N}}\right).$$

In other words, the integer s in Theorem 2.1 is 4.

Next, from [2, p. 120, Entry 9(i)]

$$q\frac{dX(q)}{dq} = (1-2\alpha(q))X(q)Z(q) \tag{2.4}$$

we conclude that $U(q) = 1-2\alpha(q)$ and that

$$b_N = 1-2\alpha_N.$$

In order to complete the proof of Theorem 1.1, it remains to verify that

$$a_N = -\frac{2}{\sqrt{N}}D_N\left(e^{-\pi/\sqrt{N}}\right). \tag{2.5}$$

This follows by observing that

$$\frac{1}{M_N(q)}q\frac{dM_N(q)}{dq} = -4\vartheta_3^2(q)\vartheta_3^2(q^N)D_N(q). \tag{2.6}$$

From (2.6) and (2.4), we deduce that

$$q\frac{dM_N(q)}{dq} = \frac{dM_N(q)}{dX(q)}q\frac{dX(q)}{dq} = \vartheta_3^4(q)U(q)X(q)\frac{dM_N(q)}{dX(q)} = -4\frac{\vartheta_3^6(q)}{\vartheta_3^2(q^N)}D_N(q).$$

Hence,

$$U(q)X(q)\frac{dM_N(q)}{dX(q)}\bigg|_{q=e^{-\pi/\sqrt{N}}} = -\frac{4\vartheta_3^2(e^{-\pi/\sqrt{N}})}{\vartheta_3^2(e^{-\pi\sqrt{N}})}D_N(e^{-\pi/\sqrt{N}}),$$

and (2.5) follows from (2.3). $\square$

We now proceed to prove Theorem 1.2. We need the following generalization of [9, Theorem 2.1].

Theorem 2.2. *Suppose $\mathcal{Z}(q)$, $\mathcal{X}(q)$ and $\mathcal{U}(q)$ are functions satisfying*

$$\mathcal{Z}\left(e^{-2\pi/\sqrt{rs}}\right) = r\mathcal{Z}\left(e^{-2\pi\sqrt{r/s}}\right)C\left(e^{-2\pi\sqrt{r/s}}\right),$$

where $C(q)$ is a certain function in q,

$$q\frac{d\mathcal{X}(q)}{dq} = \mathcal{U}(q)\mathcal{X}(q)\mathcal{Z}(q)$$

and

$$\mathcal{Z}(q) = \sum_{k=0}^{\infty}\mathcal{A}_k\mathcal{X}^k(q), \quad \mathcal{A}_k \in \mathbf{Q}.$$

Suppose

$$\mathcal{M}_N(q) = \frac{\mathcal{Z}(q)}{\mathcal{Z}(q^N)},$$

for a positive integer $N > 1$.

Let

$$\mathbf{a}_N = \frac{\mathcal{U}\left(e^{-2\pi/\sqrt{Ns}}\right)\mathcal{X}\left(e^{-2\pi/\sqrt{Ns}}\right)}{2N}\frac{d\mathcal{M}_N(q)}{d\mathcal{X}(q)}\bigg|_{q=e^{-2\pi/\sqrt{Ns}}} + \frac{\mathcal{U}\left(e^{-2\pi\sqrt{N/s}}\right)\mathcal{X}\left(e^{-2\pi\sqrt{N/s}}\right)}{2C\left(e^{-2\pi\sqrt{N/s}}\right)}\frac{dC(q)}{d\mathcal{X}(q)}\bigg|_{q=e^{-2\pi\sqrt{N/s}}},$$

$$\mathbf{b}_N = \mathcal{U}(e^{-2\pi\sqrt{N/s}}),$$

and

$$\mathbf{X}_N = \mathcal{X}(e^{-2\pi\sqrt{N/s}}).$$

If the series

$$\sum_{k=0}^{\infty}(\mathbf{b}_N k + \mathbf{a}_N)\mathcal{A}_k\mathbf{X}_N^k$$

converges, then

$$\sqrt{\frac{s}{N}}\frac{1}{2\pi} = \sum_{k=0}^{\infty}(\mathbf{b}_N k + \mathbf{a}_N)\mathcal{A}_k \mathbf{X}_N^k.$$

The differences between Theorem 2.2 and Theorem 2.1 are the transformation formulas for $\mathcal{Z}(q)$ and $Z(q)$, which resulted in a difference between $\mathbf{a}_N$ and a_N. Theorem 2.2 can be proved in exactly the same way as Theorem 2.1. Note that Theorem 2.2 is a generalization of [9, Theorem 2.1] since in the latter case, the corresponding function $C(q)$ is 1.

Proof of Theorem 1.2. It is known from Jacobi's triple product identity [2, p. 37, (22.4)] that

$$\vartheta_4(q) = \frac{\eta^2(\tau/2)}{\eta(\tau)}, \tag{2.7a}$$

and [2, p. 36, Entry 22])

$$\vartheta_2(q) = 2\frac{\eta^2(2\tau)}{\eta(\tau)}. \tag{2.7b}$$

Using (2.7) and [2, p. 43, Entry 27(iii)]

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \tag{2.8}$$

we deduce that

$$\vartheta_4^4\left(e^{-\pi/\sqrt{N}}\right) = N\vartheta_2^4\left(e^{-\pi\sqrt{N}}\right) = N\vartheta_4^4\left(e^{-\pi\sqrt{N}}\right)\frac{\vartheta_2^4\left(e^{-\pi\sqrt{N}}\right)}{\vartheta_4^4\left(e^{-\pi\sqrt{N}}\right)}. \tag{2.9}$$

Note that if we let $\mathcal{Z}(q) = \vartheta_4^4(q)$ in Theorem 2.2, then $s = 4$ and

$$C(q) = \frac{\vartheta_2^4(q)}{\vartheta_4^4(q)}.$$

Using Jacobi's identity (see [2, p. 40, Entry 25(vii)] or [5, (2.1.10)])

$$\vartheta_3^4(q) = \vartheta_2^4(q) + \vartheta_4^4(q), \tag{2.10}$$

we find that

$$C(q) = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}\frac{\vartheta_3^4(q)}{\vartheta_4^4(q)} = \frac{\alpha(q)}{1-\alpha(q)}, \tag{2.11}$$

where $\alpha(q)$ is given by (1.1).

Next, observe that

$$\mathcal{Z}(q) = \vartheta_4^4(q) = \vartheta_3^4(-q).$$

Therefore, by (2.2), we deduce that

$$\mathcal{Z}(q) = \sum_{k=0}^{\infty}\mathcal{A}_k \mathcal{X}^k(q),$$

where

$$\mathcal{A}_k = \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}$$

and

$$\mathcal{X}(q) = 4\alpha(-q)(1-\alpha(-q)).$$

Using (2.10), we observe that

$$\alpha(-q) = -\frac{\alpha(q)}{1-\alpha(q)}, \tag{2.12}$$

and hence

$$\mathcal{X}(q) = -4\frac{\alpha(q)}{1-\alpha(q)}. \tag{2.13}$$

Next, (2.1) holds with q replaced by $-q$ and therefore,

$$\mathcal{U}(q) = 1-2\alpha(-q) = \frac{1+\alpha(q)}{1-\alpha(q)}, \tag{2.14}$$

where the last equality follows from (2.12). Letting $q = e^{-\pi/\sqrt{N}}$, we deduce from (2.13) and (2.14) that

$$\mathbf{X}_N = -4\frac{\alpha_N}{1-\alpha_N}$$

and

$$\mathbf{b}_N = \frac{1+\alpha_N}{1-\alpha_N}.$$

Using the argument as in the proof of Theorem 1.1, we may write the first term of $\mathbf{a}_N$ involving $\mathcal{M}_N$ in terms of $\widehat{D}_\ell(q)$. The second term of $\mathbf{a}_N$ follows from (2.11), (2.13) and (2.14). Substituting the expressions of $\mathbf{a}_N, \mathbf{b}_N$, and $\mathbf{X}_N$ in Theorem 2.2, we complete the proof of Theorem 1.2. $\square$

The series (1.7), in a slightly different form, was discovered by the Borweins [5, p. 182, (5.5.14)].

3. The functions $D_\ell(q)$ and $\widehat{D}_\ell(q)$

In this section, instead of working with $D_\ell(q)$, we derive identities for $\widehat{D}_\ell(q)$ given by (1.8).

We first establish the following fact:

Theorem 3.1. *Let ℓ be an odd prime and let*

$$\omega_\ell = \begin{cases} 2 & \text{if } \ell \equiv 1 \pmod 4, \\ 1 & \text{if } \ell \equiv 3 \pmod 4. \end{cases}$$

Then $\widehat{D}_\ell^{\omega_\ell}(q^2)$ is a modular function on $\Gamma_0(2\ell)+W_\ell$, where $\Gamma_0(N)+W_e$ denotes the group generated by $\Gamma_0(N)$ and

$$W_e = \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e} \end{pmatrix},$$

with $e|N$, $\gcd(N/e,e)=1$ and $\det(W_e)=1$.

Proof. Let

$$T(\tau) := \vartheta_4(q^2) = \frac{\eta^2(\tau)}{\eta(2\tau)}, \tag{3.1}$$

where the product representation of $\vartheta_4(q^2)$ follows from (2.7a).

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbf{Z}, ad - bc = 1, c \equiv 0 \pmod{N} \right\}.$$

For

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2),$$

let

$$U \circ \tau := \frac{a\tau + b}{c\tau + d}.$$

It is known, using the transformation formula of the η-function (see for example [18, p. 163] or [10, Theorem 1.2]) and (3.1), that

$$T\,(U \circ \tau) = \xi(a, b, c, d)(c\tau + d)^{1/2} T(\tau) \tag{3.2}$$

where

$$\xi(a, b, c, d) = \left(\frac{c}{d}\right) e^{\pi i (d - 1 - cd/2)/4}.$$

Identity (3.2) implies that if

$$\Psi(\tau) = \frac{1}{T(\tau)} \frac{dT}{d\tau}(\tau),$$

then

$$\Psi\,(U \circ \tau) = \left(\frac{c}{2(c\tau + d)} + \Psi\,(\tau) \right) (c\tau + d)^2. \tag{3.3}$$

Next, let ℓ be an odd prime and observe that for

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2\ell),$$

$$\ell\Psi\,(\ell\,(V \circ \tau)) = \left(\frac{\gamma}{2(\gamma\tau + \delta)} + \ell\Psi\,(\ell\tau) \right) (\gamma\tau + \delta)^2.$$

Note that since

$$V \in \Gamma_0(2\ell) \subset \Gamma_0(2),$$

(3.3) also holds for the matrix V, and we find that

$$S_\ell(\tau) = \ell\Psi(\ell\tau) - \Psi(\tau)$$

is a modular form of weight 2 on $\Gamma_0(2\ell)$. By (3.2), we find that $\left(T^2(\ell\tau)T^2(\tau)\right)^{\omega_\ell}$ is a modular form of weight $2\omega_\ell$ on $\Gamma_0(2\ell)$. Therefore,

$$\left(\frac{S_\ell(\tau)}{T^2(\ell\tau)T^2(\tau)} \right)^{\omega_\ell}$$

is a modular function on $\Gamma_0(2\ell)$.

Next, by using (3.2), we conclude that

$$\left(\frac{S_\ell(W_\ell \circ \tau)}{T^2(\ell(W_\ell \circ \tau))T^2(W_\ell \circ \tau)} \right)^{\omega_\ell} = \left(\frac{S_\ell(\tau)}{T^2(\ell\tau)T^2(\tau)} \right)^{\omega_\ell}.$$

Observe that by (1.8), we find that

$$\widehat{D}_\ell^{\omega_\ell}(q^2) = \left(\frac{S_\ell(\tau)}{T^2(\ell\tau)T^2(\tau)}\right)^{\omega_\ell}.$$

This implies, from the transformation properties of $\left(\frac{S_\ell(\tau)}{T^2(\ell\tau)T^2(\tau)}\right)^{\omega_\ell}$, that $\widehat{D}_\ell^{\omega_\ell}(q^2)$ is a modular function on $\Gamma_0(2\ell)+W_\ell$. □

We now use Theorem 3.1 to derive identities for $\widehat{D}_\ell(q)$. We first determine prime numbers ℓ for which all modular functions associated with $\Gamma_0(2\ell)+W_\ell$ are rational functions of a single function, which we shall call a Hauptmodul. From the table in [12, p. 14], we find that this occurs when $\ell = 3, 5, 7, 11$ and 23. For such a prime ℓ, we construct a Hauptmodul H_l (which a priori is not unique) for the corresponding field of functions for $\Gamma_0(2\ell)+W_\ell$ and obtain the following identities:

Theorem 3.2. *Let*

$$H_\ell = H_\ell(\tau) = \left(\frac{\eta(2\tau)\eta(2\ell\tau)}{\eta(\tau)\eta(\ell\tau)}\right)^{\frac{24}{\ell+1}}.$$

Then

$$\widehat{D}_3(q^2) = 2H_3, \tag{3.4a}$$

$$\widehat{D}_5^2(q^2) = 4H_5^2(1+4H_5),$$

$$\widehat{D}_7(q^2) = 2H_7\left(1+3H_7\right),$$

$$\widehat{D}_{11}(q^2) = 2H_{11}\left(1+4H_{11}+5H_{11}^2\right), \tag{3.4b}$$

$$\widehat{D}_{23}(q^2) = 2H_{23}\left(1+5H_{23}+13H_{23}^2+20H_{23}^3+20H_{23}^4+11H_{23}^5\right). \tag{3.4c}$$

Remark 3.1. We note that since $q = e^{\pi i\tau}$, the identities given in Theorem 3.2 can all be expressed in terms of q^2. Replacing q^2 by q, we obtain identities for $\widehat{D}_\ell(q)$ in terms of

$$H_\ell(\tau/2),$$

for $\ell = 3, 5, 7, 11$ and 23, and these functions are in terms of infinite products with variable q. Replacing q by $-q$ and using

$$\prod_{k=1}^{\infty}(1-(-q)^k) = \prod_{k=1}^{\infty}\frac{(1-q^{2k})^3}{(1-q^k)(1-q^{4k})}, \tag{3.5}$$

we obtain identities from Theorem 3.2 expressing $D_\ell(q)$ in terms of Dedekind η-functions $\eta(\tau/2)$, $\eta(\tau)$, $\eta(\ell\tau/2)$ and $\eta(\ell\tau)$.

4. Explicit examples of Theorems 1.1 and 1.2

In this section, we first derive explicit series for $1/\pi$ from Theorem 3.2 for $N = 3, 5, 7, 11$ and 23. We give complete details only for the case $N = 3$. We then derive explicit series from Theorem 1.2 for $N = 6, 10, 14, 22$ and 46. We need to work harder deriving these series as our identities in Theorem 3.2 are only for $\ell = 3, 5, 7, 11$ and 23 instead of $6, 10, 14, 22$ and 46. Again we give complete details only for the case $N = 6$.

4.1. **Case $N = 3$.**

Following Remark 3.1, we deduce from (3.4a) that

$$D_3(q) = -2\frac{\eta^3(\tau)\eta^3(3\tau)}{\vartheta_3^3(q)\vartheta_3^3(q^3)}, \tag{4.1}$$

where we have used (3.5) and the product representation of $\vartheta_3(q)$ [2, p. 36, Entry 22]):

$$\vartheta_3(q) = \frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)}. \tag{4.2}$$

Let $\tau = i/\sqrt{3}$ in (4.1). Observe that

$$D_3(e^{-\pi/\sqrt{3}}) = -2\frac{\eta^6(i\sqrt{3})}{\vartheta_3^6(e^{-\pi\sqrt{3}})}, \tag{4.3}$$

where we have used (2.3) and (2.8).

Next, using (2.7a), (4.2) and (2.7b), we immediately deduce Jacobi's identity [17, pp. 515–517]

$$\eta^{24}(\tau) = \frac{1}{2^8}\vartheta_3^{24}(q)\frac{\vartheta_2^8(q)}{\vartheta_3^8(q)}\frac{\vartheta_4^8(q)}{\vartheta_3^8(q)}. \tag{4.4}$$

Letting $q = e^{-\pi\sqrt{n}}$ in (4.4), we deduce that

$$\frac{\eta^6(i\sqrt{N})}{\vartheta_3^6(e^{-\pi\sqrt{N}})} = \frac{1}{2^2}\sqrt{\alpha_N(1-\alpha_N)}, \tag{4.5}$$

where we have used (2.10). It remains to compute α_3. It is known that [2, p. 230, Entry 5(i)]

$$\left((1-\alpha(q))\left(1-\alpha(q^3)\right)\right)^{1/4} + \left(\alpha(q)\alpha(q^3)\right)^{1/4} = 1. \tag{4.6}$$

When $q = e^{-\pi/\sqrt{3}}$,

$$\alpha(e^{-3\pi/\sqrt{3}}) = \alpha(e^{-\pi\sqrt{3}}), \tag{4.7}$$

and

$$\alpha(e^{-\pi/\sqrt{r}}) = 1 - \alpha(e^{-\pi\sqrt{r}}), \tag{4.8}$$

with $r = 3$. Identity (4.8) is a consequence of (2.10) and (2.8). Substituting (4.7) and (4.8) into (4.6), we conclude that

$$\alpha_3(1-\alpha_3) = \frac{1}{2^4},$$

which implies that

$$\alpha_3 = \frac{1}{2} - \frac{\sqrt{3}}{4}. \tag{4.9}$$

Let $N = 3$ in (4.5). Substituting (4.9) in the resulting equation, we deduce using (4.3) that

$$D_3(e^{-\pi/\sqrt{3}}) = -\frac{1}{8}.$$

From (1.5), we deduce the following Ramanujan series for $1/\pi$:

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(\frac{3}{2}k+\frac{1}{4}\right)\frac{1}{4^k}.$$

We have learnt from our derivation of the series corresponding to $N=3$ that in order to derive a series for $1/\pi$ corresponding to $N=3,5,7,11,23$ from Theorem 1.1 and Theorem 3.2, we only need the value of α_N. As such, for the following derivations of the series for $1/\pi$ corresponding to $N=5,7,11,23$ we will only discuss the evaluation of α_N.

4.2. **Case** $N=5$**.**

The value of α_5 can be determined from the following modular equation of degree 5 [2, p. 280, Entry 13(i)]:

$$\begin{aligned}\left(\alpha(q)\alpha(q^5)\right)^{1/2}&+\left((1-\alpha(q))\left(1-\alpha(q^5)\right)\right)^{1/2}\\&+2\left(16\alpha(q)\alpha(q^5)\left(1-\alpha(q)\right)\left(1-\alpha(q^5)\right)\right)^{1/6}=1.\end{aligned}$$

This modular equation allows us to conclude that

$$\alpha_5=\frac{1}{2}-\sqrt{-2+\sqrt{5}}.$$

Therefore, the series we obtain from (1.5) and Theorem 3.2 is

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(\left(2\sqrt{-10+5\sqrt{5}}\right)k+\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)\left(9-4\sqrt{5}\right)^k.$$

4.3. **Case** $N=7$**.**

The value α_7 can be derived from the following modular equation of degree 7 [2, p. 314, Entry 19(i)]:

$$\left(\alpha(q)\alpha(q^7)\right)^{1/8}+\left((1-\alpha(q))\left(1-\alpha(q^7)\right)\right)^{1/8}=1. \tag{4.10}$$

This implies that

$$4\alpha_7(1-\alpha_7)=\frac{1}{64},$$

$$1-2\alpha_7=\frac{3}{8}\sqrt{7},$$

and

$$\alpha_7=\frac{1}{2}-\frac{3}{16}\sqrt{7}.$$

The series we obtain from (1.5) and Theorem 3.2 is

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(\frac{21}{8}k+\frac{5}{16}\right)\left(\frac{1}{64}\right)^k.$$

4.4. **Cases $N = 11$ and 23.**

We first observe that our series obtained in this article depend entirely on the value of $\alpha_\ell(1-\alpha_\ell)$. The degree of the polynomial satisfied by $\alpha_\ell(1-\alpha_\ell)$ increases in general with ℓ. In fact, if

$$\frac{\ell+1}{8} = \frac{\nu}{s}$$

with $(\nu, s) = 1$, then $(\alpha_\ell(1-\alpha_\ell))^{s/8}$ satisfies a polynomial equation of degree ν which can be derived from a Russell-type modular equation. For example, $(\alpha_7(1-\alpha_7))^{1/8}$ satisfies a polynomial equation of degree 1 (see (4.10)). For $N = 11$ and 23, we have to solve cubic polynomial equations since $12/8 = 3/2$ and $24/8 = 3/1$. For more discussion on the evaluations of $(\alpha_\ell(1-\alpha_\ell))^{s/8}$ and modular equations, see [13] and [20].

We now continue with $N = 11$. The modular equation given by Ramanujan is [2, p. 363, Entry 7(i)]

$$\begin{aligned}\left(\alpha(q)\alpha(q^{11})\right)^{1/4} &+ \left((1-\alpha(q))(1-\alpha(q^{11}))\right)^{1/4} \\ &+ 2\left(16\alpha(q)\alpha(q^{11})(1-\alpha(q))(1-\alpha(q^{11}))\right)^{1/12} = 1.\end{aligned}$$

This implies that

$$\alpha_{11}(1-\alpha_{11}) = -\frac{1}{12}\left(27+21\sqrt{33}\right)^{1/3} + \frac{2}{\left(27+21\sqrt{33}\right)^{1/3}} + \frac{1}{16}.$$

The series we obtain from (1.5) and Theorem 3.2 is

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac12\right)_k^3}{(k!)^3}\left(k\sqrt{11}\cdot\sqrt{1-\delta} + 2\left(\delta^{1/6} - 2\delta^{1/3} + \frac{5}{4}\delta^{1/2}\right)\right)\delta^k,$$

where

$$\delta = -\frac{1}{3}\left(27+21\sqrt{33}\right)^{1/3} + \frac{8}{\left(27+21\sqrt{33}\right)^{1/3}} + \frac{1}{4}.$$

A modular equation of degree 23 can be found in [2, p. 411, Entry 15(i)] and is given by

$$\begin{aligned}\left(\alpha(q)\alpha(q^{23})\right)^{1/8} &+ \left((1-\alpha(q))(1-\alpha(q^{23}))\right)^{1/8} \\ &+ 2^{2/3}\left(\alpha(q)\alpha(q^{23})(1-\alpha(q))(1-\alpha(q^{23}))\right)^{1/24} = 1.\end{aligned}$$

Let $X_{23} = 4\alpha_{23}(1-\alpha_{23})$. From the above modular equation of degree 23, we deduce that

$$X_{23} = \frac{1}{384}\frac{5\mu^2 - 1660 - 44\mu}{\mu},$$

where

$$\mu = \left(4724 + 924\sqrt{69}\right)^{1/3}.$$

The associated series we obtain from (1.5) and Theorem 3.2 is

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac12\right)_k^3}{(k!)^3}\left(k\sqrt{23}\cdot\sqrt{1-X_{23}} + 2d_{23}\right)X_{23}^k,$$

where

$$d_{23} = \sqrt{2}X_{23}^{1/12} - 5X_{23}^{1/6} + \frac{13\sqrt{2}}{2}X_{23}^{1/4} - 10X_{23}^{1/3} + 5\sqrt{2}X_{23}^{5/12} - \frac{11}{4}X_{23}^{1/2}.$$

As mentioned in the beginning of this article, the Borweins remarked that to derive series for $1/\pi$ given in Theorem 1.1 corresponding to $N = 11$ and 23, they needed to rely on Ramanujan's expressions for $f(11)$ and $f(23)$. We have shown here that this is not necessary and that these series can be constructed from the new identities (3.4b) and (3.4c).

Remark 4.1. We have cited [2] for modular equations of various degrees found by Ramanujan and used them to evaluate α_N. These modular equations are what we called Russell-type modular equations. They were studied systematically by R. Russell [20]. In fact, it is possible for us to construct Russell-type modular equations of any odd prime degree using the results found in [20]. For more details on how to compute such modular equations and their cubic analogues, see [13].

Remark 4.2. We observe that Russell-type modular equations of degree ℓ give us polynomials satisfied by α_ℓ. But in order to determine α_ℓ, we still face the problem of finding the zeroes of polynomials. For example, in the case of 11 and 23, we need to find roots of polynomials of degree 3. In other words, obtaining α_n using modular equations works only for relatively small composite or prime n. For certain n, especially those which are squarefree, we can compute α_n without using modular equations. This requires class field theory and explicit Shimura's reciprocity law. For more details, see [8], [11], [15], [16] and [21].

We have seen how Theorem 3.2 can be used to derive explicit series for $1/\pi$. We now use these identities to derive examples for Theorem 1.2. In [5], the Borweins provided only examples to their series for even N. As such, we will first restrict our attention to the derivation of special cases of Theorem 1.2 when N is even.

Before we proceed, we observe that if ℓ is a prime, then

$$\widehat{D}_{2\ell}(q)\vartheta_4^2(q)\vartheta_4^2(q^{2\ell}) = \widehat{D}_\ell(q)\vartheta_4^2(q)\vartheta_4^2(q^\ell) + \ell\widehat{D}_2(q^\ell)\vartheta_4^2(q^\ell)\vartheta_4^2(q^{2\ell}). \tag{4.11}$$

From the above, we know that we will need to derive a formula for $\widehat{D}_2(q)$ and this is given by

$$\widehat{D}_2^4(q) = \frac{1}{64^2}\frac{\alpha^4(q)}{(1-\alpha(q))^3}. \tag{4.12}$$

The relation (4.12) can be proved by observing that both $\widehat{D}_2(q^2)$ and $\alpha(q^2)$ are modular functions invariant under $\Gamma_0(4)$. Note that

$$\widehat{D}_2(q) \neq D_2(-q),$$

even though

$$\widehat{D}_\ell(q) = D_\ell(-q),$$

for odd prime ℓ (see (1.9)).

We are now ready to derive explicit series for $1/\pi$ arising from Theorem 1.2 for $N = 6, 10, 14, 22$ and 46.

4.5. **Case** $N = 6$.

From (4.11), we find that

$$\widehat{D}_6(e^{-\pi/\sqrt{6}}) = \widehat{D}_3(e^{-\pi/\sqrt{6}})\frac{\vartheta_4^2(e^{-\pi\sqrt{\frac{3}{2}}})}{\vartheta_4^2(e^{-\pi\sqrt{6}})} + 3\widehat{D}_2(e^{-\pi\sqrt{\frac{3}{2}}})\frac{\vartheta_4^2(e^{-\pi\sqrt{\frac{3}{2}}})}{\vartheta_4^2(e^{-\pi/\sqrt{6}})}.$$

In order to derive a series for $1/\pi$ using Theorem 1.2, we will need to derive the following identities:

$$\alpha_6 = 35 + 24\sqrt{2} - 20\sqrt{3} - 14\sqrt{6}, \tag{4.13a}$$

$$\alpha_{2/3} = 35 - 24\sqrt{2} - 20\sqrt{3} + 14\sqrt{6}, \tag{4.13b}$$

$$\widehat{D}_3^2\left(e^{-\pi/\sqrt{6}}\right) = \left(\frac{5}{2} + \frac{3}{2}\sqrt{3}\right)^2, \tag{4.13c}$$

$$\widehat{D}_2^2\left(e^{-\pi\sqrt{\frac{3}{2}}}\right) = -\frac{41}{16}\sqrt{6} + \frac{99}{16} - \frac{35}{8}\sqrt{2} + \frac{29}{8}\sqrt{3}, \tag{4.13d}$$

$$\frac{\vartheta_4^4\left(e^{-\pi\sqrt{\frac{3}{2}}}\right)}{\vartheta_4^4\left(e^{-\pi/\sqrt{6}}\right)} = 5 + \frac{8}{3}\sqrt{3} + 2\sqrt{6} + \frac{10}{3}\sqrt{2}, \tag{4.13e}$$

$$\frac{\vartheta_4^4\left(e^{-\pi\sqrt{\frac{3}{2}}}\right)}{\vartheta_4^4\left(e^{-\pi\sqrt{6}}\right)} = -3 + 2\sqrt{2} + 2\sqrt{3} - \sqrt{6}. \tag{4.13f}$$

Assuming that the above identities hold, we find that

$$\begin{aligned}\widehat{D}_6^2\left(e^{-\pi/\sqrt{6}}\right) &= \left(\widehat{D}_3\left(e^{-\pi/\sqrt{6}}\right)\frac{\vartheta_4^2\left(e^{-\pi\sqrt{\frac{3}{2}}}\right)}{\vartheta_4^2\left(e^{-\pi\sqrt{6}}\right)} + 3\widehat{D}_2\left(e^{-\pi\sqrt{\frac{3}{2}}}\right)\frac{\vartheta_4^2\left(e^{-\pi\sqrt{\frac{3}{2}}}\right)}{\vartheta_4^2\left(e^{-\pi/\sqrt{6}}\right)}\right)^2 \\ &= \frac{111}{16} + 5\sqrt{2} + \frac{33}{8}\sqrt{3} + \frac{45}{16}\sqrt{6}.\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{a}_6 &= \frac{2}{\sqrt{6}}\sqrt{\frac{\alpha_6}{1-\alpha_6}}\left(-\widehat{D}_6(e^{-\pi/\sqrt{6}})\right) + \frac{1}{2(1-\alpha_6)} \\ &= -\left(\frac{1}{4} + \frac{1}{6}\sqrt{6} - \frac{1}{6}\sqrt{3}\right) + \frac{1}{2(1-\alpha_6)} \\ &= \frac{2}{3}\sqrt{3} - \frac{5}{12}\sqrt{6}.\end{aligned}$$

Now, using the value of α_6, we immediately compute

$$\frac{1+\alpha_6}{1-\alpha_6} = \sqrt{3}\left(2 - \sqrt{2}\right) \qquad \text{and} \qquad -4\frac{\alpha_6}{(1-\alpha_6)^2} = -17 + 12\sqrt{2}.$$

Hence, by Theorem 1.2, we obtain the following identity

$$\frac{1}{\sqrt{6}\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(\sqrt{3}\left(2-\sqrt{2}\right)k+\frac{2}{3}\sqrt{3}-\frac{5}{12}\sqrt{6}\right)\left(-17+12\sqrt{2}\right)^k,$$

which is (1.10) in the introduction.

We still have to show the identities in (4.13). Observe that using (1.1), Jacobi's identity (2.10) and the product representations of $\vartheta_j(q)$ for $j=2,3,4$ given in (2.7b), (4.2) and (2.7a), we find that

$$-4\frac{\alpha(q)}{(1-\alpha(q))^2}=-64\frac{\eta^{24}(\tau)}{\eta^{24}(\tau/2)}. \tag{4.14}$$

We now recall the following modular equation of Ramanujan which is a consequence of [2, Chapter 17, Entry 12], namely,

$$U(\tau)+\frac{1}{U(\tau)}-2=V(\tau)+\frac{64}{V(\tau)}+16, \tag{4.15}$$

where

$$U(\tau)=\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}\quad\text{and}\quad V(\tau)=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^{6}.$$

Substituting $\tau=i/\sqrt{6}$ in (4.15), we find, using the evaluation formula (4.3) for the η-function, that

$$V(i/\sqrt{6})=8.$$

This implies that

$$4\frac{\alpha_6}{(1-\alpha_6)^2}=64\left(\frac{\eta(i\sqrt{6})}{\eta(i\sqrt{3/2})}\right)^{24}=17-12\sqrt{2},$$

and

$$4\frac{\alpha_{2/3}}{\left(1-\alpha_{2/3}\right)^2}=64\left(\frac{\eta(i\sqrt{2/3})}{\eta(i/\sqrt{6})}\right)^{24}=17+12\sqrt{2}.$$

This implies (4.13a) and the identity (4.13b).

The above method of deriving $\alpha_{2\ell}$ using $\alpha_{2/\ell}$ and a modular equation is due to Ramanujan. For more details, see [19, Section 2] where α_{10} is derived.

Identity (4.13c) follows from the identity

$$\begin{aligned}\widehat{D}_3^2(e^{-\pi/\sqrt{6}})=\frac{1}{16}\sqrt{16\frac{\alpha_{1/6}\alpha_{3/2}}{(1-\alpha_{1/6})^2(1-\alpha_{3/2})^2}}&=\frac{1}{16}\sqrt{16\frac{(1-\alpha_6)(1-\alpha_{2/3})}{\alpha_6^2\alpha_{2/3}^2}}\\&=\left(\frac{5}{2}+\frac{3}{2}\sqrt{3}\right)^2,\end{aligned}$$

where we have used the identity (see (4.8))

$$1-\alpha_{1/r}=\alpha_r.$$

Identity (4.13d) follows immediately from (4.12) and (4.13b).

To prove (4.13e) and (4.13f), we observe that by (2.9), it suffices to compute

$$\frac{\vartheta_4^4(e^{-\pi\sqrt{2/3}})}{\vartheta_4^4(e^{-\pi/\sqrt{6}})}.$$

To finish this final task, we recall two identities, namely (see for example [5, (2.1.6)])

$$\vartheta_3^2(q) + \vartheta_4^2(q) = 2\vartheta_3(q^2), \tag{4.16}$$

and [2, p. 214, (24.15)]

$$\frac{\vartheta_3^2(q)}{\vartheta_3^2(q^2)} = \frac{1-\sqrt{\alpha(q^2)}}{\sqrt{1-\alpha(q)}}. \tag{4.17}$$

From (1.1), (2.10), (4.16) and (4.17) we conclude that

$$\frac{\vartheta_4^2(q)}{\vartheta_4^2(q^2)} = \frac{1}{\sqrt{1-\alpha(q^2)}}\left(2-\frac{\vartheta_3^2(q)}{\vartheta_3^2(q^2)}\right) = \frac{1}{\sqrt{1-\alpha(q^2)}}\left(2-\frac{1-\sqrt{\alpha(q^2)}}{\sqrt{1-\alpha(q)}}\right).$$

This implies that

$$\begin{aligned}\frac{\vartheta_4^4(e^{-\pi/\sqrt{6}})}{\vartheta_4^4(e^{-\pi\sqrt{2/3}})} &= \frac{1}{1-\alpha_{2/3}}\left(2-\frac{1-\sqrt{\alpha_{2/3}}}{\sqrt{1-\alpha_{1/6}}}\right)^2 \\ &= -3-2\sqrt{2}+2\sqrt{3}+\sqrt{6}.\end{aligned}$$

As indicated earlier, (4.13e) and (4.13f) follow from this computation.

4.6. **Case** $N = 10$**.**

We now discuss the other cases of N, namely, $N = 10, 14, 22$ and 46. It is clear from our discussion of the case $N = 6$, to derive a series for $1/\pi$ from $\widehat{D}_N(q)$, we need, with help of the identities from Theorem 3.2, only the values for α_{2p} and $\alpha_{2/p}$. In the case of $N = 6$, we use modular equation (4.15) to derive α_6 and $\alpha_{2/3}$. We now discuss another method of deriving α_{2p} and $\alpha_{2/p}$ and we will illustrate this alternative method using the case $N = 10$. Let $\xi(q)$ be the right-hand side of (4.14), namely,

$$\xi(q) = -64\frac{\eta^{24}(\tau)}{\eta^{24}(\tau/2)}.$$

Let $\xi_n = \xi(e^{-\pi\sqrt{n}})$. Then it can be shown (see [8] and [11]) that

$$\frac{\xi_{10}}{\xi_{2/5}} + \frac{\xi_{2/5}}{\xi_{10}} = 103682.$$

Next, using (2.8), we deduce that for any positive real number n,

$$\xi_{2n}\xi_{2/n} = 1. \tag{4.18}$$

Solving the above equation and using (4.18) for any positive integer n, we deduce that

$$\xi_{10}^2 = 51841 - 23184\sqrt{5},$$

which implies that

$$\xi_{10} = -161 + 72\sqrt{5}.$$

Using (4.14), we deduce that

$$\alpha_{10} = 323 + 144\sqrt{5} - 102\sqrt{10} - 228\sqrt{2}.$$

Similarly, we obtain

$$\alpha_{2/5} = 323 - 144\sqrt{5} - 102\sqrt{10} + 228\sqrt{2}.$$

Using these values and following what we have done for $N = 6$, we deduce that

$$\frac{1}{\sqrt{10}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3} \left(\left(3\sqrt{10} - 6\sqrt{2}\right) k + \frac{23}{20}\sqrt{10} - \frac{5}{2}\sqrt{2} \right) \left(-161 + 72\sqrt{5}\right)^k.$$

4.7. **Case $N = 14$.**

The series for $1/\pi$ for $N = 14$ is not given by the Borweins. We now supply the missing series. We find, following the method illustrated in [8] and [11], that

$$\left(\frac{\xi_{14}}{\xi_{2/7}}\right)^{1/24} + \left(\frac{\xi_{2/7}}{\xi_{14}}\right)^{1/24} = 1 + \sqrt{2}.$$

This yields

$$\xi_{14} = -\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} - \frac{1}{2}\sqrt{-1 + 2\sqrt{2}}\right)^{12}.$$

Using a formula of Ramanujan [4, Theorem 1.2], we deduce that

$$\alpha_{14} = \left(-2\sqrt{2} - 2 + \sqrt{8\sqrt{2} + 11}\right)^2 \left(\sqrt{10 + 8\sqrt{2}} - \sqrt{8\sqrt{2} + 11}\right)^2.$$

Similarly, we find that

$$\alpha_{2/7} = \left(-2\sqrt{2} - 2 + \sqrt{8\sqrt{2} + 11}\right)^2 \left(\sqrt{10 + 8\sqrt{2}} + \sqrt{8\sqrt{2} + 11}\right)^2.$$

From the values of α_{14}, we should expect the series for $1/\pi$ to be very complicated. We will list the algebraic numbers needed to generate the series:

$$X_{14} = -\frac{4\alpha_{14}}{(1 - \alpha_{14})^2} = -\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} - \frac{1}{2}\sqrt{-1 + 2\sqrt{2}}\right)^{12},$$

$$b_{14} = \frac{1 + \alpha_{14}}{1 - \alpha_{14}},$$

$$V_{14} = \frac{\vartheta_4^4(e^{-\pi\sqrt{7/2}})}{\vartheta_4^4(e^{-\pi/\sqrt{14}})} \left(\widehat{D}_2(e^{-\pi\sqrt{7/2}})\right)^2 = \frac{1}{56(1 - \alpha_{14})},$$

$$U_{14} = \frac{\vartheta_4^4(e^{-\pi\sqrt{7/2}})}{\vartheta_4^4(e^{-\pi\sqrt{14}})} \left(\widehat{D}_7(e^{-\pi/\sqrt{14}})\right)^2 = 4h_{14}^2(1 + 3h_{14})^2\sqrt{\alpha_{2/7}},$$

where

$$h_{14} = \left(\frac{(1 - \alpha_{14})(1 - \alpha_{2/7})}{16^2\alpha_{14}^2\alpha_{2/7}^2}\right)^{1/8} = \sqrt{\frac{11}{4} + \frac{7}{4}\sqrt{2} + \frac{1}{4}\sqrt{217 + 154\sqrt{2}}}.$$

Then

$$\frac{1}{\sqrt{14}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3} \left(b_{14}k + \frac{2}{\sqrt{14}} \sqrt{\frac{\alpha_{14}}{1-\alpha_{14}}} \left(-\sqrt{U_{14}} - 7\sqrt{V_{14}}\right) + \frac{1}{2(1-\alpha_{14})} \right) X_{14}^k.$$

Note that in the case of $N = 14$, it is difficult to derive the series without knowing the explicit formula given by Theorem 1.2 and the corresponding identities given in Theorem 3.2. The complexity of the constants arising in this series is perhaps why the series is not given by the Borweins in their book.

4.8. **Case** $N = 22$**.**

Following the method illustrated in [8] and [11], we find that

$$\sqrt{\frac{\xi_{22}}{\xi_{2/11}}} + \sqrt{\frac{\xi_{2/11}}{\xi_{22}}} = 39202.$$

This yields

$$\xi_{22} = -\left(19601 - 13860\sqrt{2}\right),$$

and

$$\alpha_{22} = 39203 + 27720\sqrt{2} - 11820\sqrt{11} - 8358\sqrt{22}.$$

The corresponding series is

$$\frac{1}{2(-5\sqrt{2}+7)\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3} \left(-33k + \frac{17\sqrt{2}-33}{4}\right) \left(-19601 + 13860\sqrt{2}\right)^k.$$

4.9. **Case** $N = 46$**.**

The series for $1/\pi$ for $N = 46$ is not given by the Borweins. Following the method illustrated in [8] and [11], we find that

$$\left(\frac{\xi_{46}}{\xi_{2/23}}\right)^{1/24} + \left(\frac{\xi_{2/23}}{\xi_{46}}\right)^{1/24} = 3 + \sqrt{2}.$$

This implies that

$$\xi_{46} = -\left(\frac{3}{2} + \frac{\sqrt{2}}{2} - \frac{1}{2}\sqrt{7+6\sqrt{2}}\right)^{12}.$$

Therefore,

$$\alpha_{46} = \left(26 + 18\sqrt{2} - 3\sqrt{147 + 104\sqrt{2}}\right)^2 \left(\sqrt{1332 + 936\sqrt{2}} - 3\sqrt{147 + 104\sqrt{2}}\right)^2,$$

and

$$\alpha_{2/23} = \left(26 + 18\sqrt{2} - 3\sqrt{147 + 104\sqrt{2}}\right)^2 \left(\sqrt{1332 + 936\sqrt{2}} + 3\sqrt{147 + 104\sqrt{2}}\right)^2.$$

The following constants will give rise to an explicit series for $1/\pi$ associated with $N = 46$:

$$\begin{aligned}
b_{46} = {} & 78\sqrt{147+104\sqrt{2}} + 54\sqrt{2}\sqrt{147+104\sqrt{2}} \\
& - 3\sqrt{2}\sqrt{147+104\sqrt{2}}\sqrt{661+468\sqrt{2}}, \\
X_{46} = {} & -\left(\frac{3}{2}+\frac{\sqrt{2}}{2}-\frac{1}{2}\sqrt{7+6\sqrt{2}}\right)^{12}, \\
V_{46} = {} & \frac{1}{2^3\cdot 23(1-\alpha_{46})}, \\
U_{46} = {} & \left(2h_{46}\left(1+5h_{46}+13h_{46}^2+20h_{46}^3+20h_{46}^4+11h_{46}^5\right)\right)^2\sqrt{\alpha_{2/23}},
\end{aligned}$$

where

$$h_{46} = \left(\frac{(1-\alpha_{46})(1-\alpha_{2/23})}{16^2\alpha_{46}^2\alpha_{2/23}^2}\right)^{1/24}.$$

Then

$$\frac{1}{\sqrt{46}\pi} = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(b_{46}k + \frac{2}{\sqrt{46}}\sqrt{\frac{\alpha_{46}}{1-\alpha_{46}}}\left(-\sqrt{U_{46}}-23\sqrt{V_{46}}\right)+\frac{1}{2(1-\alpha_{46})}\right)X_{46}^k.$$

We have, in our attempt to prove some of the Borweins' identities [5, p. 172, Tables 5.2a, 5.2b], used (1.5) to derive series for $1/\pi$ when N is odd and (1.7) when N is even. We would like to emphasize here that these restrictions are not necessary. Indeed if we consider $N = 6, 10$ and 22, we obtain from (1.5) the following series for $1/\pi$:

The identity

$$\frac{1}{\sqrt{N}\pi} = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(b_N k + a_N\right)X_N^k$$

is true when

$$\begin{aligned}
b_6 &= -69 - 48\sqrt{2} + 40\sqrt{3} + 28\sqrt{6}, \\
a_6 &= -30 - 21\sqrt{2} + \frac{52}{3}\sqrt{3} + \frac{73}{6}\sqrt{6}, \\
X_6 &= -18872 - 13344\sqrt{2} + 10896\sqrt{3} + 7704\sqrt{6}, \\
b_{10} &= -645 + 456\sqrt{2} + 204\sqrt{10} - 288\sqrt{5}, \\
a_{10} &= -290 + 205\sqrt{2} + \frac{917}{10}\sqrt{10} - \frac{648}{5}\sqrt{5}, \\
X_{10} &= -1662776 + 1175760\sqrt{2} + 525816\sqrt{10} - 743616\sqrt{5},
\end{aligned}$$

and

$$\begin{aligned}
b_{22} &= -78405 - 55440\sqrt{2} + 23640\sqrt{11} + 16716\sqrt{22}, \\
a_{22} &= -36542 - 25839\sqrt{2} + \frac{121196}{11}\sqrt{11} + \frac{171397}{22}\sqrt{22}, \\
X_{22} &= -24589219256 - 17387203680\sqrt{2} + 7413928560\sqrt{11} + 5242439160\sqrt{22}.
\end{aligned}$$

Similarly, we also found series associated with (1.7) when N is odd. For example, when $N=3$ and 7, we have relatively simple series which were missing so far. These are respectively

$$\frac{1}{\sqrt{3}\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(\left(\sqrt{15}-8\sqrt{3}\right)k+6-\frac{10}{3}\sqrt{3}\right)\left(-416+240\sqrt{3}\right)^k,$$

and

$$\frac{1}{\sqrt{7}\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(\left(255-96\sqrt{7}\right)k+112-\frac{296}{7}\sqrt{7}\right)\left(-129536+48960\sqrt{7}\right)^k.$$

There is also a series for the case $N=5$ and it is given by

$$\frac{1}{\sqrt{N}\pi}=\sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k^3}{(k!)^3}\left(\widehat{b}_Nk+\widehat{a}_N\right)\widehat{X}_N^k,$$

where

$$\begin{aligned}\widehat{b}_5&=35+16\sqrt{5}-72\sqrt{\sqrt{5}-2}-32\sqrt{5\sqrt{5}-10},\\ \widehat{a}_5&=15+\frac{34}{5}\sqrt{5}-18\sqrt{\sqrt{5}-2}-8\sqrt{5\sqrt{5}-10}-\frac{1}{5}\sqrt{1990+890\sqrt{5}},\\ \widehat{X}_5&=-4936-2208\sqrt{5}+10160\sqrt{\sqrt{5}-2}+4544\sqrt{5\sqrt{5}-10}.\end{aligned}$$

We note that identities such as those given in Theorem 3.2 exist only when $\Gamma_0(2\ell)+W_\ell$ has genus 0, or according to [12, p. 14], when $\ell=3,5,7,11,23$. In order to compute $D_\ell(q)$ for primes other than $3,5,7,11$ and 23, we introduce modular functions similar to those used by Ramanujan in his representations of $f(\ell)$.

In Table 1, we state the value of N and in each entry, set

$$\alpha=\alpha(q),\quad \beta=\alpha(q^N).$$

Table 1: Table of identities for $D_N(q)$

$N=3$

$$D_3(q)=-\frac{(\alpha\beta(1-\alpha)(1-\beta))^{1/4}}{2}.$$

$N=5$

Let

$$X=\frac{\left(2^{10}\alpha\beta(1-\alpha)(1-\beta)\right)^{1/6}}{8},$$

then

$$D_5^2(q)=4X^2(1-4X).$$

$$N = 7$$

Let

$$X = \frac{(\alpha\beta(1-\alpha)(1-\beta))^{1/8}}{2},$$

then

$$D_7(q) = -2X(1-3X).$$

$$N = 11$$

Let

$$X = \frac{\left(2^4\alpha\beta(1-\alpha)(1-\beta)\right)^{1/12}}{2},$$

then

$$D_{11}(q) = -2X(1-4X+5X^2).$$

$$N = 13$$

Let

$$X = \frac{(\alpha\beta(1-\alpha)(1-\beta))^{1/2}}{16},$$

and

$$Y = \frac{1-(\alpha\beta)^{1/2}-((1-\alpha)(1-\beta))^{1/2}}{8},$$

then

$$\begin{aligned}&10XD_{13}^4(q) + (-116X - 404XY^2 + 528XY - Y^2 + Y^3 + 1280X^2)D_{13}^2(q)\\&- 16X - 20Y^5 - 16000X^2Y - 176XY^2 + 2112X^2 + 4Y^4\\&+ 37824X^2Y^2 - 3504XY^3 + 8240XY^4 - 23040X^3 + 192XY = 0.\end{aligned}$$

$$N = 17$$

Let

$$X = \frac{\left(2^4\alpha\beta(1-\alpha)(1-\beta)\right)^{1/6}}{4},$$

and

$$Y = \frac{1-(\alpha\beta)^{1/2}-((1-\alpha)(1-\beta))^{1/2}}{8},$$

then

$$D_{17}^2(q) = 4\,\frac{64X^3Y - 11X^2Y - 4X^2 - 24XY + 31XY^2 - 32X^3 + Y^2 + 3X - 8Y^2}{1-Y+5X}.$$

$$N = 19$$

Let

$$X = \frac{(\alpha\beta(1-\alpha)(1-\beta))^{1/4}}{4},$$

and

$$Y = \frac{1-(\alpha\beta)^{1/4}-((1-\alpha)(1-\beta))^{1/4}}{4},$$

then

$$D_{19}(q) = \frac{2X + 2Y + 16Y^3 - 10Y - 18XY}{Y - 1}.$$

$$N = 23$$

Let

$$X = \frac{\left(2^{16}\alpha\beta(1-\alpha)(1-\beta)\right)^{1/24}}{2},$$

then

$$D_{23}(q) = -2X(1 - 5X + 13X^2 - 20X^3 + 20X^4 - 11X^5).$$

$$N = 29$$

Let

$$X = \left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{256}\right)^{1/6},$$

and

$$Y = \frac{1 - (\alpha\beta)^{1/2} - ((1-\alpha)(1-\beta))^{1/2}}{8},$$

then

$$A_2 D_{29}^4(q) + A_1 D_{29}^2(q) + A_0 = 0,$$

where

$$\begin{aligned}
A_2 &= -585689508612X^2,\\
A_1 &= 123736544264XY + 3702335691264X^2 + 97491959398X\\
&\quad + 134904595824360X^4 - 44395652981864X^2Y^2 - 432321617914Y^3\\
&\quad - 42626822690432X^5 - 29875947341036X^3 - 9779263696654XY^2\\
&\quad + 41705207079730X^2Y - 8251360353152X^4Y + 5451791661904XY^3\\
&\quad - 176409878302552X^3Y,\\
A_0 &= 13753900119256887X^2Y^3 - 4877930791543X\\
&\quad + 2618284012843192X^3Y + 2305243907550368XY^3\\
&\quad - 700939761749206XY^2 + 505394444931798X^2Y\\
&\quad + 96399537859592XY - 4086296883979928X^2Y^2\\
&\quad - 14225126607270367X^3Y^2 + 4709822410848252X^5Y\\
&\quad + 25397795278722548X^3Y^3 - 16793873356376932X^4Y^2\\
&\quad + 1068896146837092XY^5 - 175149710486642X^3\\
&\quad + 5709212469240785X^4Y - 3216747114074433XY^4\\
&\quad - 16394572380315964X^2Y^4 - 1065063721978775X^5\\
&\quad - 368335914073064X^4 - 19806094957276X^2\\
&\quad + 6607217263199Y^5 - 204688220825404X^6 - 25269791081528Y^6.
\end{aligned}$$

$$N = 31$$

Let

$$X = \frac{(\alpha\beta(1-\alpha)(1-\beta))^{1/8}}{2},$$

and

$$Y = \frac{1-(\alpha\beta)^{1/8} - ((1-\alpha)(1-\beta))^{1/8}}{8},$$

then

$$D_{31}(q) = -82X^2 + 22X - 1536Y^3 - 8Y - 32XY + 160Y^2 + 896XY^2.$$

Using the identity associated with $D_{29}(q)$, we obtain Borweins' series [5, p. 172] associated with $N = 58$ in Theorem 1.2, namely,

$$\frac{1}{\sqrt{N}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}\left(\widehat{b}_N k + \widehat{a}_N\right)\widehat{X}_N^k,$$

where

$$\begin{aligned}
\widehat{b}_{58} &= -6930\sqrt{2} + 1287\sqrt{58},\\
\widehat{a}_{58} &= -\frac{6351}{2}\sqrt{2} + \frac{68403}{116}\sqrt{58},\\
\widehat{X}_{58} &= -192119201 + 35675640\sqrt{29}.
\end{aligned}$$

As in the case of $N = 13$ for Theorem 1.1, the Borweins derived the above series without the knowledge of $f(29)$ which is not listed in Ramanujan's table for $f(\ell)$.

Remark 4.3. Note that the above table contains identities analogous to Ramanujan's table for $f(\ell)$. In particular, using the expression for $D_{13}(q)$, we obtain the series given in (1.6), etc. The identities in the table were found with the assistance of computer algebra, more precisely with F.G. Garvan's `qseries` package (available at `http://qseries.org/fgarvan/qmaple/qseries/`), using suitable functions such as X and Y given in the table. Once an identity is found, the validity of the identity can be established by first deriving a modular equation from the identity and then by verifying the respective modular equation by the standard technique of comparing the q-series expansions of the modular functions which appear in the modular equation. The identity to be proved is then one of the solutions of the modular equation.

5. Series for $1/\pi$ associated with the cubic theta function $a(q)$

In this section, we consider a cubic analogue of (1.5) and (1.7). Let

$$a(q) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

and

$$\frac{1}{\alpha^\dagger(q^2)} = 1 + \frac{1}{27}\frac{\eta^{12}(\tau)}{\eta^{12}(3\tau)}.$$

The analogues of Theorems 1.1 and 1.2 are respectively given as follows:

Theorem 5.1. *Let $N \geq 2$ be a positive integer,*

$$\alpha_N^\dagger = \alpha^\dagger\Big(e^{-2\pi\sqrt{N/3}}\Big),$$

and

$$C_N(q) = \frac{1}{a^2(q)a^2(q^N)} \det\begin{pmatrix} a(q) & a(q^N) \\ q\dfrac{da(q)}{dq} & q\dfrac{da(q^N)}{dq} \end{pmatrix}.$$

Then

$$\sqrt{\frac{3}{N}}\frac{1}{2\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} \left(b_N^\dagger k + a_N^\dagger\right)\left(X_N^\dagger\right)^k, \tag{5.1}$$

where

$$b_N^\dagger = 1 - 2\alpha_N^\dagger,$$
$$a_N^\dagger = -\frac{C_N(q)}{\sqrt{N}}\bigg|_{q=e^{-2\pi/\sqrt{3N}}},$$

and

$$X_N^\dagger = 4\alpha_N^\dagger\left(1 - \alpha_N^\dagger\right).$$

Theorem 5.2. *Let $N \geq 8$ be a positive integer,*

$$\widehat{\alpha}_N^\dagger = \alpha^\dagger\Big(-e^{-\pi\sqrt{N/3}}\Big),$$

and

$$\widehat{C}_N(q) = \frac{1}{a^2(-q)a^2(-q^N)} \det\begin{pmatrix} a(-q) & a(-q^N) \\ q\dfrac{da(-q)}{dq} & q\dfrac{da(-q^N)}{dq} \end{pmatrix}.$$

Then

$$\sqrt{\frac{3}{N}}\frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} \left(\widehat{b}_N^\dagger k + \widehat{a}_N^\dagger\right)\left(\widehat{X}_N^\dagger\right)^k, \tag{5.2}$$

where

$$\widehat{b}_N^\dagger = 1 - 2\widehat{\alpha}_N^\dagger,$$
$$\widehat{a}_N^\dagger = \frac{\widehat{C}_N(q)}{\sqrt{N}}\bigg|_{q=e^{-\pi/\sqrt{3N}}},$$

and

$$\widehat{X}_N^\dagger = 4\widehat{\alpha}_N^\dagger\left(1 - \widehat{\alpha}_N^\dagger\right).$$

We now state a few identities for the cubic case similar to those in Theorem 3.2.

Theorem 5.3. *The following hold:*
Let

$$H_\ell^\dagger = H_\ell^\dagger(\tau) = \left(\frac{\eta(3\tau)\eta(3\ell\tau)}{\eta(\tau)\eta(\ell\tau)}\right)^{\frac{12}{l+1}}.$$

Then

$$C_2(q^2) = -6\frac{H_2^\dagger}{(1+9H_2^\dagger)^2}, \tag{5.3}$$

$$C_5(q^2) = -6H_5^\dagger\frac{1+4H_5^\dagger+9H_5^{\dagger\,2}}{(1+9H_5^\dagger+9H_5^{\dagger\,2})^2},$$

$$C_{11}(q^2) = -6H_{11}^\dagger(\tau)\frac{U(H_{11}^\dagger(\tau))}{V^2(H_{11}^\dagger(\tau))},$$

where

$$U(s) = 1+5s+18s^2+37s^3+54s^4+45s^5+27s^6,$$

and

$$V(s) = 1+9s+18s^2+27s^3+9s^4.$$

The examples of (5.1) which follow from Theorem 5.3 are given as follows:

5.1. **Case** $N = 2$**.**

When $N=2$, $\alpha_2^\dagger = \dfrac{\sqrt{2}-1}{2\sqrt{2}}$, $b_2^\dagger = \dfrac{\sqrt{2}}{2}$ and $X_2^\dagger = \dfrac{1}{2}$. Using (5.3) and the fact that

$$H_2^\dagger(i/\sqrt{6}) = \frac{1}{9},$$

which follows from two instances of (2.8), we deduce that

$$C_2(e^{-2\pi/\sqrt{6}}) = -\frac{1}{6}.$$

The series for $1/\pi$ in this case is

$$\frac{3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}\frac{\left(\frac12\right)_k\left(\frac13\right)_k\left(\frac23\right)_k}{(k!)^3}(6k+1)\frac{1}{2^k}.$$

For case $N=5$ and 11, identity (5.1) holds for the following values:

$$b_5^\dagger = \frac{11}{23}\sqrt{5}, \qquad a_5^\dagger = \frac{4}{75}\sqrt{5}, \qquad X_5^\dagger = \frac{4}{125}$$

and

$$b_{11}^\dagger = -\frac{5}{242}\sqrt{11}+\frac{45}{242}\sqrt{33}, \quad a_{11}^\dagger = -\frac{13}{726}\sqrt{11}+\frac{3}{121}\sqrt{33}, \quad X_{11}^\dagger = -\frac{194}{1331}+\frac{225}{2662}\sqrt{3}.$$

When $N=2$ and 5,

$$\widehat{X}_2^\dagger = -256-153\sqrt{3} \quad \text{and} \quad \widehat{X}_5^\dagger = -4,$$

which have absolute values greater than 1. This implies that the right-hand side of (5.2) diverges. In other words, the only identity from Theorem 5.3 that leads to a series for $1/\pi$ via (5.2) is when $N = 11$ and is given by

$$\frac{\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} \left(\left(\frac{45}{22}\sqrt{3} + \frac{5}{22}\right) k + \frac{13}{66} + \frac{3}{11}\sqrt{3}\right) \left(-\frac{194}{1331} - \frac{225}{2662}\sqrt{3}\right)^k .$$

We end this section with cubic analogues of Ramanujan's identities for $f(\ell)$. In Table 2, we will state the value of N and in each entry, set

$$\alpha^\dagger = \alpha^\dagger(q), \quad \beta^\dagger = \alpha^\dagger(q^N).$$

Table 2: Table of identities for $C_N(q)$

$N = 2$

Let

$$X = \frac{\left(\alpha^\dagger \beta^\dagger (1-\alpha^\dagger)(1-\beta^\dagger)\right)^{1/3}}{9},$$

then

$$C_2(q) = -6X.$$

$N = 5$

Let

$$X = \frac{\left(\alpha^\dagger \beta^\dagger (1-\alpha^\dagger)(1-\beta^\dagger)\right)^{1/6}}{3},$$

then

$$C_5(q) = -6X(1-5X).$$

$N = 11$

Let

$$X = \frac{\left(\alpha^\dagger \beta^\dagger (1-\alpha^\dagger)(1-\beta^\dagger)\right)^{1/6}}{3},$$

and

$$Y = \frac{1 - \left(\alpha^\dagger \beta^\dagger\right)^{1/3} - \left((1-\alpha^\dagger)(1-\beta^\dagger)\right)^{1/3}}{9},$$

then

$$C_{11}(q) = -33XY + 3X - 6Y + 33Y^2.$$

$N = 17$

Let

$$X = \frac{\left(\alpha^\dagger \beta^\dagger (1-\alpha^\dagger)(1-\beta^\dagger)\right)^{1/6}}{3},$$

and

$$Y = \frac{1 - \left(\alpha^\dagger \beta^\dagger\right)^{1/3} - \left((1-\alpha^\dagger)(1-\beta^\dagger)\right)^{1/3}}{9},$$

then

$$C_{17}(q) = \frac{6\left(-2X^2+34X^2Y+51XY^2-14Y^2-9XY+Y+51Y^3\right)}{8Y-1}.$$

Remark 5.1. In the above table, we give an identity for $\ell = 17$ to illustrate the fact that we can compute $C_\ell(q)$ even when the genus of $\Gamma_0(3\ell)+W_\ell$ is not zero. Applying Theorem 5.2, together with the identity given above for $C_{17}(q)$ and the value

$$\widehat{\alpha}_{17}^{\dagger} = \frac{1}{2} - \frac{\sqrt{17}}{8},$$

we obtain the series

$$\frac{12\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} (51k+7)\left(\frac{-1}{16}\right)^k,$$

which was discovered by Chan, Liaw and Tan [14, (1.15)].

6. Quartic theory and Ramanujan's most famous series for $1/\pi$

In 1985, B. Gosper brought Ramanujan's series for $1/\pi$ to the attention of the mathematical community by computing 17526200 digits of π using the series

$$\frac{1}{\pi} = 2\sqrt{2}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^3} (1103+26390k)\left(\frac{1}{99^2}\right)^{2k+1} \tag{6.1}$$

(see [1, p. 387 and p. 685]). Series (6.1) was discussed in the book by the Borweins (see [5, (5.5.23)]), where they remarked that they computed $\alpha(58)$ (see [5, (5.1.2)]) by calculating a certain number $d_0(58)$ (see [5, (5.5.16)]) to high precision. In other words, it appears that a rigorous proof has not been found for (6.1).

In this section, we will give a proof of (6.1). Identity (6.1) belongs to the quartic theory (cf. [3]) and a quartic analogue of Theorem 1.1 is given by the following Theorem:

Theorem 6.1. *Let*

$$A^2(q^2) = \frac{\eta^{16}(\tau)}{\eta^8(2\tau)}\left(1+32\frac{\eta^8(4\tau)}{\eta^8(\tau)}\right)^2 = \frac{\eta^{16}(\tau)}{\eta^8(2\tau)}\left(1+64\frac{\eta^{24}(2\tau)}{\eta^{24}(\tau)}\right),$$

and

$$\frac{1}{\alpha^{\perp}(q^2)} = 1 + \frac{1}{64}\frac{\eta^{24}(\tau)}{\eta^{24}(2\tau)}.$$

Let

$$\alpha_N^{\perp} = \alpha^{\perp}\left(e^{-\pi\sqrt{2N}}\right),$$

and

$$D_N^{\perp}(q) = \frac{1}{\sqrt{A^3(q)A^3(q^N)}} \det\begin{pmatrix} A(q) & A(q^N) \\ q\dfrac{dA(q)}{dq} & q\dfrac{dA(q^N)}{dq}\end{pmatrix}.$$

Then

$$\sqrt{\frac{2}{N}}\frac{1}{2\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(k!)^3}\left(b_N^{\perp}k + a_N^{\perp}\right)\left(X_N^{\perp}\right)^k,$$

where

$$\begin{aligned} b_N^{\perp} &= 1 - 2\alpha_N^{\perp}, \\ a_N^{\perp} &= -\left.\frac{D_N^{\perp}(q)}{2\sqrt{N}}\right|_{q=e^{-2\pi/\sqrt{2N}}}, \end{aligned}$$

and

$$X_N^{\perp} = 4\alpha_N^{\perp}\left(1 - \alpha_N^{\perp}\right),$$

In order to derive series for $1/\pi$ using Theorem (6.1), it appears that we need to construct formulas analogous to those for $D_N(q)$ given in Table 1 in Section 5 for the function $D_N^{\perp}(q)$. Fortunately, this turns out to be unnecessary. We will show that the knowledge of $D_N(q)$ is all we need in order to compute $D_N^{\perp}(q)$. We begin with our discussion with the following Theorem:

Theorem 6.2. *Let $Z(q) = \vartheta_3^4(q)$. Then*

$$\begin{aligned} -D_\ell^{\perp}(q) = \sqrt{\frac{Z(q)Z(q^\ell)}{A(q)A(q^\ell)}}\Bigg(&\frac{1}{1+\alpha(q)}\sqrt{\frac{Z(q)}{Z(q^\ell)}}\alpha(q)(1-\alpha(q)) \\ &- \frac{\ell}{1+\alpha(q^\ell)}\sqrt{\frac{Z(q^\ell)}{Z(q)}}\alpha(q^\ell)(1-\alpha(q^\ell)) - 4D_\ell(q)\Bigg). \end{aligned} \tag{6.2}$$

Proof. The proof of (6.2) follows from the identity

$$A(q) = (1 + \alpha(q))\, Z(q), \tag{6.3}$$

which follows by observing that $A(q^2)/Z(q^2)$ is a modular function on $\Gamma_0(4)$. Using (6.3), we deduce that

$$\frac{A(q)}{A(q^\ell)} = \frac{1+\alpha(q)}{1+\alpha(q^\ell)}\frac{Z(q)}{Z(q^\ell)}. \tag{6.4}$$

Logarithmically differentiating (6.4), identifying the resulting expressions with $D_\ell(q)$ and $D_\ell^{\perp}(q)$, and using the identity

$$q\frac{d\alpha(q)}{dq} = Z(q)\alpha(1-\alpha),$$

we complete the proof of (6.2). □

Identity (6.2) and Theorem 6.1 allow us to derive any series for $1/\pi$ for a positive integer N from identities for $D_N(q)$ given in Table 1. For example, when $N = 3$, we find, using the identity for $D_3(q)$ given in Section 4, that

$$\begin{aligned} \alpha_6 &= 35 + 24\sqrt{2} - 20\sqrt{3} - 14\sqrt{6}, \\ \alpha_{2/3} &= 35 - 24\sqrt{2} - 20\sqrt{3} + 14\sqrt{6}, \\ D_3(e^{-\pi\sqrt{2/3}}) &= \frac{5}{2} - \frac{3\sqrt{3}}{2}, \end{aligned}$$

$$\sqrt{\frac{Z(e^{-\pi\sqrt{2/3}})}{Z(e^{-\pi\sqrt{6}})}} = 3 - 2\sqrt{3} + 3\sqrt{2} - \sqrt{6},$$

$$\sqrt{\frac{Z(e^{-\pi\sqrt{2/3}})Z(e^{-\pi\sqrt{6}})}{A(e^{-\pi\sqrt{2/3}})A(e^{-\pi\sqrt{6}})}} = \frac{1}{\sqrt{6}} + \frac{\sqrt{2}}{4}.$$

This yields

$$-D_3^{\perp}(e^{-\pi\sqrt{2/3}}) = \frac{1}{\sqrt{6}} \quad \text{and} \quad a_6^{\perp} = \frac{\sqrt{2}}{12},$$

and we deduce the series

$$\frac{1}{\sqrt{6}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(k!)^3} \left(\frac{2\sqrt{2}}{3}k + \frac{\sqrt{2}}{12}\right) \frac{1}{9^k}.$$

Similarly, when $N = 29$, we find, using the modular equation for $D_{29}(q)$ derived in Section 4, that

$$\begin{aligned}
\alpha_{58} &= 384238403 + 71351280\sqrt{29} - 50452974\sqrt{58} - 271697580\sqrt{2}, \\
\alpha_{2/29} &= 384238403 - 71351280\sqrt{29} - 50452974\sqrt{58} + 271697580\sqrt{2}, \\
D_{29}(e^{-\pi\sqrt{2/29}}) &= 6351\sqrt{29} - 24184\sqrt{2}, \\
\sqrt{\frac{Z(e^{-\pi\sqrt{2/29}})}{Z(e^{-\pi\sqrt{58}})}} &= 37323 + 6930\sqrt{29} - 26390\sqrt{2} - 4900\sqrt{58}, \\
\sqrt{\frac{Z(e^{-\pi\sqrt{2/29}})Z(e^{-\pi\sqrt{58}})}{A(e^{-\pi\sqrt{2/29}})A(e^{-\pi\sqrt{58}})}} &= \frac{13}{198}\sqrt{29} + \frac{1}{4}\sqrt{2}.
\end{aligned}$$

This yields

$$-D_{29}^{\perp}(e^{-\pi\sqrt{2/29}}) = \frac{4412}{9801} \quad \text{and} \quad a_{29}^{\perp} = \frac{2206\sqrt{2}}{284229}.$$

Together with

$$b_{29}^{\perp} = \frac{1820}{9081}\sqrt{29} \quad \text{and} \quad X_{29}^{\perp} = \frac{1}{99^4},$$

we complete the proof of Ramanujan's series (6.1).

Remark 6.1. We were made aware that an unpublished proof of Ramanujan's series (6.1) was discovered around 2015 by Yue Zhao [22], a young Electrical Engineering student from Tsinghua University. Shaun Cooper also discovered another proof of (6.1) shortly after the discovery of our proof.

Zhao also gave a first proof of Ramanujan's series [5, p. 187]

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^3} (1123 + 21460k) \left(\frac{1}{882}\right)^{2k+1},$$

which corresponds to $N = 37$. A proof of the above identity using the method illustrated here would require an identity associated with $D_{37}(q)$ which is not present in this article.

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