SUMMATION, TRANSFORMATION, AND EXPANSION FORMULAS FOR MULTIBASIC THETA HYPERGEOMETRIC SERIES

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Abstract. After reviewing some fundamental facts from the theory of theta hypergeometric series we derive, using indefinite summation, several summation, transformation, and expansion formulas for multibasic theta hypergeometric series. Some of the identities presented here generalize corresponding formulas given in Chapter 11 of the Gasper and Rahman book [*Basic hypergeometric series*, 2^{nd} ed., Encyclopedia of Mathematics And Its Applications 96, Cambridge University Press, Cambridge, 2004].

1. Introduction

By convention, a series $\sum u_n$ is called a hypergeometric series if $g(n) = u_{n+1}/u_n$ is a rational function of n. It is called a q- (or basic) hypergeometric series if $g(n)$ is a rational function of q^n. More generally, such a series is called an *elliptic hypergeometric series* if $g(n)$ is an elliptic (doubly periodic meromorphic) function of n with n considered as a complex variable. We refer the reader to [8, Ch. 11] for some motivation for considering these three classes of series and to [8] in general for a treatise on basic hypergeometric series.

In a path-breaking paper, Frenkel and Turaev [6] in their work on elliptic $6j$-symbols (introduced by Date *et al.* [5] as elliptic solutions of the Yang-Baxter equation [1, 2]) introduced elliptic analogues of very-well-poised basic hypergeometric series. In particular, using the tetrahedral symmetry of the elliptic $6j$-symbols and the finite dimensionality of cusp forms, they derived elliptic analogues of Bailey's transformation formula (cf. [8, Eq. (2.9.1)]) for terminating ${}_{10}\phi_9$ series and of Jackson's ${}_8\phi_7$ summation formula (cf. [8, Eq. (2.6.2)]). Elliptic hypergeometric series and their extensions to theta hypergeometric series became an increasingly active area of research (see [8, Sec. 11.1] for some references). So far, many formulas for very-well-poised basic hypergeometric series have already been extended to the elliptic setting. Some formulas for *multibasic* elliptic hypergeometric series appeared in work of Warnaar [10]. Here we consider yet other

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identities involving multiple bases and theta functions, special cases of which have already been presented in [8].

We start in Section 2 with the elliptic shifted factorials, Spiridonov's [9] ${}_{r+1}E_r$ theta hypergeometric series notation and its very-well-poised ${}_{r+1}V_r$ special case, and then point out some of their main properties. We also present the Frenkel and Turaev summation and transformation formulas. In Section 3 we derive theta hypergeometric extensions of some of the summation and transformation formulas in [8, Secs. 3.6–3.8]. To give just one example, here is a transformation formula for a "split-poised" theta hypergeometric ${}_{12}E_{11}$ series

$$\begin{aligned}
&{}_{12}E_{11}\big(a, qa^{\frac12}, -qa^{\frac12}, qa^{\frac12}/p^{\frac12}, -qa^{\frac12}p^{\frac12}, b, c, a/bc, q^{-n}, B/Aq^n, C/Aq^n, 1/BCq^n;\\
&\qquad a^{\frac12}, -a^{\frac12}, a^{\frac12}p^{\frac12}, -a^{\frac12}/p^{\frac12}, aq/b, aq/c, bcq, 1/Aq^n, 1/Bq^n, 1/Cq^n, BC/Aq^n;\\
&\qquad\qquad q, p; -1\big)\\
&= \frac{(aq, bq, cq, aq/bc, Aq/B, Aq/C, BCq; q, p)_n}{(Aq, Bq, Cq, Aq/BC, aq/b, aq/c, bcq; q, p)_n}\\
&\times {}_{12}E_{11}\big(A, qA^{\frac12}, -qA^{\frac12}, qA^{\frac12}/p^{\frac12}, -qA^{\frac12}p^{\frac12}, B, C, A/BC, q^{-n}, b/aq^n, c/aq^n, 1/bcq^n;\\
&\qquad A^{\frac12}, -A^{\frac12}, A^{\frac12}p^{\frac12}, -A^{\frac12}/p^{\frac12}, Aq/B, Aq/C, BCq, 1/aq^n, 1/bq^n, 1/cq^n, bc/aq^n;\\
&\qquad\qquad q, p; -1\big)
\end{aligned} \tag{1.1}$$

for $n = 0, 1, \ldots$ (see (2.3) for the notation), a recast of (3.24), which extends the transformation formula for a split-poised ${}_{10}\phi_9$ series given in [8, Ex. 3.21]. Most of these extensions have recently been presented in [8, Ch. 11], where reference was made to an earlier (2003) version of this paper. However, some formulas in [8] (in particular, in Sec. 11.6 and Exercises 11.25–11.26) have been further generalized in the current version of this paper. The selection of formulas we give is by no means exhaustive, but they do serve to illustrate some of the possibilities for deriving summations, transformations and expansions for multibasic theta hypergeometric series. We wish to thank Mizan Rahman and Ole Warnaar for some helpful correspondences.

2. Elliptic and theta hypergeometric series

As in [8] we define a modified Jacobi theta function with argument x and nome p by

$$\theta(x;p) = (x, p/x; p)_\infty = (x;p)_\infty (p/x;p)_\infty, \qquad \theta(x_1, \ldots, x_m; p) = \prod_{k=1}^{m} \theta(x_k; p), \tag{2.1}$$

where $x, x_1, \ldots, x_m \neq 0$, $|p| < 1$, and $(x;p)_\infty = \prod_{k=0}^\infty (1 - xp^k)$. Also, following Warnaar [10], we define an *elliptic* (or *theta*) *shifted factorial* analogue of the

q-shifted factorial by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1} \theta(aq^k;p), & n = 1,2,\dots, \\ 1, & \text{n} = 0, \\ 1/\prod_{k=0}^{-n-1} \theta(aq^{n+k};p), & n = -1,-2,\dots, \end{cases} \tag{2.2}$$

and let

$$(a_1,a_2,\dots,a_m;q,p)_n = \prod_{k=1}^{m} (a_k;q,p)_n,$$

where $a, a_1, \dots, a_m \neq 0$. Notice that $\theta(x;0) = 1-x$ and, hence, $(a;q,0)_n = (a;q)_n$ is a *q-shifted factorial* in base q. Thus, the parameters q and p in $(a;q,p)_n$ are called the *base* and *nome*, respectively, and $(a;q,p)_n$ is called the *q,p-shifted factorial.* A list of useful identities for manipulating the q,p-shifted factorials (and related objects such as *q,p-binomial coefficients*, or *elliptic binomial coefficients*, and *elliptic gamma functions*) is given in [8, Sec. 11.2].

Following Spiridonov [9], an ${}_{r+1}E_r$ *theta hypergeometric series* with base q and nome p is formally defined by

$$\begin{aligned} &{}_{r+1}E_r(a_1,a_2,\dots,a_{r+1};b_1,\dots,b_r;q,p;z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_{r+1};q,p)_n}{(q,b_1,\dots,b_r;q,p)_n} z^n, \end{aligned} \tag{2.3}$$

where, as usual, it is assumed that the parameters are such that each term in the series is well-defined. If z and the a's and b's are independent of p, then it follows that

$$\begin{aligned} &\lim_{p\to 0} {}_{r+1}E_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;q,p;z) \\ &= {}_{r+1}E_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;q,0;z) \\ &= {}_{r+1}\phi_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;q,z), \end{aligned}$$

where the limit of the series is a termwise limit. See [8, Sec. 11.2] for more details and a discussion of convergence of the series in (2.3).

As in [9], a (unilateral or bilateral) series $\sum c_n$ is called an *elliptic hypergeometric series* if $g(n) = c_{n+1}/c_n$ is an elliptic function of n with n considered as a complex variable; i.e., the function $g(x)$ is a doubly periodic meromorphic function of the complex variable x. For the ${}_{r+1}E_r$ series in (2.3) it is clear that

$$g(x) = z \prod_{k=1}^{r+1} \frac{\theta(a_k q^x;p)}{\theta(b_k q^x;p)}$$

with $b_{r+1} = q$. It is not difficult to show (see [8]) that when

$$a_1a_2\dots a_{r+1} = (b_1b_2\dots b_r)q, \tag{2.4}$$

$g(x)$ is an elliptic (i.e., doubly periodic meromorphic) function of x. Therefore, (2.4) is called the *elliptic balancing condition*, and ${}_{r+1}E_r$ is said to be *elliptically balanced* (*E-balanced*) when (2.4) holds.

Corresponding to the basic hypergeometric special case (cf. [8]), the ${}_{r+1}E_r$ series in (2.3) is called *well-poised* if

$$qa_1 = a_2b_1 = a_3b_2 = \ldots = a_{r+1}b_r, \tag{2.5}$$

in which case we find that the elliptic balancing condition (2.4) reduces to

$$a_1^2 a_2^2 \cdots a_{r+1}^2 = (a_1 q)^{r+1}.$$

Using (2.2) we see that

$$\frac{\theta(aq^{2n};p)}{\theta(a;p)} = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, qa^{\frac{1}{2}}/p^{\frac{1}{2}}, -qa^{\frac{1}{2}}p^{\frac{1}{2}};q,p)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, a^{\frac{1}{2}}p^{\frac{1}{2}}, -a^{\frac{1}{2}}/p^{\frac{1}{2}};q,p)_n}(-q)^{-n}$$

is an elliptic analogue of the quotient

$$\frac{1-aq^{2n}}{1-a} = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}};q)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}};q)_n},$$

which is clearly the very-well-poised part of the very-well-poised basic hypergeometric ${}_{r+1}W_r$ series in [8, Eq. (2.1.11)]. Hence, the ${}_{r+1}E_r$ series in (2.3) is called *very-well-poised* when it is well-poised, $r \geq 4$, and

$$a_2 = qa_1^{\frac{1}{2}},\ a_3 = -qa_1^{\frac{1}{2}},\ a_4 = qa_1^{\frac{1}{2}}/p^{\frac{1}{2}},\ a_5 = -qa_1^{\frac{1}{2}}p^{\frac{1}{2}}. \tag{2.6}$$

Analogous to Spiridonov [9, Eq. (2.15)], an ${}_{r+1}V_r$ very-well-poised theta hypergeometric series is defined by

$$\begin{aligned}&{}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p;z)\\ &= \sum_{n=0}^{\infty} \frac{\theta(a_1q^{2n};p)}{\theta(a_1;p)} \frac{(a_1,a_6,a_7,\ldots,a_{r+1};q,p)_n}{(q,a_1q/a_6,a_1q/a_7,\ldots,a_1q/a_{r+1};q,p)_n}(qz)^n.\end{aligned} \tag{2.7}$$

Thus, if (2.5) and (2.6) hold, then

$$\begin{aligned}&{}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p;z)\\ &= {}_{r+1}E_r(a_1,a_2,\ldots,a_{r+1};b_1,\ldots,b_r;q,p;-z),\end{aligned}$$

and the ${}_{r+1}V_r$ series is elliptically balanced if and only if

$$(a_6^2 a_7^2 \cdots a_{r+1}^2)q^2 = (a_1 q)^{r-5}.$$

If the argument z in the ${}_{r+1}V_r$ series equals 1, then we suppress it and denote the series in (2.7) by the simpler notation ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)$. When the parameters $a_1,a_6,a_7,\ldots,a_{r+1}$ are independent of p,

$$\begin{aligned}&\lim_{p\to 0} {}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)\\ &= {}_{r-1}W_{r-2}(a_1;a_6,\ldots,a_{r+1};q,q),\end{aligned}$$

from which it follows that there is a shift $r \to r-2$ when taking the $p \to 0$ limit, and that the $p \to 0$ limit of a ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)$ series with $a_1,a_6,a_7,\ldots,a_{r+1}$ independent of p is a ${}_{r-1}W_{r-2}$ series.

Frenkel and Turaev [6] showed the following elliptic analogue of Bailey's ${}_{10}\phi_9$ transformation formula [8, Eq. (2.9.1)]

$$\begin{aligned}
&{}_{12}V_{11}(a; b, c, d, e, f, \lambda a q^{n+1}/ef, q^{-n}; q, p) \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q, p)_n} \\
&\quad \times {}_{12}V_{11}(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda a q^{n+1}/ef, q^{-n}; q, p)
\end{aligned} \tag{2.8}$$

for $n = 0, 1, \ldots$, provided that the balancing condition

$$bcdef(\lambda a q^{n+1}/ef)q^{-n}q = (aq)^3, \tag{2.9}$$

which is clearly equivalent to $\lambda = qa^2/bcd$, holds. Notice that each of the series in (2.8) is E-balanced when (2.9) holds. If we set $\lambda = a/d$ in (2.8), then we obtain a summation formula for a ${}_{10}V_9$ series which is an elliptic analogue of Jackson's ${}_8\phi_7$ summation formula [8, Eq. (2.6.2)] and of Dougall's ${}_7F_6$ summation formula [8, Eq. (2.1.6)]. After a change in parameters, this summation formula can be written in the form:

$${}_{10}V_9(a; b, c, d, e, q^{-n}; q, p) = \frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_n}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_n} \tag{2.10}$$

for $n = 0, 1, \ldots$, provided that the elliptic balancing condition $bcde = a^2 q^{n+1}$, which can be written in the form

$$(bcdeq^{-n})q = (aq)^2, \tag{2.11}$$

holds. It is obvious that if a, b, c, d, e are independent of p, then (2.10) tends to Jackson's ${}_8\phi_7$ summation formula [8, Eq. (2.6.2)] as $p \to 0$. For a further discussion of (2.8) and (2.10) including different proofs, see Gasper and Rahman [8, Secs. 11.2, 11.4, 11.5].

3. Multibasic summation and transformation formulas for theta hypergeometric series

We first observe that if the parameter a in (2.10) is replaced by a/q, then it follows that the $n = 1$ case of (2.10) is equivalent to the identity

$$1 - \frac{\theta(b, c, d, a^2/bcd; p)}{\theta(a/b, a/c, a/d, bcd/a; p)} = \frac{\theta(a, a/bc, a/bd, a/cd; p)}{\theta(a/bcd, a/d, a/c, a/b; p)}. \tag{3.1}$$

More generally, by replacing a in (2.8) by a/q it follows that the $n = 1$ case of (2.8) is equivalent to the identity

$$\begin{aligned}
&1 - \frac{\theta(b, c, d, e, f, g; p)}{\theta(a/b, a/c, a/d, a/e, a/f, a/g; p)} \\
&= \frac{\theta(a, a/ef, a^2/bcde, a^2/bcdf; p)}{\theta(a^2/bcdef, a^2/bcd, a/f, a/e; p)} \\
&\quad \times \left[1 - \frac{\theta(a/bc, a/bd, a/cd, e, f, g; p)}{\theta(a/d, a/c, a/b, a^2/bcde, a^2/bcdf, a^2/bcdg; p)}\right]
\end{aligned} \tag{3.2}$$

with $a^3 = bcdefg$, which is equivalent to the identity in [8, Ex. 5.22]. Next, define

$$\prod_{k=m}^{n} a_k = \begin{cases} a_m a_{m+1} \cdots a_n, & m \le n, \\ 1, & m = n+1, \\ (a_{n+1} a_{n+2} \cdots a_{m-1})^{-1}, & m \ge n+2, \end{cases} \tag{3.3}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, and let

$$U_n = \prod_{k=0}^{n-1} \frac{\theta(b_k, c_k, d_k, e_k, f_k, g_k; p)}{\theta(a_k/b_k, a_k/c_k, a_k/d_k, a_k/e_k, a_k/f_k, a_k/g_k; p)} \tag{3.4}$$

where $a_k^3 = b_k c_k d_k e_k f_k g_k$ for $k = 0, \pm 1, \pm 2, \ldots$, and it is assumed that the a's, b's, c's, d's, e's, f's, g's, are complex numbers such that U_n is well defined for $n = 0, \pm 1, \pm 2, \ldots$. Now use (3.2) with a, b, c, d, e, f, g replaced by a_k, b_k, c_k, d_k, e_k, f_k, g_k respectively, to get the indefinite summation formula

$$\begin{aligned} U_{-m} - U_{n+1} &= \sum_{k=-m}^{n} (U_k - U_{k+1}) \\ &= \sum_{k=-m}^{n} \frac{\theta(a_k^2/b_k c_k d_k f_k, a_k^2/b_k c_k d_k e_k, a_k, a_k/e_k f_k; p)}{\theta(a_k/e_k, a_k/f_k, a_k^2/b_k c_k d_k e_k f_k, a_k^2/b_k c_k d_k; p)} U_k \\ &\times \left[1 - \frac{\theta(a_k/c_k d_k, a_k/b_k d_k, a_k/b_k c_k, e_k, f_k, g_k; p)}{\theta(a_k/b_k, a_k/c_k, a_k/d_k, a_k^2/b_k c_k d_k e_k, a_k^2/b_k c_k d_k f_k, a_k^2/b_k c_k d_k g_k; p)}\right] \end{aligned} \tag{3.5}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, where $a_k^3 = b_k c_k d_k e_k f_k g_k$ for $k = 0, \pm 1, \pm 2, \ldots$. Since $U_0 = 1$ by (3.3) and $\theta(a_k/c_k d_k; p) = 0$ when $a_k = c_k d_k$, setting $m = 0$ and $a_k = c_k d_k$ for $k = 0, 1, \ldots, n$ in (3.5) yields after relabelling the summation formula

$$\begin{aligned} &\sum_{k=0}^{n} \frac{\theta(a_k, a_k/b_k c_k, a_k/b_k d_k, a_k/c_k d_k; p)}{\theta(a_k/b_k c_k d_k, a_k/d_k, a_k/c_k, a_k/b_k; p)} \\ &\qquad \times \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, a_j^2/b_j c_j d_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_j c_j d_j/a_j; p)} \\ &\qquad = 1 - \prod_{j=0}^{n} \frac{\theta(b_j, c_j, d_j, a_j^2/b_j c_j d_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_j c_j d_j/a_j; p)} \end{aligned} \tag{3.6}$$

for $n = 0, 1, \ldots$, which is equivalent to Warnaar's formula [10, Eq. (3.2)]. When $p = 0$ the above formula reduces to a summation formula of Macdonald that was first published in Bhatnagar and Milne [3, Thm. 2.27], and contains the summation formulas by W. Chu [4, Thms. A, B, C] as special cases.

Observe that in (3.1), (3.2), (3.4), (3.5) and (3.6) the components of each quotient of products of theta functions have been arranged so that the well-poised property of these quotients is clearly displayed; e.g., in the second sum in (3.5) the quotient of the theta functions in front of U_k is arranged so that each product of corresponding numerator and denominator parameters equals

$a_k^3/b_kc_kd_ke_kf_k$, and each of the corresponding products in the quotient of theta functions inside the square bracket equals $a_k^2/b_kc_kd_k$.

If we let

$$a_k = aw^k,\ b_k = bq^k,\ c_k = cr^k,\ d_k = ds^k,\ e_k = et^k,\ f_k = fu^k,\ g_k = gv^k,$$

with $a^3 = bcdefg$ and $w^3 = qrstuv$, then the product U_n reduces to

$$\begin{aligned}\tilde{U}_n =& \frac{(b;q,p)_n(c;r,p)_n(d;s,p)_n}{(a/b;w/q,p)_n(a/c;w/r,p)_n(a/d;w/s,p)_n}\\ &\times \frac{(e;t,p)_n(f;u,p)_n(g;v,p)_n}{(a/e;w/t,p)_n(a/f;w/u,p)_n(a/g;w/v,p)_n}\end{aligned}$$

and, by applying (3.5) and some elementary identities for q,p-shifted factorials (listed in [8, Sec. 11.2]), we obtain the following indefinite multibasic theta hypergeometric summation formula

$$\begin{aligned}\sum_{k=-m}^{n} & \frac{\theta(aw^k, a(w/tu)^k/ef, fg(uv/w)^k/a, eg(tv/w)^k/a; p)}{\theta(g(v/w)^k/a, efg(tuv/w)^k/a, a(w/u)^k/f, a(w/t)^k/e; p)}\\ &\times \frac{(b;q,p)_k(c;r,p)_k(d;s,p)_k}{(a/b;w/q,p)_k(a/c;w/r,p)_k(a/d;w/s,p)_k}\\ &\times \frac{(e;t,p)_k(f;u,p)_k(g;v,p)_k}{(a/e;w/t,p)_k(a/f;w/u,p)_k(a/g;w/v,p)_k}\\ &\times \Bigg[1 - \frac{\theta(a(w/rs)^k/cd, a(w/qs)^k/bd, a(w/qr)^k/bc; p)}{\theta(a(w/q)^k/b, a(w/r)^k/c, a(w/s)^k/d; p)}\\ &\qquad \times \frac{\theta(et^k, fu^k, gv^k; p)}{\theta(fg(uv/w)^k/a, eg(tv/w)^k/a, ef(tu/w)^k/a; p)}\Bigg]\\ =& \frac{(bw/aq;w/q,p)_m(cw/ar;w/r,p)_m(dw/as;w/s,p)_m}{(q/b;q,p)_m(r/c;r,p)_m(s/d;s,p)_m}\\ &\times \frac{(ew/at;w/t,p)_m(fw/au;w/u,p)_m(gw/av;w/v,p)_m}{(t/e;t,p)_m(u/f;u,p)_m(v/g;v,p)_m}\\ &- \frac{(b;q,p)_{n+1}(c;r,p)_{n+1}(d;s,p)_{n+1}}{(a/b;w/q,p)_{n+1}(a/c;w/r,p)_{n+1}(a/d;w/s,p)_{n+1}}\\ &\quad \times \frac{(e;t,p)_{n+1}(f;u,p)_{n+1}(g;v,p)_{n+1}}{(a/e;w/t,p)_{n+1}(a/f;w/u,p)_{n+1}(a/g;w/v,p)_{n+1}}\end{aligned} \tag{3.7}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, where $a^3 = bcdefg$ and $w^3 = qrstuv$.

If we set $p = 0$ and assume that

$$\max(|q|, |r|, |s|, |t|, |u|, |v|, |w/q|, |w/r|, |w/s|, |w/t|, |w/u|, |w/v|) < 1,$$

then letting n or m in (3.7) tend to infinity shows that this special case of (3.7) also holds with n and/or m replaced by ∞, just as in the special case [8, Eq. (3.6.14)]. Thus we have extended [8, Eq. (3.6.14)] to the bilateral multibasic summation formula

$$\sum_{k=-\infty}^{\infty} \frac{(1-aw^k)(1-a(w/tu)^k/ef)(1-fg(uv/w)^k/a)(1-eg(tv/w)^k/a)}{(1-g(v/w)^k/a)(1-efg(tuv/w)^k/a)(1-a(w/u)^k/f)(1-a(w/t)^k/e)}$$
$$\times \frac{(b;q)_k(c;r)_k(d;s)_k(e;t)_k(f;u)_k(g;v)_k}{(a/b;w/q)_k(a/c;w/r)_k(a/d;w/s)_k(a/e;w/t)_k(a/f;w/u)_k(a/g;w/v)_k}$$
$$\times \left[1-\frac{(1-a(w/rs)^k/cd)(1-a(w/qs)^k/bd)(1-a(w/qr)^k/bc)}{(1-a(w/q)^k/b)(1-a(w/r)^k/c)(1-a(w/s)^k/d)}\right.$$
$$\left.\times \frac{(1-et^k)(1-fu^k)(1-gv^k)}{(1-fg(uv/w)^k/a)(1-eg(tv/w)^k/a)(1-ef(tu/w)^k/a)}\right]$$
$$= \frac{(bw/aq;w/q)_\infty(cw/ar;w/r)_\infty(dw/as;w/s)_\infty}{(q/b;q)_\infty(r/c;r)_\infty(s/d;s)_\infty}$$
$$\times \frac{(ew/at;w/t)_\infty(fw/au;w/u)_\infty(gw/av;w/v)_\infty}{(t/e;t)_\infty(u/f;u)_\infty(v/g;v)_\infty}$$
$$- \frac{(b;q)_\infty(c;r)_\infty(d;s)_\infty(e;t)_\infty(f;u)_\infty(g;v)_\infty}{(a/b;w/q)_\infty(a/c;w/r)_\infty(a/d;w/s)_\infty(a/e;w/t)_\infty(a/f;w/u)_\infty(a/g;w/v)_\infty}, \tag{3.8}$$

where $a^3 = bcdefg$ and $w^3 = qrstuv$, and

$$\max(|q|,|r|,|s|,|t|,|u|,|v|,|w/q|,|w/r|,|w/s|,|w/t|,|w/u|,|w/v|) < 1.$$

Even though we cannot let $n \to \infty$ or $m \to \infty$ in (3.7) when $p \neq 0$ to derive summation formulas for nonterminating theta hypergeometric series (because $\lim_{a\to 0} \theta(a;p)$ does not exist when $p \neq 0$), it is possible in some special cases to let $n \to \infty$ or $m \to \infty$ in (3.5) to obtain summation formulas for nonterminating series containing products of certain theta functions. In particular, if we denote the kth factor in the product representation (3.4) for U_n by

$$z_k = \frac{\theta(b_k, c_k, d_k, e_k, f_k, a_k^3/b_kc_kd_ke_kf_k;p)}{\theta(a_k/b_k, a_k/c_k, a_k/d_k, a_k/e_k, a_k/f_k, b_kc_kd_ke_kf_k/a_k^2;p)}$$

and observe that

$$\lim_{b\to a^{\frac{1}{2}}} \frac{\theta(b;p)}{\theta(a/b;p)} = 1, \quad |p|<1,$$

when a is not an integer power of p, then it follows that there exist bilateral sequences of the a's, b's, c's, d's, e's, and f's in (3.5) such that $\Re z_k > 0$ for integer k and the series

$$\sum_{k=-\infty}^{\infty} \log z_k \quad \text{converges}, \tag{3.9}$$

where $\log z_k$ is the principal branch of the logarithm (choose, e.g., b_k, c_k, d_k, e_k, and f_k so close to $a_k^{\frac{1}{2}}$ that $|\log z_k| < 1/k^2$ for $k = \pm 1, \pm 2, \ldots$). Then both of the limits $\lim_{n\to\infty} U_n$ and $\lim_{m\to\infty} U_{-m}$ exist, and we obtain the bilateral summation formula (which extends [8, Eq. (11.6.8)])

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{\theta(a_k, a_k/e_kf_k, a_k^2/b_kc_kd_ke_k, a_k^2/b_kc_kd_kf_k; p)}{\theta(a_k^2/b_kc_kd_ke_kf_k, a_k^2/b_kc_kd_k, a_k/f_k, a_k/e_k; p)} \\
&\quad \times \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, e_j, f_j, a_j^3/b_jc_jd_je_jf_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, a_j/e_j, a_j/f_j, b_jc_jd_je_jf_j/a_j^2; p)} \\
&\quad \times \bigg[1 - \frac{\theta(a_k/b_kc_k, a_k/b_kd_k, a_k/c_kd_k; p)}{\theta(a_k/d_k, a_k/c_k, a_k/b_k; p)} \\
&\qquad \times \frac{\theta(e_k, f_k, g_k; p)}{\theta(a_k^2/b_kc_kd_ke_k, a_k^2/b_kc_kd_kf_k, a_k^2/b_kc_kd_kg_k; p)}\bigg] \\
&= \prod_{k=-\infty}^{-1} \frac{\theta(a_k/b_k, a_k/c_k, a_k/d_k, a_k/e_k, a_k/f_k, b_kc_kd_ke_kf_k/a_k^2; p)}{\theta(b_k, c_k, d_k, e_k, f_k, a_k^3/b_kc_kd_ke_kf_k; p)} \\
&\quad - \prod_{k=0}^{\infty} \frac{\theta(b_k, c_k, d_k, e_k, f_k, a_k^3/b_kc_kd_ke_kf_k; p)}{\theta(a_k/b_k, a_k/c_k, a_k/d_k, a_k/e_k, a_k/f_k, b_kc_kd_ke_kf_k/a_k^2; p)}
\end{aligned} \tag{3.10}$$

with $a_k^3 = b_kc_kd_ke_kf_kg_k$ for $k = 0, \pm1, \pm2, \ldots$, and a_k, b_k, c_k, d_k, e_k, f_k, g_k such that (3.9) holds.

However, it seems to be more useful to employ the patching

$$\begin{aligned}
&\theta(a(w/t)^k/e, a(w/u)^k/f, g(v/w)^k/a; p)(a/e; w/t, p)_k(a/f; w/u, p)_k(a/g; w/v, p)_k \\
&= \theta(a/e, a/f, g/a; p)(aw/et; w/t, p)_k(aw/fu; w/u, p)_k(aw/gv; w/v, p)_k(v/w)^k,
\end{aligned}$$

to convert the $m = 0$ case of (3.7) into the form

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(aw^k, a(w/tu)^k/ef, fg(uv/w)^k/a, eg(tv/w)^k/a, efg/a; p)}{\theta(a, a/ef, fg/a, eg/a, efg(tuv/w)^k/a; p)} \\
&\quad \times \frac{(b; q, p)_k(c; r, p)_k(d; s, p)_k}{(a/b; w/q, p)_k(a/c; w/r, p)_k(a/d; w/s, p)_k} \\
&\quad \times \frac{(e; t, p)_k(f; u, p)_k(g; v, p)_k}{(aw/et; w/t, p)_k(aw/fu; w/u, p)_k(aw/gv; w/v, p)_k}(w/v)^k \\
&\quad \times \bigg[1 - \frac{\theta(a(w/rs)^k/cd, a(w/qs)^k/bd, a(w/qr)^k/bc; p)}{\theta(a(w/q)^k/b, a(w/r)^k/c, a(w/s)^k/d; p)} \\
&\qquad \times \frac{\theta(et^k, fu^k, gv^k; p)}{\theta(fg(uv/w)^k/a, eg(tv/w)^k/a, ef(tu/w)^k/a; p)}\bigg] \\
&= \frac{\theta(a/e, a/f, g/a, efg/a; p)}{\theta(eg/a, fg/a, a, a/ef; p)} \\
&\quad \times \bigg[1 - \frac{(b; q, p)_{n+1}(c; r, p)_{n+1}(d; s, p)_{n+1}}{(a/b; w/q, p)_{n+1}(a/c; w/r, p)_{n+1}(a/d; w/s, p)_{n+1}}
\end{aligned}$$

$$\times \frac{(e;t,p)_{n+1}(f;u,p)_{n+1}(g;v,p)_{n+1}}{(a/e;w/t,p)_{n+1}(a/f;w/u,p)_{n+1}(a/g;w/v,p)_{n+1}}\Bigg] \tag{3.11}$$

where $a^3 = bcdefg$ and $w^3 = qrstuv$, and then to let $g = v^{-n}$ to obtain the following multibasic theta hypergeometric generalization of [8, Eq. (3.6.16)]

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(aw^k, a(w/tu)^k/ef, a^2(uv/w)^k/bcde, a^2(tv/w)^k/bcdf, a^2/bcd;p)}{\theta(a, a/ef, a^2/bcde, a^2/bcdf, a^2(tuv/w)^k/bcd;p)} \\
&\quad \times \frac{(b;q,p)_k(c;r,p)_k(d;s,p)_k}{(a/b;w/q,p)_k(a/c;w/r,p)_k(a/d;w/s,p)_k} \\
&\quad \times \frac{(e;t,p)_k(f;u,p)_k(v^{-n};v,p)_k}{(aw/et;w/t,p)_k(aw/fu;w/u,p)_k(awv^{n-1};w/v,p)_k}(w/v)^k \\
&\quad \times \Bigg[1 - \frac{\theta(a(w/rs)^k/cd, a(w/qs)^k/bd, a(w/qr)^k/bc;p)}{\theta(a(w/q)^k/b, a(w/r)^k/c, a(w/s)^k/d;p)} \\
&\qquad \times \frac{\theta(et^k, fu^k, v^{k-n};p)}{\theta(f(u/w)^kv^{k-n}/a, e(t/w)^kv^{k-n}/a, ef(tu/w)^k/a;p)}\Bigg] \\
&= \frac{\theta(a/e, a/f, v^{-n}/a, efv^{-n}/a;p)}{\theta(ev^{-n}/a, fv^{-n}/a, a, a/ef;p)},
\end{aligned} \tag{3.12}$$

where $a^3v^n = bcdef$ and $w^3 = qrstuv$, and $n = 0, 1, \ldots$. By letting $f \to a$ in (3.12) we obtain

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(aw^k, (w/tu)^k/e, v^{-n}(uv/w)^k, a(tv/w)^k/bcd, a^2/bcd;p)}{\theta(a, 1/e, a^2/bcde, a/bcd, a^2(tuv/w)^k/bcd;p)} \\
&\quad \times \frac{(b;q,p)_k(c;r,p)_k(d;s,p)_k}{(a/b;w/q,p)_k(a/c;w/r,p)_k(a/d;w/s,p)_k} \\
&\quad \times \frac{(e;t,p)_k(a;u,p)_k(v^{-n};v,p)_k}{(aw/et;w/t,p)_k(w/u;w/u,p)_k(awv^{n-1};w/v,p)_k}(w/v)^k \\
&\quad \times \Bigg[1 - \frac{\theta(a(w/rs)^k/cd, a(w/qs)^k/bd, a(w/qr)^k/bc;p)}{\theta(a(w/q)^k/b, a(w/r)^k/c, a(w/s)^k/d;p)} \\
&\qquad \times \frac{\theta(et^k, au^k, v^{k-n};p)}{\theta((u/w)^kv^{k-n}, e(t/w)^kv^{k-n}/a, e(tu/w)^k;p)}\Bigg] \\
&= \delta_{n,0}
\end{aligned} \tag{3.13}$$

for $n = 0, 1, \ldots$, where $a^2v^n = bcde$, $w^3 = qrstuv$, and $\delta_{n,m}$ is the Kronecker delta function.

Setting $w = rs$ and $d = a/c$ in (3.13), we have $e = av^n/b$ and obtain (after doing the simultaneous replacements $q \mapsto r$, $rs \mapsto rst/q$, $u \mapsto rst/q^2$ and $v \mapsto s$) the identity (see [8, Eq. (11.6.11)])

$$\sum_{k=0}^{n} \frac{\theta(a(rst/q)^k, br^k/q^k, s^{k-n}/q^k, as^nt^k/bq^k;p)}{\theta(a, b, s^{-n}, as^n/b;p)}$$

$$\times \frac{(a;rst/q^2,p)_k(b;r,p)_k(s^{-n};s,p)_k(as^n/b;t,p)_k}{(q;q,p)_k(ast/bq;st/q,p)_k(as^nrt/q;rt/q,p)_k(brs^{1-n}/q;rs/q,p)_k}q^k$$
$$= \delta_{n,0}, \tag{3.14}$$

where $n = 0, 1, \ldots$, which generalizes [8, Eq. (3.6.17)]. In particular, if we replace n, a, b, and k in the $s = t = q$ case of (3.14) by $n - m$, ar^mq^m, br^mq^{-m}, and $j - m$, respectively, we obtain the orthogonality relation

$$\sum_{j=m}^{n} a_{nj}b_{jm} = \delta_{n,m} \tag{3.15}$$

with

$$a_{nj} = \frac{(-1)^{n+j}\theta(ar^jq^j, br^jq^{-j};p)(arq^n, brq^{-n};r,p)_{n-1}}{(q;q,p)_{n-j}(arq^n, brq^{-n};r,p)_j(bq^{1-2n}/a;q,p)_{n-j}},$$
$$b_{jm} = \frac{(ar^mq^m, br^mq^{-m};r,p)_{j-m}}{(q, aq^{1+2m}/b;q,p)_{j-m}}\Big(-\frac{a}{b}q^{1+2m}\Big)^{j-m}q^{2\binom{j-m}{2}}.$$

This shows that the triangular matrix $A = (a_{nj})$ is the inverse of the triangular matrix $B = (b_{jm})$, and yields a theta hypergeometric analogue of [8, Eqs. (3.6.18)–(3.6.20)]. It should be noted, on the contrary, that by replacing n and k in (3.13) by $n - m$ and $j - m$ one does not obtain a sum of the form (3.15).

By proceeding as in the derivation of Eq. (3.6.22) in [8], we find that the latter extends to the bibasic theta hypergeometric summation formula

$$\theta(a/r, b/r;p)\sum_{k=0}^{n}\frac{(aq^k, bq^{-k};r,p)_{n-1}\theta(aq^{2k}/b;p)}{(q;q,p)_k(q;q,p)_{n-k}(aq^k/b;q,p)_{n+1}}(-1)^kq^{\binom{k}{2}} = \delta_{n,0} \tag{3.16}$$

for $n = 0, 1, \ldots$, which when $r = q$ reduces to

$${}_8V_7(a/b;q/b, aq^{n-1}, q^{-n}, q^{-2n};q,p) = \delta_{n,0}.$$

Special cases of the summation formula (3.12), combined with the argument applied in [8, Sec. 3.8], can be used to extend equations (3.8.14) and (3.8.15) of [8] to the quadratic theta hypergeometric transformation formulas

$$\sum_{k=0}^{n}\frac{\theta(acq^{3k};p)}{\theta(ac;p)}\frac{(a, b, cq/b;q,p)_k(f, a^2c^2q^{2n+1}/f, q^{-2n};q^2,p)_k}{(cq^2, acq^2/b, abq;q^2,p)_k(acq/f, f/acq^{2n}, acq^{2n+1};q,p)_k}q^k$$
$$= \frac{(acq;q,p)_{2n}(ac^2q^2/bf, abq/f;q^2,p)_n}{(acq/f;q,p)_{2n}(abq, ac^2q^2/b;q^2,p)_n}$$
$$\times {}_{12}V_{11}(ac^2/b;f, ac/b, c, cq/b, cq^2/b, a^2c^2q^{2n+1}/f, q^{-2n};q^2,p) \tag{3.17}$$

and

$$\sum_{k=0}^{2n}\frac{\theta(acq^{3k};p)}{\theta(ac;p)}\frac{(d, f, a^2c^2q/df;q^2,p)_k(a, cq^{2n+1}, q^{-2n};q,p)_k}{(acq/d, acq/f, df/ac;q,p)_k(cq^2, aq^{1-2n}, acq^{2n+2};q^2,p)_k}q^k$$
$$= \frac{(acq, acq/df;q,p)_n(acq^{1-n}/d, acq^{1-n}/f;q^2,p)_n}{(acq/d, acq/f;q,p)_n(acq^{1-n}, acq^{1-n}/df;q^2,p)_n}$$

$$\times\, {}_{12}V_{11}(acq^{-2n-1}; c, d, f, a^2c^2q/df, aq^{-2n-1}, q^{1-2n}, q^{-2n}; q^2, p) \tag{3.18}$$

for $n = 0, 1, \ldots$; see Thms. 4.2 and 4.7 in Warnaar [10].

Also of interest is the special case of (3.11) that is obtained by setting $w \mapsto rs$, $c \mapsto a/d$, and $f \mapsto a/d$ (hence $g \to ad/be$), which after the simultaneous replacements $q \mapsto r$, $rs \mapsto rst/q$, $u \mapsto rst/q^2$, $v \mapsto s$, $a \mapsto ad$, and $e \mapsto ad^2/bc$ gives the identity (see also [8, Eq. (11.6.9)])

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(ad(rst/q)^k, br^k/dq^k, cs^k/dq^k, adt^k/bcq^k; p)}{\theta(ad, b/d, c/d, ad/bc; p)} \\
&\quad \times \frac{(a; rst/q^2, p)_k (b; r, p)_k (c; s, p)_k (ad^2/bc; t, p)_k}{(dq; q, p)_k (adst/bq; st/q, p)_k (adrt/cq; rt/q, p)_k (bcrs/dq; rs/q, p)_k} q^k \\
&= \frac{\theta(a, b, c, ad^2/bc; p)}{d\,\theta(ad, b/d, c/d, ad/bc; p)} \\
&\quad \times \frac{(arst/q^2; rst/q^2, p)_n (br; r, p)_n (cs; s, p)_n (ad^2t/bc; t, p)_n}{(dq; q, p)_n (adst/bq; st/q, p)_n (adrt/cq; rt/q, p)_n (bcrs/dq; rs/q, p)_n} \\
&\quad - \frac{\theta(d, ad/b, ad/c, bc/d; p)}{d\,\theta(ad, b/d, c/d, ad/bc; p)}.
\end{aligned} \tag{3.19}$$

Just as in the derivation in Gasper [7] of the quadbasic transformation formula in [8, Ex. 3.21], one can extend indefinite summation formulas (such as in (3.6) and (3.19)) to transformation formulas by applying the identity

$$\sum_{k=0}^{n} \lambda_k \sum_{j=0}^{n-k} \Lambda_j = \sum_{k=0}^{n} \Lambda_k \sum_{j=0}^{n-k} \lambda_j,$$

which follows by a reversing the order of summation. For example, by taking λ_k to be the kth term in the series in (3.6) and Λ_k to be this term with a_k, b_k, c_k, d_k, and p replaced by A_k, B_k, C_k, D_k, and P, respectively, we obtain the rather general transformation formula

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(a_k, a_k/b_kc_k, a_k/b_kd_k, a_k/c_kd_k; p)}{\theta(a_k/b_kc_kd_k, a_k/d_k, a_k/c_k, a_k/b_k; p)} \\
&\quad \times \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, a_j^2/b_jc_jd_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_jc_jd_j/a_j; p)} \\
&\quad \times \left\{1 - \prod_{j=0}^{n-k} \frac{\theta(B_j, C_j, D_j, A_j^2/B_jC_jD_j; P)}{\theta(A_j/B_j, A_j/C_j, A_j/D_j, B_jC_jD_j/A_j; P)}\right\} \\
&= \sum_{k=0}^{n} \frac{\theta(A_k, A_k/B_kC_k, A_k/B_kD_k, A_k/C_kD_k; P)}{\theta(A_k/B_kC_kD_k, A_k/D_k, A_k/C_k, A_k/B_k; P)} \\
&\quad \times \prod_{j=0}^{k-1} \frac{\theta(B_j, C_j, D_j, A_j^2/B_jC_jD_j; P)}{\theta(A_j/B_j, A_j/C_j, A_j/D_j, B_jC_jD_j/A_j; P)}
\end{aligned}$$

$$\times \left\{ 1 - \prod_{j=0}^{n-k} \frac{\theta(b_j, c_j, d_j, a_j^2/b_jc_jd_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_jc_jd_j/a_j; p)} \right\}. \tag{3.20}$$

The special case of (3.20) that is obtained by using (3.19) instead of (3.6) is

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(ad(rst/q)^k, br^k/dq^k, cs^k/dq^k, adt^k/bcq^k; p)}{\theta(ad, b/d, c/d, ad/bc; p)} \\
&\quad \times \frac{(a; rst/q^2, p)_k (b; r, p)_k (c; s, p)_k (ad^2/bc; t, p)_k}{(dq; q, p)_k (adst/bq; st/q, p)_k (adrt/cq; rt/q, p)_k (bcrs/dq; rs/q, p)_k} q^k \\
&\quad \times \left(\frac{\theta(A, B, C, AD^2/BC; P)(Q^{-n}/D; Q, P)_k (B(Q/ST)^n/AD; ST/Q, P)_k}{D\,\theta(AD, B/D, C/D, AD/BC; P)((Q^2/RST)^n/A; RST/Q^2, P)_k} \right. \\
&\qquad \times \frac{(C(Q/RT)^n/AD; RT/Q, P)_k (D(Q/RS)^n/BC; RS/Q, P)_k}{(R^{-n}/B; R, P)_k (S^{-n}/C; S, P)_k (BCT^{-n}/AD^2; T, P)_k} \\
&\qquad - \frac{\theta(D, AD/B, AD/C, BC/D; P)(DQ; Q, P)_n (ADST/BQ; ST/Q, P)_n}{D\,\theta(AD, B/D, C/D, AD/BC; P)(ARST/Q^2; RST/Q^2, P)_n (BR; R, P)_n} \\
&\qquad \left. \times \frac{(ADRT/CQ; RT/Q, P)_n (BCRS/DQ; RS/Q, P)_n}{(CS; S, P)_n (AD^2T/BC; T, P)_n} \right) \\
&= \frac{(arst/q^2; rst/q^2, p)_n (br; r, p)_n (cs; s, p)_n (ad^2t/bc; t, p)_n}{(dq; q, p)_n (adst/bq; st/q, p)_n (adrt/cq; rt/q, p)_n (bcrs/dq; rs/q, p)_n} \\
&\quad \times \frac{(DQ; Q, P)_n (ADST/BQ; ST/Q, P)_n}{(ARST/Q^2; RST/Q^2, P)_n (BR; R, P)_n} \\
&\quad \times \frac{(ADRT/CQ; RT/Q, P)_n (BCRS/DQ; RS/Q, P)_n}{(CS; S, P)_n (AD^2T/BC; T, P)_n} \\
&\times \sum_{k=0}^{n} \frac{\theta(AD(RST/Q)^k, BR^k/DQ^k, CS^k/DQ^k, ADT^k/BCQ^k; P)}{\theta(AD, B/D, C/D, AD/BC; P)(DQ; Q, P)_k} Q^k \\
&\quad \times \frac{(A; RST/Q^2, P)_k (B; R, P)_k (C; S, P)_k (AD^2/BC; T, P)_k}{(ADST/BQ; ST/Q, P)_k (ADRT/CQ; RT/Q, P)_k (BCRS/DQ; RS/Q, P)_k} \\
&\quad \times \left(\frac{\theta(a, b, c, ad^2/bc; p)(q^{-n}/d; q, p)_k (b(q/st)^n/ad; st/q, p)_k}{d\,\theta(ad, b/d, c/d, ad/bc; p)((q^2/rst)^n/a; rst/q^2, p)_k (r^{-n}/b; r, p)_k} \right. \\
&\qquad \times \frac{(c(q/rt)^n/ad; rt/q, p)_k (d(q/rs)^n/bc; rs/q, p)_k}{(s^{-n}/c; s, p)_k (bct^{-n}/ad^2; t, p)_k} \\
&\qquad - \frac{\theta(d, ad/b, ad/c, bc/d; p)(dq; q, p)_n (adst/bq; st/q, p)_n}{d\,\theta(ad, b/d, c/d, ad/bc; p)(arst/q^2; rst/q^2, p)_n (br; r, p)_n} \\
&\qquad \left. \times \frac{(adrt/cq; rt/q, p)_n (bcrs/dq; rs/q, p)_n}{(cs; s, p)_n (ad^2t/bc; t, p)_n} \right).
\end{aligned} \tag{3.21}$$

The $d, D \to 1$ special case of (3.21) is

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(a(rst/q)^k, br^kq^{-k}, cs^kq^{-k}, at^k/bcq^k;p)}{\theta(a,b,c,a/bc;p)} \\
&\quad\times \frac{(a;rst/q^2,p)_k(b;r,p)_k(c;s,p)_k(a/bc;t,p)_k}{(q;q,p)_k(ast/bq;st/q,p)_k(art/cq;rt/q,p)_k(bcrs/q;rs/q,p)_k} \\
&\quad\times \frac{(Q^{-n};Q,P)_k(B(Q/ST)^n/A;ST/Q,P)_k(C(Q/RT)^n/A;RT/Q,P)_k}{((Q^2/RST)^n/A;RST/Q^2,P)_k(R^{-n}/B;R,P)_k(S^{-n}/C;S,P)_k} \\
&\quad\times \frac{((Q/RS)^n/BC;RS/Q,P)_k}{(BC/AT^n;T,P)_k} q^k \\
&= \frac{(arst/q^2;rst/q^2,p)_n(br;r,p)_n(cs;s,p)_n(at/bc;t,p)_n}{(q;q,p)_n(ast/bq;st/q,p)_n(art/cq;rt/q,p)_n(bcrs/q;rs/q,p)_n} \\
&\quad\times \frac{(Q;Q,P)_n(AST/BQ;ST/Q,P)_n}{(ARST/Q^2;RST/Q^2,P)_n(BR;R,P)_n} \\
&\quad\times \frac{(ART/CQ;RT/Q,P)_n(BCRS/Q;RS/Q,P)_n}{(CS;S,P)_n(AT/BC;T,P)_n} \\
&\quad\times \sum_{k=0}^{n} \frac{\theta(A(RST/Q)^k, BR^kQ^{-k}, CS^kQ^{-k}, AT^k/BCQ^k;P)}{\theta(A,B,C,A/BC;P)} \\
&\qquad\times \frac{(A;RST/Q^2,P)_k(B;R,P)_k}{(Q;Q,P)_k(AST/BQ;ST/Q,P)_k} \\
&\qquad\times \frac{(C;S,P)_k(A/BC;T,P)_k}{(ART/CQ;RT/Q,P)_k(BCRS/Q;RS/Q,P)_k} \\
&\qquad\times \frac{(q^{-n};q,p)_k(b(q/st)^n/a;st/q,p)_k}{((q^2/rst)^n/a;rst/q^2,p)_k} \\
&\qquad\times \frac{(c(q/rt)^n/a;rt/q,p)_k((q/rs)^n/bc;rs/q,p)_k}{(r^{-n}/b;r,p)_k(s^{-n}/c;s,p)_k(bc/at^n;t,p)_k} Q^k,
\end{aligned} \tag{3.22}$$

for $n = 0, 1, \ldots$. For $s = t = q$ and $S = T = Q$ this reduces to the elliptic quadbasic transformation formula

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(ar^kq^k, br^kq^{-k};p)}{\theta(a,b;p)} \frac{(a,b;r,p)_k(c,a/bc;q,p)_k}{(q,aq/b;q,p)_k(ar/c,bcr;r,p)_k} \\
&\quad\times \frac{(CR^{-n}/A, R^{-n}/BC;R,P)_k(Q^{-n}, BQ^{-n}/A;Q,P)_k}{(Q^{-n}/C, BCQ^{-n}/A;Q,P)_k(R^{-n}/A, R^{-n}/B;R,P)_k} q^k \\
&= \frac{(ar,br;r,p)_n(cq,aq/bc;q,p)_n(Q,AQ/B;Q,P)_n(AR/C,BCR;R,P)_n}{(q,aq/b;q,p)_n(arc,bc/r;r,p)_n(AR,BR;R,P)_n(CQ,AQ/BC;Q,P)_n} \\
&\quad\times \sum_{k=0}^{n} \frac{\theta(AR^kQ^k, BR^kQ^{-k};P)}{\theta(A,B;P)} \frac{(A,B;R,P)_k(C,A/BC;Q,P)_k}{(Q,AQ/B;Q,P)_k(AR/C,BCR;R,P)_k}
\end{aligned}$$

$$\times \frac{(cr^{-n}/a, r^{-n}/bc; r, p)_k (q^{-n}, bq^{-n}/a; q, p)_k}{(q^{-n}/c, bcq^{-n}/a; q, p)_k (r^{-n}/a, r^{-n}/b; r, p)_k} Q^k, \tag{3.23}$$

which is an extension of the second identity in [8, Ex. 3.21] (see also [8, Ex. 11.25]). If we now set $R = Q = r = q$, we obtain the following transformation formula for a "split-poised" theta hypergeometric series

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(aq^{2k}; p)}{\theta(a; p)} \frac{(a, b, c, a/bc; q, p)_k}{(q, aq/b, aq/c, bcq; q, p)_k} \\
&\quad \times \frac{(q^{-n}, B/Aq^n, C/Aq^n, 1/BCq^n; q, p)_k}{(1/Aq^n, 1/Bq^n, 1/Cq^n, BC/Aq^n; q, p)_k} q^k \\
&= \frac{(aq, bq, cq, aq/bc, Aq/B, Aq/C, BCq; q, p)_n}{(Aq, Bq, Cq, Aq/BC, aq/b, aq/c, bcq; q, p)_n} \\
&\quad \times \sum_{k=0}^{n} \frac{\theta(Aq^{2k}; p)}{\theta(A; p)} \frac{(A, B, C, A/BC; q, p)_k}{(q, Aq/B, Aq/C, BCq; q, p)_k} \\
&\quad\quad \times \frac{(q^{-n}, b/aq^n, c/aq^n, 1/bcq^n; q, p)_k}{(1/aq^n, 1/bq^n, 1/cq^n, bc/aq^n; q, p)_k} q^k
\end{aligned} \tag{3.24}$$

for $n = 0, 1, \ldots$, which is an extension of the transformation formula for a split-poised ${}_{10}\phi_9$ series given in [8, Ex. 3.21]. This formula may also be written as a transformation formula for a split-poised ${}_{12}E_{11}$ series, see (1.1).

We will now use (3.14) to derive multibasic extensions of the Fields and Wimp, Verma, and Gasper expansion formulas in [8, Eqs. (3.7.1)–(3.7.3) & (3.7.6)–(3.7.9)], and multibasic theta hypergeometric extensions of [8, Eqs. (3.7.6)–(3.7.8)]. Let $a = \gamma(rst/q)^j$ and $b = \sigma(r/q)^j$ in (3.14) and replace the summation index k by $n-k$. For $j, n = 0, 1, \ldots$, we assume that $B_n(p)$ and $C_{j,n}$ are complex numbers such that $C_{j,0} = 1$ and the sequence $\big\{B_n(p)\big\}$ has finite support when $p \neq 0$. Then, for $j = 0, 1, \ldots$,

$$\begin{aligned}
B_j(p)x^j &= \sum_{n=0}^{\infty} \frac{\theta(\gamma\sigma^{-1}(st)^{n+j}; p)(\gamma\sigma^{-1}(st)^j; st/q, p)_n}{\theta(\gamma\sigma^{-1}(st)^j; p)(s; s, p)_n} \\
&\quad \times \frac{(\gamma rs^j tq^{-1}; rt/q, p)_j (\sigma rs^{1-j}q^{-1}; rs/q, p)_j}{(\gamma rs^{n+j}tq^{-1}; rt/q, p)_j (\sigma rs^{1-n-j}q^{-1}; rs/q, p)_j} \\
&\quad \times s^{\binom{n+1}{2}} q^{-\binom{n+1}{2}-nj} B_{j+n}(p) C_{j,n} x^{j+n} \delta_{n,0} \\
&= \sum_{k=0}^{\infty} \sum_{n=j}^{\infty} \frac{\theta(\gamma(rst/q)^n, \sigma(r/q)^n, \gamma\sigma^{-1}(st)^{n+k}, \gamma\sigma^{-1}s^{n+k}t^n q^{j-n}, s^{-k}q^{j-n}; p)}{\theta(s^{j-n-k}; p)(s; s, p)_k (q; q, p)_n} \\
&\quad \times \frac{(\gamma\sigma^{-1}(st)^{n+1}q^{j-n-1}; st/q, p)_{k-1} (\gamma\sigma^{-1}s^{n+k}t^{j+1}; t, p)_{n-j-1}}{(\gamma rs^{n+k}tq^{-1}; rt/q, p)_n (\sigma rs^{1-n-k}q^{-1}; rs/q, p)_n} \\
&\quad \times (\gamma rs^j tq^{-1}; rt/q, p)_j (\gamma(rst)^{j+1}q^{-j-2}; rst/q^2, p)_{n-j-1} \\
&\quad \times (\sigma rs^{1-j}q^{-1}; rs/q, p)_j (\sigma r^{j+1}q^{-j}; r, p)_{n-j-1} (q^{-n}; q, p)_j
\end{aligned}$$

$$\times (-1)^n B_{n+k}(p) C_{j,n+k-j} x^{n+k} s^{\binom{k+1}{2}} q^{-\binom{k+1}{2}+\binom{n}{2}+n(1+j-n-k)} \tag{3.25}$$

by interchanging sums and setting $n \mapsto n+k-j$ (this extension of [8, Eq. (3.7.5)] corrects [8, Eq. (11.6.20)]).

By multiplying both sides of (3.25) by $A_j w^j/(q;q,p)_j$ and summing from $j=0$ to ∞ we get that the following multibasic expansion formula (this corrects [8, Eq. (11.6.21)])

$$\begin{aligned}
&\sum_{n=0}^{\infty} A_n B_n(p) \frac{(xw)^n}{(q;q,p)_n} \\
&= \sum_{n=0}^{\infty} \frac{\theta(\gamma(rst/q)^n, \sigma(r/q)^n; p)}{(q;q,p)_n} (-x)^n q^{n+\binom{n}{2}} \\
&\quad \times \sum_{k=0}^{\infty} \frac{\theta(\gamma\sigma^{-1}(st)^{n+k}; p)}{(\gamma r s^{n+k} t q^{-1}; rt/q, p)_n (\sigma r s^{1-n-k} q^{-1}; rs/q, p)_n} \frac{B_{n+k}(p) x^k}{(s;s,p)_k} s^{\binom{k+1}{2}} q^{-\binom{k+1}{2}} \\
&\quad \times \sum_{j=0}^{n} \frac{\theta(\gamma\sigma^{-1} s^{n+k} t^n q^{j-n}, s^{-k} q^{j-n}; p)(q^{-n}; q,p)_j}{\theta(s^{j-n-k}; p)(q;q,p)_j} \\
&\quad \times (\gamma\sigma^{-1}(st)^{n+1} q^{j-n-1}; st/q, p)_{k-1} (\gamma\sigma^{-1} s^{n+k} t^{j+1}; t, p)_{n-j-1} \\
&\quad \times (\gamma r s^j t q^{-1}; rt/q, p)_j (\gamma (rst)^{j+1} q^{-j-2}; rst/q^2, p)_{n-j-1} \\
&\quad \times (\sigma r s^{1-j} q^{-1}; rs/q, p)_j (\sigma r^{j+1} q^{-j}; r, p)_{n-j-1} A_j C_{j,n+k-j} w^j q^{n(j-n-k)},
\end{aligned} \tag{3.26}$$

which reduces to [8, Eq. (3.7.6)] by letting $p=0$ and then setting $r=p$ and $s=t=q$.

If we set $r=s=t=q$ and $C_{j,m} \equiv 1$ in (3.26) we obtain an expansion formula that is equivalent to the following extension of [8, Eq. (3.7.7)] (which corrects a slight misprint in [8, Eq. (11.6.22)])

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n B_n(p) \frac{(xw)^n}{(q;q,p)_n} &= \sum_{n=0}^{\infty} \frac{(\sigma, \gamma q^{n+1}/\sigma, \alpha, \beta; q,p)_n}{(q, \gamma q^n; q,p)_n} \left(\frac{x}{\sigma}\right)^n \\
&\quad \times \sum_{k=0}^{\infty} \frac{\theta(\gamma q^{2n+2k}/\sigma; p)(\gamma q^{2n}/\sigma, \sigma^{-1}, \alpha q^n, \beta q^n; q,p)_k}{\theta(\gamma q^{2n}/\sigma; p)(q, \gamma q^{2n+1}; q,p)_k} B_{n+k}(p) x^k \\
&\quad \times \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n; q,p)_j}{(q, \gamma q^{n+1}/\sigma, q^{1-n}/\sigma, \alpha, \beta; q,p)_j} A_j (wq)^j,
\end{aligned} \tag{3.27}$$

where, as previously, it is assumed that $\{B_n(p)\}$ has finite support when $p \neq 0$. Clearly, one cannot let $\sigma \to \infty$ in (3.27) to obtain an extension of [8, Eq. (3.7.3)] that holds when $p \neq 0$.

Corresponding to the q-analogue of the Fields and Wimp expansion formula displayed in [8, Eq. (3.7.8)], (3.27) gives the rather general theta hypergeometric expansion formula (see [8, Eq. (11.6.23)])

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a_R, c_T; q, p)_n}{(q, b_S, d_U; q, p)_n} A_n B_n(p)(xw)^n \\
&= \sum_{n=0}^{\infty} \frac{(c_T, e_K, \sigma, \gamma q^{n+1}/\sigma; q, p)_n}{(q, d_U, f_M, \gamma q^n; q, p)_n} \Big(\frac{x}{\sigma}\Big)^n \\
&\quad \times \sum_{k=0}^{\infty} \frac{\theta(\gamma q^{2n+2k}/\sigma; p)(\gamma q^{2n}/\sigma, \sigma^{-1}, c_T q^n, e_K q^n; q, p)_k}{\theta(\gamma q^{2n}/\sigma; p)(q, \gamma q^{2n+1}, d_U q^n, f_M q^n; q, p)_k} B_{n+k}(p) x^k \\
&\quad \times \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n, a_R, f_M; q, p)_j}{(q, \gamma q^{n+1}/\sigma, q^{1-n}/\sigma, b_S, e_K; q, p)_j} A_j (wq)^j, \qquad (3.28)
\end{aligned}$$

where we used the contracted notation that was used in [8, Eq. (3.7.8)], and, in order to avoid convergence problems, it is assumed that $\{B_n\}$ has finite support when $p \neq 0$.

By using (3.13) instead of its special case (3.14) and proceeding as above, one can derive even more general extensions of the multibasic Fields and Wimp, Verma, and Gasper expansions. Since they are rather lengthy and do not seem to be of any particular interest at this time, we will not give them here.

References

[1] R. J. Baxter, "Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain, II: Equivalence to a generalized ice-type model", *Ann. Phys.* **76** (1973), 193–228.

[2] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York, 1982.

[3] G. Bhatnagar and S. C. Milne, "Generalized bibasic hypergeometric series and their $U(n)$ extensions", *Adv. Math.* **131** (1997), 188–252, 1997.

[4] W. Chu, "Inversion techniques and combinatorial identities", *Boll. Un. Mat. Ital.* (7) **7**-B (1993), 737–760.

[5] E. Date, M. Jimbo, A. Kuniba, T. Miwa, M. Okado, "Exactly solvable SOS models: local height probabilities and theta function identities", *Nuclear Phys. B* **290** (1987), 231–273.

[6] I. B. Frenkel and V. G. Turaev, "Elliptic solutions of the Yang–Baxter equation and modular hypergeometric functions", in V. I. Arnold et al. (eds.), *The Arnold–Gelfand Mathematical Seminars*, 171–204, Birkhäuser, Boston, 1997.

[7] G. Gasper, "Summation, transformation, and expansion formulas for bibasic series", *Trans. Amer. Math. Soc.* **312** (1989), 257–277.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series*, 2^{nd} ed., Encyclopedia of Mathematics And Its Applications 96, Cambridge University Press, Cambridge, 2004.

[9] V. P. Spiridonov, "Theta hypergeometric series", in V. A. Malyshev and A. M. Vershik (eds.), *Asymptotic Combinatorics with Applications to Mathematical Physics*, 307–327, Kluwer Acad. Publ., Dordrecht, 2002.

[10] S. O. Warnaar, "Summation and transformation formulas for elliptic hypergeometric series", *Constr. Approx.* **18** (2002), 479–502.

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