MULTILATERAL TRANSFORMATIONS OF q-SERIES WITH QUOTIENTS OF PARAMETERS THAT ARE NONNEGATIVE INTEGRAL POWERS OF q

MICHAEL SCHLOSSER

ABSTRACT. We give multidimensional generalizations of several transformation formulae for basic hypergeometric series of a specific type. Most of the upper parameters of the series differ multiplicatively from corresponding lower parameters by a nonnegative integral power of the base q. In one dimension, formulae for such series have been found, in the $q \to 1$ case, by B. M. Minton and P. W. Karlsson, and in the basic case by G. Gasper, by W. C. Chu, and more recently by the author. Our identities involve multilateral basic hypergeometric series associated to the root system A_r (or equivalently, the unitary group $U(r+1)$).

1. INTRODUCTION

The theory of hypergeometric and basic hypergeometric (or q-hypergeometric) series (cf. L. J. Slater [33], and G. Gasper and M. Rahman [13]) contains numerous summation and transformation formulae. Many of these appear in applications including number theory, combinatorics, physics, representation theory, and computer algebra (see e.g. G. E. Andrews [1]).

One particular example is B. M. Minton's [26] summation formula, found in 1970, which is useful for simplifying sums that arise in certain problems in theoretical physics (such as Racah coefficients). B. M. Minton's formula is of special interest since it sums a specific hypergeometric series with an arbitrary number of parameters. B. M. Minton derived his formula by expanding a hypergeometric series in terms of other hypergeometric series, exploiting an identity already obtained by C. Fox [9] in 1925 (but published in 1927). B. M. Minton iterated this expansion and suitably specialized the parameters to successively evaluate the (inner) sums. A condition on the parameters of the specific hypergeometric series considered by B. M. Minton is that most of the upper parameters differ from corresponding lower ones by a nonnegative integer. B. M. Minton's result was slightly extended by P. W. Karlsson [19] who was using the same method.

In the early 1980's, G. Gasper [10] found q-analogues of Karlsson and Minton's results. In the basic case, the condition on the parameters is that most of the upper parameters differ *multiplicatively* from corresponding lower ones by a nonnegative integral power of q. G. Gasper even extended his results to a transformation formula [10, Eq. (19)]. For the above material, see the exposition in G. Gasper and M. Rahman [13], in particular Section 1.9, and Exercises 1.30 and 1.34.

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Note that G. Gasper and M. Rahman [13] use the terminology "Karlsson–Minton" and "q-Karlsson–Minton", respectively, to denote the type of the series in question. We are dropping this terminology in the present paper, since the work is based on expanding a hypergeometric function in terms of another, which has a longer history. Instead, we introduce the acronyms IPD and q-IPD, respectively, where IPD stands for "**I**ntegral **P**arameter **D**ifferences" (motivated by the title of P. W. Karlsson's [19] article), see Section 2. It should be mentioned that expansions of hypergeometric series in terms of other hypergeometric series have also been obtained by J. L. Fields and J. Wimp [8], by A. Verma [35], and in more generality (concerning identities between general sequences), by J. L. Fields and M. E. H. Ismail [7]. Thus, as pointed out to us by Mourad Ismail [18], one may easily write generalizations of the Karlsson–Minton formulae to series involving partly hypergeometric coefficients and partly general sequences.

By using an essentially different method, namely by partial fraction expansions, W. C. Chu [5] generalized G. Gasper's q-IPD type identities further to a bilateral series transformation. In another article, G. Gasper [11, Eq. (5.13)] found a new summation for a very-well-poised basic hypergeometric series of q-IPD type. Again, W. C. Chu [6] extended G. Gasper's result to a summation for a very-well-poised *bilateral* basic hypergeometric series.

Very recently, the author [31, Sec. 8] found even more general identities of q-IPD type, by elementary manipulations of series, using L. J. Slater's [32] general transformations for bilateral basic hypergeometric series. Already earlier J. Haglund [17, pp. 415–416] had discovered that W. C. Chu's [5] bilateral transformation formula can be obtained by specializing L. J. Slater's [32] general transformation for ${}_t\psi_t$ series.

In this article, we provide *multidimensional* extensions of several specific transformation formulae of q-IPD type, in particular, multivariate extensions of the identities in Propositions 2.1, 2.2, 2.3 and 2.4. These multivariate extensions involve multiple basic hypergeometric series associated to the root system A_{r-1} (or equivalently, the unitary group $U(r)$). Such type of series are considered in the work of R. A. Gustafson, S. C. Milne, and several other authors, see e.g. [4], [14] [15], [16], [20], [21], [22], [23], [24], [25], [27], [28], and [29].

As a matter of fact, there are unfortunately no suitable *multidimensional* extensions of L. J. Slater's [32] general transformation formulae known (yet). Thus, in higher dimensions we cannot specialize down from such higher level identities. Instead we proceed from lower level identities to systematically derive the upper level ones. In this fashion, using certain A_{r-1} summation theorems (from R. A. Gustafson [15] and S. C. Milne [22]), elementary manipulation of series, and induction, we prove two multilateral transformations of q-IPD type, namely Theorems 4.2 and 4.6. The first one of these, Theorem 4.2, involves *very-well-poised* multilateral series (over A_{r-1}), and contains r-dimensional generalizations of W. C. Chu's [6, Theorem 2] and G. Gasper's [11, Eq. (5.13)] summations as special cases, see Corollaries 4.3 and 4.4, respectively. The other transformation formula in Theorem 4.6, involves multilateral series with an arbitrary argument z. Four other multilateral transformations of q-IPD type are derived by simpler means, using tools developed in [25], see Theorems 4.7, 4.8, 4.9, and 4.10.

In [29, Theorem 6.4], we already gave some multiple series generalizations (associated to the root systems of classical type) of W. C. Chu's [5] bilateral transformation. The multiple series identities in [29] were derived by using one-dimensional identities, combined with certain determinant evaluations. Using the same determinantal mathod, one could also deduce multilateral generalizations of L. J. Slater's [32] general transformation formulae, and in particular of the q-IPD type transformations which were found in [31, Sec. 8]. However, most of the results of this article, in particular Theorems 4.2 and 4.6, are deeper since they are derived by using genuine multidimensional summation theorems.

Our article is organized as follows. In Section 2, we introduce some standard notation for q-series and basic hypergeometric series, and state several important one-dimensional results. In Section 3, we consider multiple series and recollect some specific ingredients which we need in Section 4 to state and prove our multilateral identities of q-IPD type.

2. Notation and one-dimensional results

In order to state and prove our theorems, we employ some standard q-series notation (cf. G. Gasper and M. Rahman [13]). For a complex number q with $0<|q|<1$, define the *q-shifted factorial* by

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j),$$

and

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}, \qquad \text{where } k \text{ is an integer.} \tag{2.1}$$

Further, for brevity, we also employ the notation

$$(a_1,\dots,a_m;q)_k \equiv (a_1;q)_k \dots (a_m;q)_k,$$

where k is an integer or infinity. Further, we utilize the notations

$${}_t\phi_{t-1}\left[\begin{matrix}a_1,a_2,\dots,a_t\\ b_1,b_2,\dots,b_{t-1}\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_t;q)_k}{(q,b_1,\dots,b_{t-1};q)_k}z^k, \tag{2.2}$$

and

$${}_t\psi_t\left[\begin{matrix}a_1,a_2,\dots,a_t\\ b_1,b_2,\dots,b_t\end{matrix};q,z\right] := \sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\dots,a_t;q)_k}{(b_1,b_2,\dots,b_t;q)_k}z^k, \tag{2.3}$$

for *basic hypergeometric ${}_t\phi_{t-1}$ series*, and *bilateral basic hypergeometric ${}_t\psi_t$ series*, respectively. Note that G. Gasper and M. Rahman [13] have more general definitions for ${}_r\phi_s$ series and for ${}_r\psi_s$ series, but in this article we are only really concerned with the case where $s=r-1$ for the ${}_r\phi_s$ series, and where $r=s$ for the ${}_r\psi_s$ series.

Clearly, a bilateral ${}_t\psi_t$ series becomes a unilateral ${}_t\phi_{t-1}$ series if one of the lower parameters, say b_t, is q (or more generally, q^j where j is a positive integer). This is because $(q;q)_k^{-1}=0$, for $k=-1,-2,\dots$, by definition (2.1). In this case, the ${}_t\psi_t$ series terminates naturally from below. On the other hand, if in a ${}_t\phi_{t-1}$ series one of the upper parameters, say a_t, equals q^{-n}, where n is a nonnegative integer, then the ${}_t\phi_{t-1}$ series terminates naturally from above. This is because $(q^{-n};q)_k=0$, for $k=n+1,n+2,\dots$, by definition (2.1). Such a ${}_t\phi_{t-1}$ series terminates after $n+1$ terms.

The ratio test gives simple criteria of when the above series converge, if they do not terminate. Remember that we assume $0<|q|<1$. The ${}_t\phi_{t-1}$ series in (2.2) converges absolutely in the radius $|z|<1$, while the ${}_t\psi_t$ series in (2.3) converges absolutely in the annulus $|b_1\dots b_t/a_1\dots a_t|<|z|<1$.

The classical theory of basic hypergeometric series consists of several summation and transformation formulae involving ${}_t\phi_{t-1}$ series. The classical summation theorems for terminating ${}_3\phi_2$, ${}_6\phi_5$, and ${}_8\phi_7$ series require that the parameters satisfy the additional condition of being either balanced and/or very-well-poised. A ${}_t\phi_{t-1}$ basic hypergeometric series is called *balanced* if $b_1\cdots b_{t-1}=a_1\cdots a_tq$ and $z=q$. An ${}_t\phi_{t-1}$ series is *well-poised* if $a_1q=a_2b_1=\cdots=a_tb_{t-1}$. It is called *very-well-poised* if it is well-poised and if $a_2=q\sqrt{a_1}$ and $a_3=-q\sqrt{a_1}$. Note that the factor

$$\frac{(q\sqrt{a_1},-q\sqrt{a_1};q)_k}{(\sqrt{a_1},-\sqrt{a_1};q)_k}=\frac{1-a_1q^{2k}}{1-a_1} \tag{2.4}$$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series, and we call (2.4) the *very-well-poised term* of the series. Similarly, a bilateral ${}_t\psi_t$ basic hypergeometric series is well-poised if $a_1b_1=a_2b_2\cdots=a_tb_t$ and very-well-poised if, in addition, $a_1=-a_2=qb_1=-qb_2$.

In our proofs in Section 4, we often make use of some elementary identities involving q-shifted factorials, listed in G. Gasper and M. Rahman [13, Appendix I].

With the above notations for basic hypergeometric and bilateral basic hypergeometric series, we are ready to state some important (one-dimensional) summation formulae.

One of the most fundamental summation theorems in the theory of (bilateral) basic hypergeometric series is W. N. Bailey's [2] very-well-poised ${}_6\psi_6$ summation,

$$\begin{aligned}{}_6\psi_6&\left[\begin{matrix}q\sqrt{a},-q\sqrt{a},b,c,d,e\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e\end{matrix};q,\frac{a^2q}{bcde}\right]\\ &=\frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,a^2q/bcde;q)_\infty},\end{aligned} \tag{2.5}$$

provided the series either terminates, or $|q|<1$ and $|a^2q/bcde|<1$, for convergence. For a simple proof of (2.5) using elementary manipulations of series, see [30].

Another important summation is the terminating balanced q-Pfaff–Saalschütz summation (cf. [13, Eq. (II.12)]),

$${}_3\phi_2\left[\begin{matrix}a,b,q^{-n}\\ c,abq^{1-n}/c\end{matrix};q,q\right]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{2.6}$$

S. Ramanujan's ${}_1\psi_1$ summation (cf. [13, Eq. (5.2.1)]) reads as follows,

$${}_1\psi_1\left[\begin{matrix}a\\ b\end{matrix};q,z\right]=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \tag{2.7}$$

provided the series either terminates, or $|q|<1$ and $|b/a|<|z|<1$, for convergence.

Finally, the terminating q-binomial theorem is (cf. [13, Eq. (II.4)])

$${}_1\phi_0\left[\begin{matrix}q^{-n}\\ -\end{matrix};q,z\right]=(zq^{-n};q)_n. \tag{2.8}$$

Note that (2.8) is just the special case $a\to q^{-n}$, $b\to q$ of (2.7).

In this article, we prove multidimensional extensions (associated to the root system A_{r-1}) of four transformations of q-IPD type, namely Propositions 2.1, 2.2, 2.3, and 2.4. We need to explain our terminology first.

We say that a basic hypergeometric series is of *q-IPD type* if there are s upper parameters $a_1, \dots, a_s$ and s lower parameters $b_1, \dots, b_s$ such that each a_i differs from b_i multiplicatively by a nonnegative integral power of q, i.e. $a_i = b_i q^{m_i}$, $m_i \ge 0$. G. Gasper [10] found some summation formulae for particular basic hypergeometric series of such type. These were q-analogues of formulae originally discovered by B. M. Minton [26] and P. W. Karlsson [19], using C. Fox' [9] expansion of a hypergeometric function in terms of other hypergeometric functions. We call the series considered by B. M. Minton and P. W. Karlsson to be of *IPD type*, where IPD stands for "**I**ntegral **P**arameter **D**ifferences", motivated by the title of P. W. Karlsson's [19] article. G. Gasper [10, Eq. (19)] also extended his summations to a transformation formula. Later, W. C. Chu [5] found *bilateral* summations and transformations of q-IPD type, generalizing G. Gasper's identities of [10]. In an expository paper, G. Gasper [11, Eq. (5.13)] derived a summation formula for a specific *very-well-poised* basic hypergeometric series of q-IPD type. His result was then generalized to a summation for bilateral series, again by W. C. Chu [6, Theorem 2]. (It is maybe interesting that as application W. C. Chu [6, Eq. (5.25)] applied an inverse relation to his bilateral summation and (re-)derived an important bibasic identity, actually due to G. Gasper and M. Rahman [12, Eq. (2.8)]. This shows how strongly seemingly different aspects in q-series are interconnected.)

In a recent article [31, Sec. 8], the author found formulae of q-IPD type covering all of the above q-IPD type identities as special cases. In the following, we list the four transformation formulae from [31, Sec. 8] which we extend to higher dimensions. The first one of these involves *very-well-poised* bilateral basic hypergeometric series.

Proposition 2.1 (A bilateral very-well-poised q-IPD type transformation)**.** *Let a, b, c, d, e, f, and $h_1, \dots, h_s$ be indeterminate, let $m_1, \dots, m_s$ be nonnegative integers, let $|m| = \sum_{i=1}^{s} m_i$, and suppose that the series in* (2.9) *are well-defined. Then*

$$
\begin{aligned}
(2.9)\quad {}_{6+2s}\psi_{6+2s}&\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \end{matrix}\right. \\
&\qquad \left.\begin{matrix} h_1, \dots, h_s, \frac{aq^{1+m_1}}{h_1}, \dots, \frac{aq^{1+m_s}}{h_s} \\ \frac{aq}{h_1}, \dots, \frac{aq}{h_s}, h_1 q^{-m_1}, \dots, h_s q^{-m_s} \end{matrix}; q, \frac{a^2 q^{1-|m|}}{bcde}\right] \\
&= \frac{(a, \frac{q}{a}, \frac{fq}{b}, \frac{fq}{c}, \frac{fq}{d}, \frac{fq}{e}, \frac{aq}{bf}, \frac{aq}{cf}, \frac{aq}{df}, \frac{aq}{ef}; q)_\infty}{(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{f^2 q}{a}, \frac{aq}{f^2}; q)_\infty} \prod_{i=1}^{s} \frac{(\frac{fq}{h_i}, \frac{aq}{fh_i}; q)_{m_i}}{(\frac{aq}{h_i}, \frac{q}{h_i}; q)_{m_i}} \\
&\qquad \times {}_{6+2s}\psi_{6+2s}\left[\begin{matrix} \frac{qf}{\sqrt{a}}, -\frac{qf}{\sqrt{a}}, \frac{bf}{a}, \frac{cf}{a}, \frac{df}{a}, \frac{ef}{a}, \\ \frac{f}{\sqrt{a}}, -\frac{f}{\sqrt{a}}, \frac{fq}{b}, \frac{fq}{c}, \frac{fq}{d}, \frac{fq}{e}, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} \frac{fh_1}{a}, \dots, \frac{fh_s}{a}, \frac{fq^{1+m_1}}{h_1}, \dots, \frac{fq^{1+m_s}}{h_s} \\ \frac{fq}{h_1}, \dots, \frac{fq}{h_s}, \frac{fh_1 q^{-m_1}}{a}, \dots, \frac{fh_s q^{-m_s}}{a} \end{matrix}; q, \frac{a^2 q^{1-|m|}}{bcde}\right],
\end{aligned}
$$

where the series either terminate, or $|a^2 q^{1-|m|}/bcde| < 1$, for convergence.

Note that f does not appear on the left side of (2.9).

The special case $f \mapsto b$, $c \mapsto a/b$ of Proposition 2.1 is exactly W. C. Chu's summation in [6, Theorem 2]. If we specialize this summation then further by setting $e \mapsto a$ we arrive at G. Gasper's [11, Eq. (5.13)] summation.

The following transformation formula involves bilateral basic hypergeometric series with an independent argument z.

Proposition 2.2 (A bilateral q-IPD type transformation). *Let a, b, c, z, and $h_1,\dots,h_s$ be indeterminate, let $m_1,\dots,m_s$ be nonnegative integers, let $|m| = \sum_{i=1}^s m_i$, and suppose that the series in (2.10) are well-defined. Then*

$$
\begin{aligned}
(2.10)\quad {}_{1+s}\psi_{1+s}&\left[\begin{matrix}a, h_1q^{m_1},\dots,h_sq^{m_s}\\ b, h_1,\dots,h_s\end{matrix};q,z\right]\\
&=\frac{(c/a,bq/c,az,q/az;q)_\infty}{(q/a,b,azq/c,c/az;q)_\infty}\prod_{i=1}^s\frac{(h_iq/c;q)_{m_i}}{(h_i;q)_{m_i}}\\
&\quad\times {}_{1+s}\psi_{1+s}\left[\begin{matrix}aq/c, h_1q^{1+m_1}/c,\dots,h_sq^{1+m_s}/c\\ bq/c, h_1q/c,\dots,h_sq/c\end{matrix};q,z\right],
\end{aligned}
$$

where the series either terminate, or $|bq^{-|m|}/a| < |z| < 1$, for convergence.

Note that c does not appear on the left side of (2.10).

The next two transformations involve series whose argument depends on the parameters.

Proposition 2.3 (A bilateral q-IPD type transformation). *Let a, b, c, d, e, and $h_1,\dots,h_s$ be indeterminate, let N be an arbitrary integer, $m_1,\dots,m_s$ be nonnegative integers, let $|m| = \sum_{i=1}^s m_i$, and suppose that the series in (2.11) are well-defined. Then*

$$
\begin{aligned}
(2.11)\quad {}_{2+s}\psi_{2+s}&\left[\begin{matrix}a,b, h_1q^{m_1},\dots,h_sq^{m_s}\\ c,d, h_1,\dots,h_s\end{matrix};q,\frac{eq^{-N}}{ab}\right]\\
&=\left(\frac{e}{q}\right)^N\frac{(e/a,e/b,cq/e,dq/e;q)_\infty}{(q/a,q/b,c,d;q)_\infty}\prod_{i=1}^s\frac{(h_iq/e;q)_{m_i}}{(h_i;q)_{m_i}}\\
&\quad\times {}_{2+s}\psi_{2+s}\left[\begin{matrix}aq/e,bq/e, h_1q^{1+m_1}/e,\dots,h_sq^{1+m_s}/e\\ cq/e,dq/e, h_1q/e,\dots,h_sq/e\end{matrix};q,\frac{eq^{-N}}{ab}\right],
\end{aligned}
$$

where the series either terminate, or $|e/ab| < |q^N| < |eq^{|m|}/cd|$, for convergence.

If we reverse the ${}_{2+s}\psi_{2+s}$ series on the right side of (2.11), we obtain

Proposition 2.4 (A bilateral q-IPD type transformation). *Let a, b, c, d, e, and $h_1,\dots,h_s$ be indeterminate, let N be an arbitrary integer, $m_1,\dots,m_s$ be nonnegative integers, let $|m| = \sum_{i=1}^s m_i$, and suppose that the series in (2.12) are well-defined. Then*

$$
\begin{aligned}
(2.12)\quad {}_{2+s}\psi_{2+s}&\left[\begin{matrix}a,b, h_1q^{m_1},\dots,h_sq^{m_s}\\ c,d, h_1,\dots,h_s\end{matrix};q,\frac{eq^{-N}}{ab}\right]\\
&=\left(\frac{e}{q}\right)^N\frac{(e/a,e/b,cq/e,dq/e;q)_\infty}{(q/a,q/b,c,d;q)_\infty}\prod_{i=1}^s\frac{(h_iq/e;q)_{m_i}}{(h_i;q)_{m_i}}\\
&\quad\times {}_{2+s}\psi_{2+s}\left[\begin{matrix}e/c,e/d, e/h_1,\dots,e/h_s\\ e/a,e/b, eq^{-m_1}/h_1,\dots,eq^{-m_s}/h_s\end{matrix};q,\frac{cdq^{N-|m|}}{e}\right],
\end{aligned}
$$

where the series either terminate, or $|e/ab| < |q^N| < |eq^{|m|}/cd|$, *for convergence.*

The $e = aq$ case of Proposition 2.4 reduces to W. C. Chu's [5, Eq. (15)] transformation. If we specialize the resulting transformation further by setting $c = q$ we obtain G. Gasper's [10, Eq. (19)] transformation.

Propositions 2.1, 2.2, 2.3, and 2.4 appeared as Corollaries 8.6, 8.3, 8.2 and Equation (8.8) in [31]. They were originally derived as special cases from even more general transformations for bilateral basic hypergeometric series of q-IPD type.

3. Preliminaries on multiple series

In general, we consider multiple series of the form

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} S(\mathbf{k}), \tag{3.1}$$

where $\mathbf{k} = (k_1, \dots, k_r)$, which reduce to classical (bilateral) basic hypergeometric series when $r = 1$. We call such a multiple basic hypergeometric series *balanced* if it reduces to a balanced series when $r = 1$. We define *well-poised* and *very-well-poised* series analogously. In case these series do not terminate from below, we also call such series *multilateral* basic hypergeometric series.

In our particular cases, we also have

$$\prod_{1\le i<j<r} \left(\frac{z_i q^{k_i} - z_j q^{k_j}}{z_i - z_j} \right) \tag{3.2}$$

(or something similar), as a factor of $S(\mathbf{k})$. A typical example is the right side of (3.5). Since we may associate (3.2) with the product side of the Weyl denominator formula for the root system A_{r-1} (see e.g. D. Stanton [34]), we call our series A_{r-1} basic hypergeometric series, in accordance with I. M. Gessel and C. Krattenthaler [14, Eq. (7.1)]. Note that often in the literature (e.g. [3], [23], [25], [27], [28]) these r-dimensional series are called A_r series instead of A_{r-1} series.

For convenience, we frequently use the notation $|\mathbf{k}| := k_1 + \cdots + k_r$. Note that on the right side of (3.5) we have (in addition to (3.2))

$$\prod_{i=1}^{r} \left(\frac{1 - az_i q^{k_i+|\mathbf{k}|}}{1 - az_i} \right) \tag{3.3}$$

appearing as a factor in the summand of the series. It is easy to see that the $r = 1$ case of (3.3) essentially reduces to (2.4). To clarify the special appearance of the very-well-poised term in the multidimensional case (and even in the one-dimensional) case, it is useful to view the series in one higher dimension. In particular, we can write

$$\begin{aligned} &\prod_{1\le i<j<r} \left(\frac{z_i q^{k_i} - z_j q^{k_j}}{z_i - z_j} \right) \prod_{i=1}^{r} \left(\frac{1 - az_i q^{k_i+(k_1+\cdots+k_r)}}{1 - az_i} \right) \\ &\qquad = q^{k_1+\cdots+k_r} \prod_{1\le i<j\le r+1} \left(\frac{z_i q^{k_i} - z_j q^{k_j}}{z_i - z_j} \right), \end{aligned} \tag{3.4}$$

where $z_{r+1} = 1/a$ and $k_{r+1} = -(k_1 + \cdots + k_r)$. Thus, some A_{r-1} basic hypergeometric series identities are sometimes better viewed as identites associated to the *affine* root system $\tilde{A}_r$ (or, equivalently, the special unitary group $SU(r+1)$). For such an example, see Remark 3.2.

Let a, $b_1,\dots,b_r$, c, d, $e_1,\dots,e_r$, $z_1,\dots,z_r$, and w be indeterminate. For purpose of compact notation, we define for $r\ge 1$

$$
\begin{aligned}
(3.5)\quad & {}_6\Psi_6^{(r)}\left[a;b_1,\dots,b_r;c,d;e_1,\dots,e_r;z_1,\dots,z_r\,\middle|\,q,w\right] \\
&:= \sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^r\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right) \\
&\qquad\times\prod_{i,j=1}^r\frac{(b_jz_i/z_j;q)_{k_i}}{(az_iq/e_jz_j;q)_{k_i}}\prod_{i=1}^r\frac{(e_iz_i;q)_{|\mathbf{k}|}}{(az_iq/b_i;q)_{|\mathbf{k}|}} \\
&\qquad\qquad\times\prod_{i=1}^r\frac{(cz_i;q)_{k_i}}{(az_iq/d;q)_{k_i}}\cdot\frac{(d;q)_{|\mathbf{k}|}}{(aq/c;q)_{|\mathbf{k}|}}\,w^{|\mathbf{k}|}\Bigg).
\end{aligned}
$$

The above ${}_6\Psi_6^{(r)}$ series is an r-dimensional ${}_6\psi_6$ series (which reduces to a classical very-well-posied ${}_6\psi_6$ when $r=1$).

For convenience, we sometimes use capital letters to abbreviate the (r-fold) products of certain variables. Specifically, in this article we use $A\equiv a_1\cdots a_r$, $B\equiv b_1\cdots b_r$, $C\equiv c_1\cdots c_r$, $E\equiv e_1\cdots e_r$, and $F\equiv f_1\cdots f_r$, respectively.

In our derivation of the multilateral q-IPD type transformation in Theorem 4.2 we utilize the following r-dimensional generalization of W. N. Bailey's summation formula in (2.5).

Theorem 3.1 ((Gustafson) An A_{r-1} ${}_6\psi_6$ summation)**.** *Let a, $b_1,\dots,b_r$, c, d, $e_1,\dots,e_r$, and $z_1,\dots,z_r$ be indeterminate, let $r\ge 1$, and suppose that none of the denominators in* (3.6) *vanishes. Then*

$$
\begin{aligned}
(3.6)\quad & {}_6\Psi_6^{(r)}\left[a;b_1,\dots,b_r;c,d;e_1,\dots,e_r;z_1,\dots,z_r\,\middle|\,q,\frac{a^{r+1}q}{BcdE}\right] \\
&=\frac{(aq/Bc,a^rq/dE,aq/cd;q)_\infty}{(a^{r+1}q/BcdE,aq/c,q/d;q)_\infty}\prod_{i,j=1}^r\frac{(az_iq/b_ie_jz_j,qz_i/z_j;q)_\infty}{(qz_i/b_iz_j,az_iq/e_jz_j;q)_\infty} \\
&\qquad\times\prod_{i=1}^r\frac{(aq/ce_iz_i,az_iq/b_id,az_iq,q/az_i;q)_\infty}{(az_iq/b_i,q/e_iz_i,q/cz_i,az_iq/d;q)_\infty},
\end{aligned}
$$

provided $|a^{r+1}q/BcdE|<1$.

Remark 3.2. Using (3.4), the multilateral identity in (3.6) can also be written in a more compact form. We then have R. A. Gustafson's [15, Theorem 1.15] $\tilde{A}_r$ ${}_6\psi_6$ summation: Let $a_1,\dots,a_{r+1}$, $b_1,\dots,b_{r+1}$, and $z_1,\dots,z_{r+1}$ be indeterminate, let $r\ge 1$, and suppose that none of the denominators in (3.7) vanishes. Then

$$
\begin{aligned}
(3.7)\quad & \sum_{\substack{-\infty\le k_1,\dots,k_{r+1}\le\infty\\ k_1+\cdots+k_{r+1}=0}}\prod_{1\le i<j\le r+1}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i,j=1}^{r+1}\frac{(a_jz_i/z_j;q)_{k_i}}{(b_jz_i/z_j;q)_{k_i}} \\
&=\frac{(b_1\dots b_{r+1}q^{-r},q/a_1\dots a_{r+1};q)_\infty}{(q,b_1\dots b_{r+1}q^{-r}/a_1\dots a_{r+1};q)_\infty}\prod_{i,j=1}^{r+1}\frac{(qz_i/z_j,b_jz_i/a_iz_j;q)_\infty}{(b_jz_i/z_j,z_iq/a_iz_j;q)_\infty},
\end{aligned}
$$

provided $|b_1 \dots b_{r+1} q^{-r}/a_1 \dots a_{r+1}| < 1$. It is not difficult to see that (3.7) and (3.6) are equivalent.

We also need the following r-dimensional generalization of the terminating q-Pfaff–Saalschütz summation from S. C. Milne [22, Theorem 4.15].

Theorem 3.3 ((Milne) An A_{r-1} terminating ${}_3\phi_2$ summation)**.** *Let $a_1, \dots, a_r$, b, c, and $x_1, \dots, x_r$, be indeterminate, let N be a nonnegative integer, let $r \geq 1$, and suppose that none of the denominators in* (3.8) *vanishes. Then*

$$\sum_{\substack{k_1,\dots,k_r \geq 0 \\ 0 \leq |\mathbf{k}| \leq N}} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{r} \frac{(a_j x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \times \prod_{i=1}^{r} \frac{(b x_i; q)_{k_i}}{(c x_i; q)_{k_i}} \cdot \frac{(q^{-N}; q)_{|\mathbf{k}|}}{(a_1 \dots a_r b q^{1-N}/c; q)_{|\mathbf{k}|}} q^{|\mathbf{k}|} \Bigg) = \frac{(c/b; q)_N}{(c/a_1 \dots a_r b; q)_N} \prod_{i=1}^{r} \frac{(c x_i/a_i; q)_N}{(c x_i; q)_N}. \tag{3.8}$$

The $r = 1$ case of (3.8) clearly reduces to (2.6).

In our derivation of the multilateral q-IPD type transformation in Theorem 4.6 we utilize R. A. Gustafson's [15, Theorem 1.17] A_{r-1} extension of S. Ramanujan's ${}_1\psi_1$ summation (2.7).

Theorem 3.4 ((Gustafson) An A_{r-1} ${}_1\psi_1$ summation)**.** *Let $a_1, \dots, a_r$, $b_1, \dots, b_r$, $x_1, \dots, x_r$, and z be indeterminate, let $r \geq 1$, and suppose that none of the denominators in* (3.9) *vanishes. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1 \leq i < j \leq r} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{r} \frac{(a_j x_i/x_j; q)_{k_i}}{(b_j x_i/x_j; q)_{k_i}} z^{|\mathbf{k}|} = \frac{(Az, q/Az; q)_\infty}{(z, Bq^{1-r}/Az; q)_\infty} \prod_{i,j=1}^{r} \frac{(b_j x_i/a_i x_j, q x_i/x_j; q)_\infty}{(q x_i/a_i x_j, b_j x_i/x_j; q)_\infty}, \tag{3.9}$$

where $|Bq^{1-r}/A| < |z| < 1$.

Further, we make use of the following terminating q-binomial theorem from S. C. Milne [22, Theorem 5.46], which is a multiple extension of (2.8).

Theorem 3.5 ((Milne) An A_{r-1} terminating q-binomial theorem)**.** *Let $x_1, \dots$, x_r, and z be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers, let $r \geq 1$, and suppose that none of the denominators in* (3.10) *vanishes. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,\dots,r}} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-n_j} x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \prod_{i=1}^{r} x_i^{k_i} \times q^{-\binom{|\mathbf{k}|}{2} + \sum_{i=1}^{r} \binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg) = \prod_{i=1}^{r} (z x_i q^{-|\mathbf{n}|}; q)_{n_i}. \tag{3.10}$$

In Section 4, we also give two multiple series extensions each of Propositions 2.3 and 2.4, see Theorems 4.7, 4.8, 4.9, and 4.10. These A_{r-1} extensions are not as deep as those in Theorems 4.2 or 4.6. In our derivations, we make use of Lemmas 4.3 and 4.9 from [25], displayed as follows:

Lemma 3.6. *Let $b_1,\dots,b_r$ and $x_1,\dots,x_r$ be indeterminate, let $r\ge 1$, and suppose that none of the denominators in (3.11) vanishes. Then, if $f(n)$ is an arbitrary function of integers n, we have*

$$\sum_{n=-\infty}^{\infty}\frac{f(n)}{(Bq^{1-r};q)_n}=\frac{(q;q)_\infty}{(Bq^{1-r};q)_\infty}\prod_{i,j=1}^{r}\frac{(b_jx_i/x_j;q)_\infty}{(qx_i/x_j;q)_\infty}$$
$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{r}(b_jx_i/x_j;q)_{k_i}^{-1}\prod_{i=1}^{r}x_i^{rk_i-|\mathbf{k}|}$$
$$\times(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^{r}\binom{k_i}{2}}\cdot f(|\mathbf{k}|)\Bigg),\tag{3.11}$$

provided the series converge.

Lemma 3.7. *Let $a_1,\dots,a_r$, $b_1,\dots,b_r$, and $x_1,\dots,x_r$ be indeterminate, let $r\ge 1$, and suppose that none of the denominators in (3.12) vanishes. Then, if $g(n)$ is an arbitrary function of integers n, we have*

$$\sum_{n=-\infty}^{\infty}\frac{(A;q)_n}{(Bq^{1-r};q)_n}g(n)$$
$$=\frac{(q,Bq^{1-r}/A;q)_\infty}{(Bq^{1-r},q/A;q)_\infty}\prod_{i,j=1}^{r}\frac{(b_jx_i/x_j,x_iq/a_ix_j;q)_\infty}{(qx_i/x_j,b_jx_i/a_ix_j;q)_\infty}$$
$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{r}\frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}\cdot g(|\mathbf{k}|),\tag{3.12}$$

provided the series converge.

4. Multilateral identities of q-IPD type

Here we give six new multilateral transformations of q-IPD type, extending the q-IPD type transformations of Propositions 2.1, 2.2, 2.3, and 2.4 to higher dimensions. The transformation formula in Theorem 4.2, which generalizes Proposition 2.1, involves multiple series very-well-poised over the root system A_{r-1}. A special case of that theorem is given as Corollary 4.3, which is a multilateral summation formula extending W. C. Chu's [6, Theorem 2] bilateral summation to r-dimensions. A further specialization gives a multiple extension of G. Gasper's [11, Eq. (5.13)] very-well-poised summation, see Corollary 4.4. In Theorem 4.6 we provide an A_{r-1} extension of Proposition 2.2. The interesting feature about that transformation is that it involves multilateral series with an *independent* argument z (subject to convergence), similar to the case of S. Ramanujan's ${}_1\psi_1$ summation (2.7) and its extension in Theorem 3.4. We were, unfortunately, not able to give multidimensional extensions of Propositions 2.3 or 2.4 which are as deep as Theorems 4.2 and 4.6. Instead, we derive multiple extensions of a *simpler* type, using Lemmas 3.6 and 3.7. Theorems 4.7 and 4.8 are simple A_{r-1} extensions of Proposition 2.3, while Theorems 4.9 and 4.10 are simple A_{r-1} extensions of Proposition 2.4. Of course, by the same method one could also derive simple multilateral generalizations of the

q-IPD type transformations in Propositions 2.1 and 2.2. However, we decided to derive the identities of simpler type only in the cases were we were unable to find corresponding deeper ones.

For our derivation of Theorem 4.2, we need the following lemma, which is easily established by applying Theorem 3.1 twice.

Lemma 4.1. *Let a, $b_1, \dots, b_r$, c, d, $e_1, \dots, e_r$, $f_1, \dots, f_r$, and $z_1, \dots, z_r$ be indeterminate, let $r \geq 1$, and suppose that none of the denominators in* (4.1) *vanishes. Then*

$$
\begin{aligned}
(4.1)\quad & {}_6\Psi_6^{(r)}\left[a; b_1, \dots, b_r; c, d; e_1, \dots, e_r; z_1, \dots, z_r \,\middle|\, q, \frac{a^{r+1}q}{BcdE}\right] \\
&= \frac{(Fq/d, aq/cF; q)_\infty}{(aq/c, q/d; q)_\infty} \prod_{i,j=1}^{r} \frac{(qz_i/z_j, az_iq/e_jf_iz_j, f_jz_iq/b_iz_j; q)_\infty}{(qf_jz_i/f_iz_j, qz_i/b_iz_j, az_iq/e_jz_j; q)_\infty} \\
&\quad \times \prod_{i=1}^{r} \frac{(az_iq, q/az_i, Fq/e_iz_i, az_iq/b_iF, az_iq/df_i, f_iq/cz_i; q)_\infty}{(Ff_iq/az_i, az_iq/Ff_i, az_iq/b_i, q/e_iz_i, q/cz_i, az_iq/d; q)_\infty} \\
&\quad \times {}_6\Psi_6^{(r)}\left[\frac{F}{a}; \frac{e_1f_1}{a}, \dots, \frac{e_rf_r}{a}; \frac{d}{a}, \frac{cF}{a}; \frac{b_1F}{af_1}, \dots, \frac{b_rF}{af_r}; \frac{f_1}{z_1}, \dots, \frac{f_r}{z_r} \,\middle|\, q, \frac{a^{r+1}q}{BcdE}\right],
\end{aligned}
$$

provided $|a^{r+1}q/BcdE| < 1$.

Now, for compact notation, let us extend definition (3.5) by introducing additional indeterminates $g_1, \dots, g_s$ and $h_1, \dots, h_s$:

$$
\begin{aligned}
(4.2)\quad & {}_{6+2s}\Psi_{6+2s}^{(r)}\big[a; b_1, \dots, b_r; c, d; e_1, \dots, e_r; z_1, \dots, z_r; \\
&\qquad\qquad g_1, \dots, g_s; h_1, \dots, h_s \,\big|\, q, w\big] \\
&:= \sum_{k_1, \dots, k_r = -\infty}^{\infty} \Bigg(\prod_{1 \leq i < j \leq r} \left(\frac{z_iq^{k_i} - z_jq^{k_j}}{z_i - z_j}\right) \prod_{i=1}^{r} \left(\frac{1 - az_iq^{k_i + |\mathbf{k}|}}{1 - az_i}\right) \\
&\qquad \times \prod_{i,j=1}^{r} \frac{(b_jz_i/z_j; q)_{k_i}}{(az_iq/e_jz_j; q)_{k_i}} \prod_{i=1}^{r} \frac{(e_iz_i; q)_{|\mathbf{k}|}}{(az_iq/b_i; q)_{|\mathbf{k}|}} \\
&\qquad \times \prod_{i=1}^{r} \frac{(cz_i, g_1z_i, \dots, g_sz_i; q)_{k_i}}{(az_iq/d, az_iq/h_1, \dots, az_iq/h_s; q)_{k_i}} \\
&\qquad\qquad \times \frac{(d, h_1, \dots, h_s; q)_{|\mathbf{k}|}}{(aq/c, aq/g_1, \dots, aq/g_s; q)_{|\mathbf{k}|}}\, w^{|\mathbf{k}|} \Bigg).
\end{aligned}
$$

We have

Theorem 4.2 (A multilateral very-well-poised A_{r-1} q-IPD type transformation)**.** *Let a, $b_1, \dots, b_r$, c, d, $e_1, \dots, e_r$, $f_1, \dots, f_r$, $z_1, \dots, z_r$, and $h_1, \dots, h_s$ be indeterminate, let $N_1, \dots, N_s$ be nonnegative integers, let $|N| = \sum_{i=1}^{s} N_i$, $r \geq 1$, and suppose that none of the denominators in* (4.3) *vanishes. Then*

$$
\begin{aligned}
(4.3)\quad & {}_{6+2s}\Psi_{6+2s}^{(r)}\Bigg[a; b_1, \dots, b_r; c, d; e_1, \dots, e_r; z_1, \dots, z_r; \\
&\qquad \frac{aq^{1+N_1}}{h_1}, \dots, \frac{aq^{1+N_s}}{h_s}; h_1, \dots, h_s \,\Bigg|\, q, \frac{a^{r+1}q^{1-|N|}}{BcdE}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=1}^{s}\left[\frac{(Fq/h_j;q)_{N_j}}{(q/h_j;q)_{N_j}}\prod_{i=1}^{r}\frac{(az_iq/f_ih_j;q)_{N_j}}{(az_iq/h_j;q)_{N_j}}\right]\\
&\times\frac{(Fq/d,aq/cF;q)_\infty}{(aq/c,q/d;q)_\infty}\prod_{i,j=1}^{r}\frac{(qz_i/z_j,az_iq/e_jf_iz_j,f_jz_iq/b_iz_j;q)_\infty}{(qf_jz_i/f_iz_j,qz_i/b_iz_j,az_iq/e_jz_j;q)_\infty}\\
&\times\prod_{i=1}^{r}\frac{(az_iq,q/az_i,Fq/e_iz_i,az_iq/b_iF,az_iq/df_i,f_iq/cz_i;q)_\infty}{(Ff_iq/az_i,az_iq/Ff_i,az_iq/b_i,q/e_iz_i,q/cz_i,az_iq/d;q)_\infty}\\
&\times {}_{6+2s}\Psi_{6+2s}^{(r)}\left[\frac{F}{a};\frac{e_1f_1}{a},\dots,\frac{e_rf_r}{a};\frac{d}{a},\frac{cF}{a};\frac{b_1F}{af_1},\dots,\frac{b_rF}{af_r};\frac{f_1}{z_1},\dots,\frac{f_r}{z_r};\right.\\
&\qquad\left.\frac{h_1}{a},\dots,\frac{h_s}{a};\frac{Fq^{1+N_1}}{h_1},\dots,\frac{Fq^{1+N_s}}{h_s}\,\middle|\,q,\frac{a^{r+1}q^{1-|N|}}{BcdE}\right],
\end{aligned}
$$

provided $|a^{r+1}q^{1-|N|}/BcdE|<1$.

Proof. We proceed by induction on s. For $s=0$ (4.3) is true by Lemma 4.1. So, suppose that the transformation is already shown for $s\mapsto s-1$. Then, by using some elementary identities from [13, Appendix I],

$$
\begin{aligned}
&{}_{6+2s}\Psi_{6+2s}^{(r)}\left[a;b_1,\dots,b_r;c,d;e_1,\dots,e_r;z_1,\dots,z_r;\right.\\
&\qquad\left.\frac{aq^{1+N_1}}{h_1},\dots,\frac{aq^{1+N_s}}{h_s};h_1,\dots,h_s\,\middle|\,q,\frac{a^{r+1}q^{1-|N|}}{BcdE}\right]\\
&=\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^{r}\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)\\
&\qquad\times\prod_{i,j=1}^{r}\frac{(b_jz_i/z_j;q)_{k_i}}{(az_iq/e_jz_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(e_iz_i;q)_{|\mathbf{k}|}}{(az_iq/b_i;q)_{|\mathbf{k}|}}\\
&\qquad\times\prod_{i=1}^{r}\frac{(cz_i,az_iq^{1+N_1}/h_1,\dots,az_iq^{1+N_{s-1}}/h_{s-1};q)_{k_i}}{(az_iq/d,az_iq/h_1,\dots,az_iq/h_{s-1};q)_{k_i}}\\
&\qquad\times\frac{(d,h_1,\dots,h_{s-1};q)_{|\mathbf{k}|}}{(aq/c,h_1q^{-N_1},\dots,h_{s-1}q^{-N_{s-1}};q)_{|\mathbf{k}|}}\left(\frac{a^{r+1}q^{1-|N|}}{BcdE}\right)^{|\mathbf{k}|}\\
&\qquad\times\frac{(h_s;q)_{|\mathbf{k}|}}{(h_sq^{-N_s};q)_{|\mathbf{k}|}}\prod_{i=1}^{r}\frac{(az_iq^{1+N_s}/h_s;q)_{k_i}}{(az_iq/h_s;q)_{k_i}}\Bigg)\\
&=\frac{(a^rq/Eh_s;q)_{N_s}}{(q/h_s;q)_{N_s}}\prod_{i=1}^{r}\frac{(e_iz_iq/h_s;q)_{N_s}}{(az_iq/h_s;q)_{N_s}}\\
&\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^{r}\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)\\
&\qquad\times\prod_{i,j=1}^{r}\frac{(b_jz_i/z_j;q)_{k_i}}{(az_iq/e_jz_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(e_iz_i;q)_{|\mathbf{k}|}}{(az_iq/b_i;q)_{|\mathbf{k}|}}\\
&\qquad\times\prod_{i=1}^{r}\frac{(cz_i,az_iq^{1+N_1}/h_1,\dots,az_iq^{1+N_{s-1}}/h_{s-1};q)_{k_i}}{(az_iq/d,az_iq/h_1,\dots,az_iq/h_{s-1};q)_{k_i}}
\end{aligned}
$$

$$\times \frac{(d,h_1,\dots,h_{s-1};q)_{|\mathbf{k}|}}{(aq/c,h_1q^{-N_1},\dots,h_{s-1}q^{-N_{s-1}};q)_{|\mathbf{k}|}}\left(\frac{a^{r+1}q^{1-(N_1+\cdots+N_{s-1})}}{BcdE}\right)^{|\mathbf{k}|} \times \frac{(q^{1-|\mathbf{k}|}/h_s;q)_{N_s}}{(a^rq/Eh_s;q)_{N_s}}\prod_{i=1}^r\frac{(az_iq^{1+k_i}/h_s;q)_{N_s}}{(e_iz_iq/h_s;q)_{N_s}}\Bigg).$$

Now we expand the last factors (those involving $(\cdot;q)_{N_s}$) by applying the $a_i\mapsto e_iq^{-k_i}/a$, $b\mapsto q^{|\mathbf{k}|}$, $c\mapsto q/h_s$, $x_i\mapsto e_iz_i$, $i=1,\dots,r$, and $N\mapsto N_s$ case of the A_{r-1} q-Pfaff–Saalschütz summation in Theorem 3.3. We obtain

$$\frac{(a^rq/Eh_s;q)_{N_s}}{(q/h_s;q)_{N_s}}\prod_{i=1}^r\frac{(e_iz_iq/h_s;q)_{N_s}}{(az_iq/h_s;q)_{N_s}}
\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{z_iq^{k_i}-z_jq^{k_j}}{z_i-z_j}\right)\prod_{i=1}^r\left(\frac{1-az_iq^{k_i+|\mathbf{k}|}}{1-az_i}\right)
\times\prod_{i,j=1}^r\frac{(b_jz_i/z_j;q)_{k_i}}{(az_iq/e_jz_j;q)_{k_i}}\prod_{i=1}^r\frac{(e_iz_i;q)_{|\mathbf{k}|}}{(az_iq/b_i;q)_{|\mathbf{k}|}}
\times\prod_{i=1}^r\frac{(cz_i,az_iq^{1+N_1}/h_1,\dots,az_iq^{1+N_{s-1}}/h_{s-1};q)_{k_i}}{(az_iq/d,az_iq/h_1,\dots,az_iq/h_{s-1};q)_{k_i}}
\times\frac{(d,h_1,\dots,h_{s-1};q)_{|\mathbf{k}|}}{(aq/c,h_1q^{-N_1},\dots,h_{s-1}q^{-N_{s-1}};q)_{|\mathbf{k}|}}\left(\frac{a^{r+1}q^{1-(N_1+\cdots+N_{s-1})}}{BcdE}\right)^{|\mathbf{k}|}
\times\sum_{\substack{l_1,\dots,l_r\ge0\\0\le|\mathbf{l}|\le N_s}}\prod_{1\le i<j\le r}\left(\frac{e_iz_iq^{l_i}-e_jz_jq^{l_j}}{e_iz_i-e_jz_j}\right)\prod_{i,j=1}^r\frac{(e_iz_iq^{-k_j}/az_j;q)_{l_i}}{(qe_iz_i/e_jz_j;q)_{l_i}}
\times\prod_{i=1}^r\frac{(ez_iq^{|\mathbf{k}|};q)_{l_i}}{(e_iz_iq/h_s;q)_{l_i}}\cdot\frac{(q^{-N_s};q)_{|\mathbf{l}|}}{(Eh_sq^{-N_s}/a^r;q)_{|\mathbf{l}|}}q^{|\mathbf{l}|}\Bigg)$$

$$=\frac{(a^rq/Eh_s;q)_{N_s}}{(q/h_s;q)_{N_s}}\prod_{i=1}^r\frac{(e_iz_iq/h_s;q)_{N_s}}{(az_iq/h_s;q)_{N_s}}
\times\sum_{\substack{l_1,\dots,l_r\ge0\\0\le|\mathbf{l}|\le N_s}}\Bigg(\prod_{1\le i<j\le r}\left(\frac{e_iz_iq^{l_i}-e_jz_jq^{l_j}}{e_iz_i-e_jz_j}\right)\prod_{i,j=1}^r\frac{(e_iz_i/az_j;q)_{l_i}}{(qe_iz_i/e_jz_j;q)_{l_i}}
\times\prod_{i=1}^r\frac{(ez_i;q)_{l_i}}{(e_iz_iq/h_s;q)_{l_i}}\cdot\frac{(q^{-N_s};q)_{|\mathbf{l}|}}{(Eh_sq^{-N_s}/a^r;q)_{|\mathbf{l}|}}q^{|\mathbf{l}|}
\times{}_{6+2(s-1)}\Psi^{(r)}_{6+2(s-1)}\left[a;b_1,\dots,b_r;c,d;e_1q^{l_1},\dots,e_rq^{l_r};z_1,\dots,z_r;\right.
\left.\frac{aq^{1+N_1}}{h_1},\dots,\frac{aq^{1+N_{s-1}}}{h_{s-1}};h_1,\dots,h_{s-1}\middle|\,q,\frac{a^{r+1}q^{1-(N_1+\cdots+N_{s-1})-|\mathbf{l}|}}{BcdE}\right]\Bigg).$$

By the $e_i\mapsto e_iq^{l_i}$, $i=1,\dots,r$, case of the inductive hypothesis we obtain

$$\frac{(a^rq/Eh_s;q)_{N_s}}{(q/h_s;q)_{N_s}}\prod_{i=1}^r\frac{(e_iz_iq/h_s;q)_{N_s}}{(az_iq/h_s;q)_{N_s}}$$

$$\times \sum_{\substack{l_1,\dots,l_r\geq 0\\ 0\leq|\mathbf{l}|\leq N_s}} \Bigg(\prod_{1\leq i<j\leq r} \left(\frac{e_iz_iq^{l_i}-e_jz_jq^{l_j}}{e_iz_i-e_jz_j} \right) \prod_{i,j=1}^r \frac{(e_iz_i/az_j;q)_{l_i}}{(qe_iz_i/e_jz_j;q)_{l_i}}$$

$$\times \prod_{i=1}^r \frac{(ez_i;q)_{l_i}}{(e_iz_iq/h_s;q)_{l_i}} \cdot \frac{(q^{-N_s};q)_{|\mathbf{l}|}}{(Eh_sq^{-N_s}/a^r;q)_{|\mathbf{l}|}} q^{|\mathbf{l}|}$$

$$\times \prod_{j=1}^{s-1} \left[\frac{(Fq/h_j;q)_{N_j}}{(q/h_j;q)_{N_j}} \prod_{i=1}^r \frac{(az_iq/f_ih_j;q)_{N_j}}{(az_iq/h_j;q)_{N_j}} \right]$$

$$\times \frac{(Fq/d,aq/cF;q)_\infty}{(aq/c,q/d;q)_\infty} \prod_{i,j=1}^r \frac{(qz_i/z_j, az_iq^{1-l_j}/e_jf_iz_j, f_jz_iq/b_iz_j;q)_\infty}{(qf_jz_i/f_iz_j, qz_i/b_iz_j, az_iq^{1-l_j}/e_jz_j;q)_\infty}$$

$$\times \prod_{i=1}^r \frac{(az_iq, q/az_i, Fq^{1-l_i}/e_iz_i, az_iq/b_iF, az_iq/df_i, f_iq/cz_i;q)_\infty}{(Ff_iq/az_i, az_iq/Ff_i, az_iq/b_i, q^{1-l_i}/e_iz_i, q/cz_i, az_iq/d;q)_\infty}$$

$$\times \sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\leq i<j\leq r} \left(\frac{f_iq^{k_i}/z_i-f_jq^{k_j}/z_j}{f_i/z_i-f_j/z_j} \right) \prod_{i=1}^r \left(\frac{1-Ff_iq^{k_i+|\mathbf{k}|}/az_i}{1-Ff_i/az_i} \right)$$

$$\times \prod_{i,j=1}^r \frac{(e_jf_iz_jq^{l_j}/az_i;q)_{k_i}}{(f_iz_jq/b_jz_i;q)_{k_i}} \prod_{i=1}^r \frac{(b_iF/az_i;q)_{|\mathbf{k}|}}{(Fq^{1-l_i}/e_iz_i;q)_{|\mathbf{k}|}}$$

$$\times \prod_{i=1}^r \frac{(df_i/az_i, f_ih_1/az_i,\dots,f_ih_{s-1}/az_i;q)_{k_i}}{(f_iq/cz_i, f_ih_1q^{-N_1}/az_i,\dots,f_ih_{s-1}q^{-N_{s-1}}/az_i;q)_{k_i}}$$

$$\times \frac{(cF/a, Fq^{1+N_1}/h_1,\dots,Fq^{1+N_{s-1}}/h_{s-1};q)_{|\mathbf{k}|}}{(Fq/d, Fq/h_1,\dots,Fq/h_{s-1};q)_{|\mathbf{k}|}} \left(\frac{a^{r+1}q^{1-(N_1+\cdots+N_{s-1})-|\mathbf{l}|}}{BcdE} \right)^{|\mathbf{k}|} \Bigg)$$

$$= \frac{(a^rq/Eh_s;q)_{N_s}}{(q/h_s;q)_{N_s}} \prod_{i=1}^r \frac{(e_iz_iq/h_s;q)_{N_s}}{(az_iq/h_s;q)_{N_s}} \prod_{j=1}^{s-1} \left[\frac{(Fq/h_j;q)_{N_j}}{(q/h_j;q)_{N_j}} \prod_{i=1}^r \frac{(az_iq/f_ih_j;q)_{N_j}}{(az_iq/h_j;q)_{N_j}} \right]$$

$$\times \frac{(Fq/d,aq/cF;q)_\infty}{(aq/c,q/d;q)_\infty} \prod_{i,j=1}^r \frac{(qz_i/z_j, az_iq/e_jf_iz_j, f_jz_iq/b_iz_j;q)_\infty}{(qf_jz_i/f_iz_j, qz_i/b_iz_j, az_iq/e_jz_j;q)_\infty}$$

$$\times \prod_{i=1}^r \frac{(az_iq, q/az_i, Fq/e_iz_i, az_iq/b_iF, az_iq/df_i, f_iq/cz_i;q)_\infty}{(Ff_iq/az_i, az_iq/Ff_i, az_iq/b_i, q/e_iz_i, q/cz_i, az_iq/d;q)_\infty}$$

$$\times \sum_{k_1,\dots,k_r=-\infty}^{\infty} \Bigg(\prod_{1\leq i<j\leq r} \left(\frac{f_iq^{k_i}/z_i-f_jq^{k_j}/z_j}{f_i/z_i-f_j/z_j} \right) \prod_{i=1}^r \left(\frac{1-Ff_iq^{k_i+|\mathbf{k}|}/az_i}{1-Ff_i/az_i} \right)$$

$$\times \prod_{i,j=1}^r \frac{(e_jf_iz_j/az_i;q)_{k_i}}{(f_iz_jq/b_jz_i;q)_{k_i}} \prod_{i=1}^r \frac{(b_iF/az_i;q)_{|\mathbf{k}|}}{(Fq/e_iz_i;q)_{|\mathbf{k}|}}$$

$$\times \prod_{i=1}^r \frac{(df_i/az_i, f_ih_1/az_i,\dots,f_ih_{s-1}/az_i;q)_{k_i}}{(f_iq/cz_i, f_ih_1q^{-N_1}/az_i,\dots,f_ih_{s-1}q^{-N_{s-1}}/az_i;q)_{k_i}}$$

$$\times \frac{(cF/a, Fq^{1+N_1}/h_1,\dots,Fq^{1+N_{s-1}}/h_{s-1};q)_{|\mathbf{k}|}}{(Fq/d, Fq/h_1,\dots,Fq/h_{s-1};q)_{|\mathbf{k}|}} \left(\frac{a^{r+1}q^{1-(N_1+\cdots+N_{s-1})}}{BcdE} \right)^{|\mathbf{k}|}$$

$$\times \sum_{\substack{l_1,\dots,l_r\geq 0\\ 0\leq |\mathbf{k}|\leq N_s}} \prod_{1\leq i<j\leq r} \left(\frac{e_iz_iq^{l_i}-e_jz_jq^{l_j}}{e_iz_i-e_jz_j}\right) \prod_{i,j=1}^r \frac{(e_if_jz_iq^{k_j}/az_j;q)_{l_i}}{(qe_iz_i/e_jz_j;q)_{l_i}}$$

$$\times \prod_{i=1}^r \frac{(ez_iq^{-|\mathbf{k}|}/F;q)_{l_i}}{(e_iz_iq/h_s;q)_{l_i}} \cdot \frac{(q^{-N_s};q)_{|\mathbf{l}|}}{(Eh_sq^{-N_s}/a^r;q)_{|\mathbf{l}|}}\, q^{|\mathbf{l}|} \Bigg).$$

Now, we evaluate the inner multiple sum by the $a_i \mapsto e_if_iq^{k_i}/a$, $b \mapsto q^{-|\mathbf{k}|}/F$, $c \mapsto q/h_s$, $x_i \mapsto e_iz_i$, $i=1,\dots,r$, and $N \mapsto N_s$ case of the A_{r-1} q-Pfaff–Saalschütz summation in Theorem 3.3, and obtain

$$\prod_{j=1}^s \left[\frac{(Fq/h_j;q)_{N_j}}{(q/h_j;q)_{N_j}} \prod_{i=1}^r \frac{(az_iq/f_ih_j;q)_{N_j}}{(az_iq/h_j;q)_{N_j}}\right]$$

$$\times \frac{(Fq/d,aq/cF;q)_\infty}{(aq/c,q/d;q)_\infty} \prod_{i,j=1}^r \frac{(qz_i/z_j,az_iq/e_jf_iz_j,f_jz_iq/b_iz_j;q)_\infty}{(qf_jz_i/f_iz_j,qz_i/b_iz_j,az_iq/e_jz_j;q)_\infty}$$

$$\times \prod_{i=1}^r \frac{(az_iq,q/az_i,Fq/e_iz_i,az_iq/b_iF,az_iq/df_i,f_iq/cz_i;q)_\infty}{(Ff_iq/az_i,az_iq/Ff_i,az_iq/b_i,q/e_iz_i,q/cz_i,az_iq/d;q)_\infty}$$

$$\times \sum_{k_1,\dots,k_r=-\infty}^{\infty} \Bigg(\prod_{1\leq i<j\leq r} \left(\frac{f_iq^{k_i}/z_i-f_jq^{k_j}/z_j}{f_i/z_i-f_j/z_j}\right) \prod_{i=1}^r \left(\frac{1-Ff_iq^{k_i+|\mathbf{k}|}/az_i}{1-Ff_i/az_i}\right)$$

$$\times \prod_{i,j=1}^r \frac{(e_jf_iz_j/az_i;q)_{k_i}}{(f_iz_jq/b_jz_i;q)_{k_i}} \prod_{i=1}^r \frac{(b_iF/az_i;q)_{|\mathbf{k}|}}{(Fq/e_iz_i;q)_{|\mathbf{k}|}}$$

$$\times \prod_{i=1}^r \frac{(df_i/az_i,f_ih_1/az_i,\dots,f_ih_s/az_i;q)_{k_i}}{(f_iq/cz_i,f_ih_1q^{-N_1}/az_i,\dots,f_ih_sq^{-N_s}/az_i;q)_{k_i}}$$

$$\times \frac{(cF/a,Fq^{1+N_1}/h_1,\dots,Fq^{1+N_s}/h_s;q)_{|\mathbf{k}|}}{(Fq/d,Fq/h_1,\dots,Fq/h_s;q)_{|\mathbf{k}|}} \left(\frac{a^{r+1}q^{1-|N|}}{BcdE}\right)^{|\mathbf{k}|} \Bigg),$$

which is the right side of (4.3). □

A special case of Theorem 4.2 immediately gives the following summation formula as a corollary. It is an A_{r-1} extension of an identity due to W. C. Chu [6, Theorem 2].

Corollary 4.3 (A multilateral very-well-poised A_{r-1} q-IPD type summation). *Let $a, b_1,\dots,b_r, d, e_1,\dots,e_r, z_1,\dots,z_r$, and $h_1,\dots,h_s$ be indeterminate, let $N_1,\dots,N_s$ be nonnegative integers, let $|N|=\sum_{i=1}^s N_i$, $r\geq 1$, and suppose that none of the denominators in* (4.4) *vanishes. Then*

$$(4.4)\quad {}_{6+2s}\Psi_{6+2s}^{(r)}\Bigg[a;b_1,\dots,b_r;\frac{a}{B},d;e_1,\dots,e_r;z_1,\dots,z_r;$$

$$\frac{aq^{1+N_1}}{h_1},\dots,\frac{aq^{1+N_s}}{h_s};h_1,\dots,h_s\,\Bigg|\,q,\frac{a^rq^{1-|N|}}{dE}\Bigg]$$

$$= \frac{(Bq/d,q;q)_\infty}{(Bq,q/d;q)_\infty} \prod_{i=1}^r \frac{(az_iq,q/az_i,Bq/e_iz_i,az_iq/db_i;q)_\infty}{(az_iq/b_i,q/e_iz_i,Bq/az_i,az_iq/d;q)_\infty}$$

$$\times \prod_{i,j=1}^{r} \frac{(qz_i/z_j, az_iq/e_jb_iz_j;q)_\infty}{(qz_i/b_iz_j, az_iq/e_jz_j;q)_\infty} \prod_{j=1}^{s} \left[\frac{(Bq/h_j;q)_{N_j}}{(q/h_j;q)_{N_j}} \prod_{i=1}^{r} \frac{(az_iq/b_ih_j;q)_{N_j}}{(az_iq/h_j;q)_{N_j}} \right],$$

provided $|a^r q^{1-|N|}/dE| < 1$.

Proof. In (4.3), we let $c \to a/B$ and $f_i \to b_i$, for $i = 1, \dots, r$. In this case the ${}_{6+2s}\Psi_{6+2s}^{(r)}$ series on the right side terminates from below, and from above, and evalutes to one. In particular, the appearance of the factor

$$\prod_{i,j=1}^{r} (b_iz_jq/b_jz_i;q)_{k_i}^{-1}$$

makes the terms in the series vanish unless $k_i \geq 0$, $i = 1, \dots, r$. Similarly, the appearance of the factor

$$(1;q)_{|\mathbf{k}|}$$

ensures that if $|\mathbf{k}| > 0$, the terms of the series are zero. In total, only the term where $k_1 = \cdots = k_r = 0$ survives, and that term is just one. □

A further specialization of Corollary 4.3, namely the case $e_i \to a$, $i = 1, \dots, r$, yields an r-dimensional generalization of G. Gasper's [11, Eq. (5.13)] very-well-poised ${}_{6+2s}\phi_{5+2s}$ summation.

Corollary 4.4 (A very-well-poised A_{r-1} q-IPD type summation)**.** *Let* a, $b_1, \dots, b_r$, d, $z_1, \dots, z_r$, *and* $h_1, \dots, h_s$ *be indeterminate, let* $N_1, \dots, N_s$ *be nonnegative integers, let* $|N| = \sum_{i=1}^{s} N_i$, $r \geq 1$, *and suppose that none of the denominators in* (4.5) *vanishes. Then*

$$\sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{z_iq^{k_i} - z_jq^{k_j}}{z_i - z_j} \right) \prod_{i=1}^{r} \left(\frac{1 - az_iq^{k_i+|\mathbf{k}|}}{1 - az_i} \right)$$
$$\times \prod_{i,j=1}^{r} \frac{(b_jz_i/z_j;q)_{k_i}}{(qz_i/z_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(az_i;q)_{|\mathbf{k}|}}{(az_iq/b_i;q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{r} \frac{(az_i/B, az_iq^{1+N_1}/h_1, \dots, az_iq^{1+N_s}/h_s;q)_{k_i}}{(az_iq/d, az_iq/h_1, \dots, az_iq/h_s;q)_{k_i}}$$
$$\times \frac{(d, h_1, \dots, h_s;q)_{|\mathbf{k}|}}{(Bq, h_1q^{-N_1}, \dots, h_sq^{-N_s};q)_{|\mathbf{k}|}} \left(\frac{q^{1-|N|}}{d} \right)^{|\mathbf{k}|} \Bigg)$$
$$= \frac{(Bq/d, q;q)_\infty}{(Bq, q/d;q)_\infty} \prod_{i=1}^{r} \frac{(az_iq, az_iq/b_id;q)_\infty}{(az_iq/b_i, az_iq/d;q)_\infty}$$
$$\times \prod_{j=1}^{s} \left[\frac{(Bq/h_j;q)_{N_j}}{(q/h_j;q)_{N_j}} \prod_{i=1}^{r} \frac{(az_iq/b_ih_j;q)_{N_j}}{(az_iq/h_j;q)_{N_j}} \right], \tag{4.5}$$

provided $|q^{1-|N|}/d| < 1$.

To derive the multilateral q-IPD type transformation in Theorem 4.6 we need the following lemma, which is easily established by applying Theorem 3.4 twice.

Lemma 4.5. *Let* $a_1, \dots, a_r$, $b_1, \dots, b_r$, $c_1, \dots, c_r$, $x_1, \dots, x_r$, $y_1, \dots, y_r$, *and* z *be indeterminate, let* $r \geq 1$, *and suppose that none of the denominators in* (4.6) *vanishes. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right) \prod_{i,j=1}^{r} \frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}} z^{|\mathbf{k}|} \tag{4.6}$$
$$= \frac{(Az,q/Az;q)_\infty}{(Azq^r/C,Cq^{1-r}/Az;q)_\infty} \prod_{i,j=1}^{r} \frac{(qx_i/x_j,b_jx_i/a_ix_j,c_iy_i/a_iy_j,b_jy_iq/c_jy_j;q)_\infty}{(qy_i/y_j,b_jc_iy_i/a_ic_jy_j,qx_i/a_ix_j,b_jx_i/x_j;q)_\infty}$$
$$\times \sum_{k_1,\dots,k_r=-\infty}^{\infty} \prod_{1\le i<j\le r} \left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right) \prod_{i,j=1}^{r} \frac{(a_jy_iq/c_jy_j;q)_{k_i}}{(b_jy_iq/c_jy_j;q)_{k_i}} z^{|\mathbf{k}|},$$

where $|Bq^{1-r}/A|<|z|<1$.

We have

Theorem 4.6 (A multilateral A_{r-1} q-IPD type transformation)**.** *Let $a_1,\dots,a_r$, $b_1,\dots,b_r$, $c_1,\dots,c_r$, $x_1,\dots,x_r$, $y_1,\dots,y_r$, $h_{11},\dots,h_{rs}$, and z be indeterminate, let $N_{11},\dots,N_{rs}$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (4.7) *vanishes. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \left(\prod_{1\le i<j\le r} \left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right) \prod_{i,j=1}^{r} \frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}\right.$$
$$\left.\times \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{(h_{ij}q^{N_{ij}};q)_{|\mathbf{k}|}}{(h_{ij};q)_{|\mathbf{k}|}} z^{|\mathbf{k}|}\right) \tag{4.7}$$
$$= \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{(h_{ij}q^r/C;q)_{N_{ij}}}{(h_{ij};q)_{N_{ij}}} \cdot \frac{(Az,q/Az;q)_\infty}{(Azq^r/C,Cq^{1-r}/Az;q)_\infty}$$
$$\times \prod_{i,j=1}^{r} \frac{(qx_i/x_j,b_jx_i/a_ix_j,c_iy_i/a_iy_j,b_jy_iq/c_jy_j;q)_\infty}{(qy_i/y_j,b_jc_iy_i/a_ic_jy_j,qx_i/a_ix_j,b_jx_i/x_j;q)_\infty}$$
$$\times \sum_{k_1,\dots,k_r=-\infty}^{\infty} \left(\prod_{1\le i<j\le r} \left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right) \prod_{i,j=1}^{r} \frac{(a_jy_iq/c_jy_j;q)_{k_i}}{(b_jy_iq/c_jy_j;q)_{k_i}}\right.$$
$$\left.\times \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{(h_{ij}q^{r+N_{ij}}/C;q)_{|\mathbf{k}|}}{(h_{ij}q^r/C;q)_{|\mathbf{k}|}} z^{|\mathbf{k}|}\right),$$

provided $\left|Bq^{1-r-\sum_{i,j}N_{ij}}/A\right|<|z|<1$.

Proof. We proceed by induction on s. For $s=0$ (4.7) is true by Lemma 4.5. So, suppose that the transformation is already shown for $s\mapsto s-1$. Then (again using some elementary identities from [13, Appendix I]),

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \left(\prod_{1\le i<j\le r} \left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right) \prod_{i,j=1}^{r} \frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}\right.$$
$$\left.\times \prod_{i=1}^{r}\prod_{j=1}^{s-1} \frac{(h_{ij}q^{N_{ij}};q)_{|\mathbf{k}|}}{(h_{ij};q)_{|\mathbf{k}|}} z^{|\mathbf{k}|} \cdot \prod_{i=1}^{r} \frac{(h_{is}q^{N_{is}};q)_{|\mathbf{k}|}}{(h_{is};q)_{|\mathbf{k}|}}\right)$$
$$= \prod_{i=1}^{r} \frac{1}{(h_{is};q)_{N_i}} \sum_{k_1,\dots,k_r=-\infty}^{\infty} \left(\prod_{1\le i<j\le r} \left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right) \prod_{i,j=1}^{r} \frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}\right.$$

$$\times \prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{N_{ij}};q)_{|\mathbf{k}|}}{(h_{ij};q)_{|\mathbf{k}|}}\,z^{|\mathbf{k}|}\cdot\prod_{i=1}^{r}(h_{is}q^{|\mathbf{k}|};q)_{N_{is}}\Bigg).$$

Now we expand the last factors (those involving $(\cdot;q)_{N_{is}}$) by applying the $x_i\mapsto h_{is}$, $n_i\mapsto N_{is}$, $i=1,\dots,r$, and $z\mapsto q^{|\mathbf{k}|+(N_{1s}+\cdots+N_{rs})}$ case of the A_{r-1} summation in Theorem 3.5. We obtain

$$\prod_{i=1}^{r}\frac{1}{(h_{is};q)_{N_i}}\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{r}\frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}$$

$$\times\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{N_{ij}};q)_{|\mathbf{k}|}}{(h_{ij};q)_{|\mathbf{k}|}}\,z^{|\mathbf{k}|}$$

$$\times\sum_{\substack{0\le l_i\le N_{is}\\ i=1,\dots,r}}\prod_{1\le i<j\le r}\left(\frac{h_{is}q^{l_i}-h_{js}q^{l_j}}{h_{is}-h_{js}}\right)\prod_{i,j=1}^{r}\frac{(q^{-N_{js}}h_{is}/h_{js};q)_{l_i}}{(qh_{is}/h_{js};q)_{l_i}}$$

$$\times\prod_{i=1}^{r}h_{is}^{l_i}\cdot q^{|\mathbf{k}||\mathbf{l}|+(N_{1s}+\cdots+N_{rs})|\mathbf{l}|-\binom{|\mathbf{l}|}{2}+\sum_{i=1}^{r}\binom{l_i}{2}}\Bigg)$$

$$=\prod_{i=1}^{r}\frac{1}{(h_{is};q)_{N_i}}\sum_{\substack{0\le l_i\le N_{is}\\ i=1,\dots,r}}\Bigg(\prod_{1\le i<j\le r}\left(\frac{h_{is}q^{l_i}-h_{js}q^{l_j}}{h_{is}-h_{js}}\right)\prod_{i,j=1}^{r}\frac{(q^{-N_{js}}h_{is}/h_{js};q)_{l_i}}{(qh_{is}/h_{js};q)_{l_i}}$$

$$\times\prod_{i=1}^{r}h_{is}^{l_i}\cdot q^{(N_{1s}+\cdots+N_{rs})|\mathbf{l}|-\binom{|\mathbf{l}|}{2}+\sum_{i=1}^{r}\binom{l_i}{2}}$$

$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{r}\frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}$$

$$\times\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{N_{ij}};q)_{|\mathbf{k}|}}{(h_{ij};q)_{|\mathbf{k}|}}\left(zq^{|\mathbf{l}|}\right)^{|\mathbf{k}|}\Bigg).$$

By the $z\mapsto zq^{|\mathbf{l}|}$ case of the inductive hypothesis we obtain

$$\prod_{i=1}^{r}\frac{1}{(h_{is};q)_{N_i}}\sum_{\substack{0\le l_i\le N_{is}\\ i=1,\dots,r}}\Bigg(\prod_{1\le i<j\le r}\left(\frac{h_{is}q^{l_i}-h_{js}q^{l_j}}{h_{is}-h_{js}}\right)\prod_{i,j=1}^{r}\frac{(q^{-N_{js}}h_{is}/h_{js};q)_{l_i}}{(qh_{is}/h_{js};q)_{l_i}}$$

$$\times\prod_{i=1}^{r}h_{is}^{l_i}\cdot q^{(N_{1s}+\cdots+N_{rs})|\mathbf{l}|-\binom{|\mathbf{l}|}{2}+\sum_{i=1}^{r}\binom{l_i}{2}}$$

$$\times\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{r}/C;q)_{N_{ij}}}{(h_{ij};q)_{N_{ij}}}\cdot\frac{(Azq^{|\mathbf{l}|},q^{1-|\mathbf{l}|}/Az;q)_\infty}{(Azq^{r+|\mathbf{l}|}/C,Cq^{1-r-|\mathbf{l}|}/Az;q)_\infty}$$

$$\times\prod_{i,j=1}^{r}\frac{(qx_i/x_j,b_jx_i/a_ix_j,c_iy_i/a_iy_j,b_jy_iq/c_jy_j;q)_\infty}{(qy_i/y_j,b_jc_iy_i/a_ic_jy_j,qx_i/a_ix_j,b_jx_i/x_j;q)_\infty}$$

$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\prod_{1\le i<j\le r}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^{r}\frac{(a_jy_iq/c_jy_j;q)_{k_i}}{(b_jy_iq/c_jy_j;q)_{k_i}}$$

$$\times \prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{r+N_{ij}}/C;q)_{|\mathbf{k}|}}{(h_{ij}q^{r}/C;q)_{|\mathbf{k}|}}\left(zq^{|\mathbf{l}|}\right)^{|\mathbf{k}|}\Bigg)$$

$$=\prod_{i=1}^{r}\frac{1}{(h_{is};q)_{N_i}}\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{r}/C;q)_{N_{ij}}}{(h_{ij};q)_{N_{ij}}}\cdot\frac{(Az,q/Az;q)_\infty}{(Azq^r/C,Cq^{1-r}/Az;q)_\infty}$$

$$\times\prod_{i,j=1}^{r}\frac{(qx_i/x_j,b_jx_i/a_ix_j,c_iy_i/a_iy_j,b_jy_iq/c_jy_j;q)_\infty}{(qy_i/y_j,b_jc_iy_i/a_ic_jy_j,qx_i/a_ix_j,b_jx_i/x_j;q)_\infty}$$

$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^{r}\frac{(a_jy_iq/c_jy_j;q)_{k_i}}{(b_jy_iq/c_jy_j;q)_{k_i}}\right.$$

$$\times\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{r+N_{ij}}/C;q)_{|\mathbf{k}|}}{(h_{ij}q^{r}/C;q)_{|\mathbf{k}|}}z^{|\mathbf{k}|}$$

$$\times\sum_{\substack{0\le l_i\le N_{is}\\ i=1,\dots,r}}\prod_{1\le i<j\le r}\left(\frac{h_{is}q^{l_i}-h_{js}q^{l_j}}{h_{is}-h_{js}}\right)\prod_{i,j=1}^{r}\frac{(q^{-N_{js}}h_{is}/h_{js};q)_{l_i}}{(qh_{is}/h_{js};q)_{l_i}}$$

$$\left.\times\prod_{i=1}^{r}h_{is}^{l_i}\cdot\left(\frac{q^r}{C}\right)^{|\mathbf{l}|}q^{|\mathbf{k}||\mathbf{l}|+(N_{1s}+\dots+N_{rs})|\mathbf{l}|-\binom{|\mathbf{l}|}{2}+\sum_{i=1}^{r}\binom{l_i}{2}}\right).$$

Now, we evaluate the inner multiple sum by the $x_i\mapsto h_{is}$, $n_i\mapsto N_{is}$, $i=1,\dots,r$, and $z\mapsto q^{r+|\mathbf{k}|+(N_{1s}+\dots+N_{rs})}/C$ case of the A_{r-1} summation in Theorem 3.5. We obtain

$$\prod_{i=1}^{r}\frac{1}{(h_{is};q)_{N_i}}\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{r}/C;q)_{N_{ij}}}{(h_{ij};q)_{N_{ij}}}\cdot\frac{(Az,q/Az;q)_\infty}{(Azq^r/C,Cq^{1-r}/Az;q)_\infty}$$

$$\times\prod_{i,j=1}^{r}\frac{(qx_i/x_j,b_jx_i/a_ix_j,c_iy_i/a_iy_j,b_jy_iq/c_jy_j;q)_\infty}{(qy_i/y_j,b_jc_iy_i/a_ic_jy_j,qx_i/a_ix_j,b_jx_i/x_j;q)_\infty}$$

$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^{r}\frac{(a_jy_iq/c_jy_j;q)_{k_i}}{(b_jy_iq/c_jy_j;q)_{k_i}}\right.$$

$$\left.\times\prod_{i=1}^{r}\prod_{j=1}^{s-1}\frac{(h_{ij}q^{r+N_{ij}}/C;q)_{|\mathbf{k}|}}{(h_{ij}q^{r}/C;q)_{|\mathbf{k}|}}z^{|\mathbf{k}|}\cdot\prod_{i=1}^{r}(h_iq^{r+|\mathbf{k}|}/C;q)_{N_{is}}\right),$$

which, after an elementary manipulation of q-shifted factorials, gives us the right side of (4.7), as desired. □

Finally, we provide four more multilateral transformations of q-IPD type. Unfortunately, we were not able to find multiple extensions of Propositions 2.3 or 2.4 which are as deep as the identities in Theorems 4.2 and 4.6. The following theorems are obtained by combining Propositions 2.3 and 2.4 each with Lemmas 3.6 and 3.7, thus giving rise to four different multilateral transformations.

Theorem 4.7 (A multilateral A_{r-1} q-IPD type transformation)**.** *Let a, b, $c_1,\dots,c_r$, d, $e_1,\dots,e_r$, $x_1,\dots,x_r$, $y_1,\dots,y_r$, and $h_1,\dots,h_s$ be indeterminate, let N be an integer, let $m_1,\dots,m_s$ be nonnegative integers, let $|m|=\sum_{i=1}^{s}m_i$, $r\ge 1$, and suppose that none of the denominators in (4.8) vanishes. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{r}(c_jx_i/x_j;q)_{k_i}^{-1}\prod_{i=1}^{r}x_i^{rk_i-|\mathbf{k}|}$$
$$\times(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^{r}\binom{k_i}{2}}$$
$$\times\frac{(a,b,h_1q^{m_1},\dots,h_sq^{m_s};q)_{|\mathbf{k}|}}{(d,h_1,\dots,h_s;q)_{|\mathbf{k}|}}\left(\frac{Eq^{1-r-N}}{ab}\right)^{|\mathbf{k}|}\Bigg)$$
$$=(Eq^{-r})^N\frac{(Eq^{1-r}/a,Eq^{1-r}/b,dq^r/E;q)_\infty}{(q/a,q/b,d;q)_\infty}$$
$$\times\prod_{i=1}^{s}\frac{(h_iq^r/E;q)_{m_i}}{(h_i;q)_{m_i}}\prod_{i,j=1}^{r}\frac{(qx_i/x_j,c_jy_iq/e_jy_j;q)_\infty}{(qy_i/y_j,c_jx_i/x_j;q)_\infty}$$
$$\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^{r}(c_jy_iq/e_jy_j;q)_{k_i}^{-1}\prod_{i=1}^{r}y_i^{rk_i-|\mathbf{k}|}$$
$$\times(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^{r}\binom{k_i}{2}}$$
$$\times\frac{(aq^r/E,bq^r/E,h_1q^{r+m_1}/E,\dots,h_sq^{r+m_s}/E;q)_{|\mathbf{k}|}}{(dq^r/E,h_1q^r/E,\dots,h_sq^r/E;q)_{|\mathbf{k}|}}\left(\frac{Eq^{1-r-N}}{ab}\right)^{|\mathbf{k}|}\Bigg), \tag{4.8}$$

provided $|Eq^{1-r}/ab|<|q^N|<|Eq^{|m|}/Cd|$.

Proof. We have, for $|Eq^{1-r}/ab|<|q^N|<|Eq^{|m|}/Cd|$,

$$ {}_{2+s}\psi_{2+s}\left[\begin{matrix}a,b,h_1q^{m_1},\dots,h_sq^{m_s}\\ Cq^{1-r},d,h_1,\dots,h_s\end{matrix};q,\frac{Eq^{1-r-N}}{ab}\right] \tag{4.9}$$
$$=(Eq^{-r})^N\frac{(Eq^{1-r}/a,Eq^{1-r}/b,Cq/E,dq^r/E;q)_\infty}{(q/a,q/b,Cq^{1-r},d;q)_\infty}\prod_{i=1}^{s}\frac{(h_iq^r/E;q)_{m_i}}{(h_i;q)_{m_i}}$$
$$\times{}_{2+s}\psi_{2+s}\left[\begin{matrix}aq^r/E,bq^r/E,h_1q^{r+m_1}/E,\dots,h_sq^{r+m_s}/E\\ Cq/E,dq^r/E,h_1q^r/E,\dots,h_sq^r/E\end{matrix};q,\frac{Eq^{1-r-N}}{ab}\right],$$

by the q-IPD type transformation in (2.11). Now we apply Lemma 3.6 to the ${}_{2+s}\psi_{2+s}$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_{2+s}\psi_{2+s}$ on left side of (4.9) by the $b_i\mapsto c_i$, $i=1,\dots,r$, and

$$f(n)=\frac{(a,b,h_1q^{m_1},\dots,h_sq^{m_s};q)_n}{(d,h_1,\dots,h_s;q)_n}\left(\frac{Eq^{1-r-N}}{ab}\right)^n$$

case of Lemma 3.6. The ${}_{2+s}\psi_{2+s}$ on the right side of (4.9) is rewritten by the $b_i\mapsto c_iq/e_i$, $x_i\mapsto y_i$, $i=1,\dots,r$, and

$$f(n)=\frac{(aq^r/E,bq^r/E,h_1q^{r+m_1}/E,\dots,h_sq^{r+m_s}/E;q)_n}{(dq^r/E,h_1q^r/E,\dots,h_sq^r/E;q)_n}\left(\frac{Eq^{1-r-N}}{ab}\right)^n$$

case of Lemma 3.6. Finally, we divide both sides of the resulting equation by

$$\frac{(q;q)_\infty}{(Cq^{1-r};q)_\infty}\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j;q)_\infty}{(qx_i/x_j;q)_\infty} \tag{4.10}$$

and simplify to obtain (4.8). □

Theorem 4.8 (A multilateral A_{r-1} q-IPD type transformation). *Let $a_1,\dots,a_r$, b, $c_1,\dots,c_r$, d, $e_1,\dots,e_r$, $x_1,\dots,x_r$, $y_1,\dots,y_r$, and $h_1,\dots,h_s$ be indeterminate, let N be an integer, let $m_1,\dots,m_s$ be nonnegative integers, let $|m|=\sum_{i=1}^s m_i$, $r\geq 1$, and suppose that none of the denominators in (4.11) vanishes. Then*

$$\begin{aligned}
(4.11)\qquad &\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{r}\frac{(a_jx_i/x_j;q)_{k_i}}{(c_jx_i/x_j;q)_{k_i}}\\
&\qquad\times\frac{(b,h_1q^{m_1},\dots,h_sq^{m_s};q)_{|\mathbf{k}|}}{(d,h_1,\dots,h_s;q)_{|\mathbf{k}|}}\left(\frac{Eq^{1-r-N}}{Ab}\right)^{|\mathbf{k}|}\Bigg)\\
&=(Eq^{-r})^N\frac{(Eq^{1-r}/b,dq^r/E;q)_\infty}{(q/b,d;q)_\infty}\prod_{i=1}^{s}\frac{(h_iq^r/E;q)_{m_i}}{(h_i;q)_{m_i}}\\
&\qquad\times\prod_{i,j=1}^{r}\frac{(qx_i/x_j,c_jx_i/a_ix_j,c_jy_iq/e_jy_j,e_iy_i/a_iy_j;q)_\infty}{(qy_i/y_j,c_je_iy_i/a_ie_jy_j,c_jx_i/x_j,x_iq/a_ix_j;q)_\infty}\\
&\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^{r}\frac{(a_jy_iq/e_jy_j;q)_{k_i}}{(c_jy_iq/e_jy_j;q)_{k_i}}\\
&\qquad\times\frac{(bq^r/E,h_1q^{r+m_1}/E,\dots,h_sq^{r+m_s}/E;q)_{|\mathbf{k}|}}{(dq^r/E,h_1q^r/E,\dots,h_sq^r/E;q)_{|\mathbf{k}|}}\left(\frac{Eq^{1-r-N}}{Ab}\right)^{|\mathbf{k}|}\Bigg),
\end{aligned}$$

provided $|Eq^{1-r}/Ab|<|q^N|<|Eq^{|m|}/Cd|$.

Proof. We have, for $|Eq^{1-r}/Ab|<|q^N|<|Eq^{|m|}/Cd|$,

$$\begin{aligned}
(4.12)\qquad &{}_{2+s}\psi_{2+s}\left[\begin{matrix}A,b,h_1q^{m_1},\dots,h_sq^{m_s}\\ Cq^{1-r},d,h_1,\dots,h_s\end{matrix};q,\frac{Eq^{1-r-N}}{Ab}\right]\\
&=(Eq^{-r})^N\frac{(Eq^{1-r}/A,Eq^{1-r}/b,Cq/E,dq^r/E;q)_\infty}{(q/A,q/b,Cq^{1-r},d;q)_\infty}\prod_{i=1}^{s}\frac{(h_iq^r/E;q)_{m_i}}{(h_i;q)_{m_i}}\\
&\times{}_{2+s}\psi_{2+s}\left[\begin{matrix}Aq^r/E,bq^r/E,h_1q^{r+m_1}/E,\dots,h_sq^{r+m_s}/E\\ Cq/E,dq^r/E,h_1q^r/E,\dots,h_sq^r/E\end{matrix};q,\frac{Eq^{1-r-N}}{Ab}\right],
\end{aligned}$$

by the q-IPD type transformation in (2.11). Now we apply Lemma 3.7 to the ${}_{2+s}\psi_{2+s}$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_{2+s}\psi_{2+s}$ on left side of (4.12) by the $b_i\mapsto c_i$, $i=1,\dots,r$, and

$$g(n)=\frac{(b,h_1q^{m_1},\dots,h_sq^{m_s};q)_n}{(d,h_1,\dots,h_s;q)_n}\left(\frac{Eq^{1-r-N}}{Ab}\right)^n$$

case of Lemma 3.7. The ${}_{2+s}\psi_{2+s}$ on the right side of (4.12) is rewritten by the $a_i\mapsto a_iq/e_i$, $b_i\mapsto c_iq/e_i$, $x_i\mapsto y_i$, $i=1,\dots,r$, and

$$g(n)=\frac{(bq^r/E,h_1q^{r+m_1}/E,\dots,h_sq^{r+m_s}/E;q)_n}{(dq^r/E,h_1q^r/E,\dots,h_sq^r/E;q)_n}\left(\frac{Eq^{1-r-N}}{Ab}\right)^n$$

case of Lemma 3.7. Finally, we divide both sides of the resulting equation by

$$(4.13)\qquad \frac{(q,Cq^{1-r}/A;q)_\infty}{(Cq^{1-r},q/A;q)_\infty}\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j,x_iq/a_ix_j;q)_\infty}{(qx_i/x_j,c_jx_i/a_ix_j;q)_\infty}$$

and simplify to obtain (4.11). □

Theorem 4.9 (A multilateral A_{r-1} q-IPD type transformation). *Let $a_1,\dots,a_r$, b, $c_1,\dots,c_r$, d, $e_1,\dots,e_r$, $x_1,\dots,x_r$, $y_1,\dots,y_r$, and $h_1,\dots,h_s$ be indeterminate, let N be an integer, let $m_1,\dots,m_s$ be nonnegative integers, let $|m|=\sum_{i=1}^s m_i$, $r\ge 1$, and suppose that none of the denominators in (4.14) vanishes. Then*

$$
\begin{aligned}
(4.14)\qquad &\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^r(c_jx_i/x_j;q)_{k_i}^{-1}\prod_{i=1}^r x_i^{rk_i-|\mathbf{k}|}\\
&\qquad\times(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}}\\
&\qquad\times\frac{(A,b,h_1q^{m_1},\dots,h_sq^{m_s};q)_{|\mathbf{k}|}}{(d,h_1,\dots,h_s;q)_{|\mathbf{k}|}}\left(\frac{Eq^{1-r-N}}{Ab}\right)^{|\mathbf{k}|}\Bigg)\\
&=(Eq^{-r})^N\frac{(Eq^{1-r}/b,Cq/E,dq^r/E;q)_\infty}{(q/A,q/b,d;q)_\infty}\\
&\qquad\times\prod_{i=1}^s\frac{(h_iq^r/E;q)_{m_i}}{(h_i;q)_{m_i}}\prod_{i,j=1}^r\frac{(qx_i/x_j,e_jy_i/a_jy_j;q)_\infty}{(qy_i/y_j,c_jx_i/x_j;q)_\infty}\\
&\times\sum_{k_1,\dots,k_r=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^r(e_jy_i/a_jy_j;q)_{k_i}^{-1}\prod_{i=1}^r y_i^{rk_i-|\mathbf{k}|}\\
&\qquad\times(-1)^{(r-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+r\sum_{i=1}^r\binom{k_i}{2}}\\
&\qquad\times\frac{(E/C,Eq^{1-r}/d,Eq^{1-r}/h_1,\dots,Eq^{1-r}/h_s;q)_{|\mathbf{k}|}}{(Eq^{1-r}/b,Eq^{1-r-m_1}/h_1,\dots,Eq^{1-r-m_s}/h_s;q)_{|\mathbf{k}|}}\left(\frac{Cdq^{N-|m|}}{E}\right)^{|\mathbf{k}|}\Bigg),
\end{aligned}
$$

provided $|Eq^{1-r}/Ab|<|q^N|<|Eq^{|m|}/Cd|$.

Proof. We have, for $|Eq^{1-r}/Ab|<|q^N|<|Eq^{|m|}/Cd|$,

$$
\begin{aligned}
(4.15)\qquad &{}_{2+s}\psi_{2+s}\left[\begin{matrix}A,b,h_1q^{m_1},\dots,h_sq^{m_s}\\ Cq^{1-r},d,h_1,\dots,h_s\end{matrix};q,\frac{Eq^{1-r-N}}{Ab}\right]\\
&=(Eq^{-r})^N\frac{(Eq^{1-r}/A,Eq^{1-r}/b,Cq/E,dq^r/E;q)_\infty}{(q/A,q/b,Cq^{1-r},d;q)_\infty}\prod_{i=1}^s\frac{(h_iq^r/E;q)_{m_i}}{(h_i;q)_{m_i}}\\
&\times{}_{2+s}\psi_{2+s}\left[\begin{matrix}E/C,Eq^{1-r}/d,Eq^{1-r}/h_1,\dots,Eq^{1-r}/h_s\\ Eq^{1-r}/A,Eq^{1-r}/b,Eq^{1-r-m_1}/h_1,\dots,Eq^{1-r-m_s}/h_s\end{matrix};q,\frac{Cdq^{N-|m|}}{E}\right],
\end{aligned}
$$

by the q-IPD type transformation in (2.12). Now we apply Lemma 3.6 to the ${}_{2+s}\psi_{2+s}$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_{2+s}\psi_{2+s}$ on left side of (4.15) by the $b_i\mapsto c_i$, $i=1,\dots,r$, and

$$
f(n)=\frac{(A,b,h_1q^{m_1},\dots,h_sq^{m_s};q)_n}{(d,h_1,\dots,h_s;q)_n}\left(\frac{Eq^{1-r-N}}{Ab}\right)^n
$$

case of Lemma 3.6. The ${}_{2+s}\psi_{2+s}$ on the right side of (4.15) is rewritten by the $b_i\mapsto c_i/a_i$, $x_i\mapsto y_i$, $i=1,\dots,r$, and

$$
f(n)=\frac{(E/C,Eq^{1-r}/d,Eq^{1-r}/h_1,\dots,Eq^{1-r}/h_s;q)_n}{(Eq^{1-r}/b,Eq^{1-r-m_1}/h_1,\dots,Eq^{1-r-m_s}/h_s;q)_n}\left(\frac{Cdq^{N-|m|}}{E}\right)^n
$$

case of Lemma 3.6. Finally, we divide both sides of the resulting equation by (4.10) and simplify to obtain (4.14). □

Theorem 4.10 (A multilateral A_{r-1} q-IPD type transformation). *Let $a_1, \ldots, a_r$, b, $c_1, \ldots, c_r$, d, $e_1, \ldots, e_r$, $x_1, \ldots, x_r$, $y_1, \ldots, y_r$, and $h_1, \ldots, h_s$ be indeterminate, let N be an integer, let $m_1, \ldots, m_s$ be nonnegative integers, let $|m| = \sum_{i=1}^{s} m_i$, $r \geq 1$, and suppose that none of the denominators in (4.16) vanishes. Then*

$$
\begin{aligned}
\sum_{k_1,\ldots,k_r=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{r} \frac{(a_j x_i/x_j;q)_{k_i}}{(c_j x_i/x_j;q)_{k_i}} \\
\times \frac{(b, h_1 q^{m_1}, \ldots, h_s q^{m_s};q)_{|\mathbf{k}|}}{(d, h_1, \ldots, h_s;q)_{|\mathbf{k}|}} \left(\frac{Eq^{1-r-N}}{Ab} \right)^{|\mathbf{k}|} \Bigg) \\
= (Eq^{-r})^N \frac{(Eq^{1-r}/b, dq^r/E;q)_\infty}{(q/b, d;q)_\infty} \prod_{i=1}^{s} \frac{(h_i q^r/E;q)_{m_i}}{(h_i;q)_{m_i}} \\
\times \prod_{i,j=1}^{r} \frac{(qx_i/x_j, c_j x_i/a_i x_j, e_j y_i/a_j y_j, c_i y_i q/e_i y_j;q)_\infty}{(qy_i/y_j, c_i e_j y_i/a_j e_i y_j, c_j x_i/x_j, x_i q/a_i x_j;q)_\infty} \\
\times \sum_{k_1,\ldots,k_r=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \prod_{i,j=1}^{r} \frac{(e_j y_i/c_j y_j;q)_{k_i}}{(e_j y_i/a_j y_j;q)_{k_i}} \\
\times \frac{(Eq^{1-r}/d, Eq^{1-r}/h_1, \ldots, Eq^{1-r}/h_s;q)_{|\mathbf{k}|}}{(Eq^{1-r}/b, Eq^{1-r-m_1}/h_1, \ldots, Eq^{1-r-m_s}/h_s;q)_{|\mathbf{k}|}} \left(\frac{Cdq^{N-|m|}}{E} \right)^{|\mathbf{k}|} \Bigg),
\end{aligned}
\tag{4.16}
$$

provided $|Eq^{1-r}/Ab| < |q^N| < |Eq^{|m|}/Cd|$.

Proof. We have, for $|Eq^{1-r}/Ab| < |q^N| < |Eq^{|m|}/Cd|$, (4.15) by the q-IPD type transformation in (2.12). Now we apply Lemma 3.7 to the ${}_{2+s}\psi_{2+s}$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_{2+s}\psi_{2+s}$ on left side of (4.15) by the $b_i \mapsto c_i$, $i = 1, \ldots, r$, and

$$
g(n) = \frac{(b, h_1 q^{m_1}, \ldots, h_s q^{m_s};q)_n}{(d, h_1, \ldots, h_s;q)_n} \left(\frac{Eq^{1-r-N}}{Ab} \right)^n
$$

case of Lemma 3.7. The ${}_{2+s}\psi_{2+s}$ on the right side of (4.15) is rewritten by the $a_i \mapsto e_i/c_i$, $b_i \mapsto e_i/a_i$, $x_i \mapsto y_i$, $i = 1, \ldots, r$, and

$$
g(n) = \frac{(Eq^{1-r}/d, Eq^{1-r}/h_1, \ldots, Eq^{1-r}/h_s;q)_n}{(Eq^{1-r}/b, Eq^{1-r-m_1}/h_1, \ldots, Eq^{1-r-m_s}/h_s;q)_n} \left(\frac{Cdq^{N-|m|}}{E} \right)^n
$$

case of Lemma 3.7. Finally, we divide both sides of the resulting equation by (4.13) and simplify to obtain (4.16). □

The $e_i = a_i q$, $i = 1, \ldots, r$, cases of Theorems 4.9 and 4.10 yield two A_{r-1} extensions of W. C. Chu's [5, Eq. 15] bilateral transformation. If we specialize these identities further by setting $c_i = q$, $i = 1, \ldots, r$, we obtain two A_{r-1} extensions of G. Gasper's [10, Eq. (19)] q-IPD type transformation.

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Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210, USA

Current address: Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: schlosse@ap.univie.ac.at

URL: http://www.mat.univie.ac.at/People/mschloss