A SIMPLE PROOF OF BAILEY'S VERY-WELL-POISED ${}_6\psi_6$ SUMMATION

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ABSTRACT. We give elementary derivations of some classical summation formulae for bilateral (basic) hypergeometric series. In particular, we apply Gauß' ${}_2F_1$ summation and elementary series manipulations to give a simple proof of Dougall's ${}_2H_2$ summation. Similarly, we apply Rogers' nonterminating ${}_6\phi_5$ summation and elementary series manipulations to give a simple proof of Bailey's very-well-poised ${}_6\psi_6$ summation. Our method of proof extends M. Jackson's first elementary proof of Ramanujan's ${}_1\psi_1$ summation.

1. INTRODUCTION

The theories of unilateral (or one-sided) hypergeometric and basic (q-)hypergeometric series have quite a rich history dating back to at least Euler. Formulae for *bilateral* (basic) hypergeometric series were not discovered until 1907 when Dougall [10], using residue calculus, derived summations for the bilateral ${}_2H_2$ and very-well-poised ${}_5H_5$ series. Ramanujan [15] extended the q-binomial theorem by finding a summation formula for the bilateral ${}_1\psi_1$ series. Later, Bailey [6],[7] carried out systematical investigations of summations and transformations for bilateral basic hypergeometric series. Further significant contributions were made by Slater [25],[26], a student of Bailey. See [11] and [26] for an excellent survey of the above classical material.

Bailey's [6, Eq. (4.7)] very-well-poised ${}_6\psi_6$ summation (cf. [11, Eq. (5.3.1)]) is a very powerful identity, as it stands at the top of the classical hierarchy of summation formulae for bilateral series. Some of the applications of the ${}_6\psi_6$ summation to partitions and number theory are given in Andrews [1]. Though several proofs of Bailey's ${}_6\psi_6$ summation are already known (see, e. g., Bailey [6], Slater and Lakin [27], Andrews [1], Askey and Ismail [5], and Askey [4]), none of them is entirely elementary. Here we provide a new simple proof of the very-well-poised ${}_6\psi_6$ summation formula, directly from three applications of Rogers' [22, p. 29, second eq.] nonterminating ${}_6\phi_5$ summation (cf. [11, Eq. (2.7.1)]) and elementary manipulations of series.

The method of proof we apply extends that already used by M. Jackson [19, Sec. 4] in her first elementary proof (as pointed out to us by George Andrews [2]) of Ramanujan's ${}_1\psi_1$ summation formula [15] (cf. [11, Eq. (5.2.1)]). Jackson's proof essentially derives the ${}_1\psi_1$ summation from the q-Gauß summation, by manipulation of series. In view of this background, it is surprising that this method has not been further applied for half a century. A possible explanation is that the applicability

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of her method was viewed as too limited. In fact, only after changing the order of steps in Jackson's proof, we were able to extend her proof to a "method".

Indeed, the method can also be applied to derive other summations. After recalling some notation for (basic) hypergeometric series in Section 2, we review Jackson's elementary proof of the ${}_1\psi_1$ summation in Section 3. In Section 4, we apply our extension of Jackson's method to give an elementary proof of Dougall's [10] ${}_2H_2$ summation. Finally, in Section 5, we give an elementary derivation of Bailey's very-well-poised ${}_6\psi_6$ summation.

We want to point out that by using a similar but slightly different method, the author [24] has found elementary derivations of transformations for bilateral basic hypergeometric series. In fact, in [24] we use Bailey's [6] nonterminating very-well-poised ${}_8\phi_7$ summation theorem combined with bilateral series identities to derive a very-well-poised ${}_8\psi_8$ transformation, a very-well-poised ${}_{10}\psi_{10}$ transformation, and by induction, Slater's [25] general transformation for very-well-poised ${}_{2r}\psi_{2r}$ series. Similarly, some other bilateral series identities are also elementarily derived in [24].

In the near future, we plan to apply the methods of this article and of [24] to the settings of multiple basic hypergeometric series. See Milne [21], Gustafson [13], v. Diejen [9], and Schlosser [23], for several of these different settings. We are quite confident that we may not only get simpler proofs for already known results but should also obtain derivations of new formulae.

Finally, we wish to gratefully acknowledge the helpful comments and suggestions of George Andrews, Mourad Ismail, and Stephen Milne.

2. Background and notation

Here we recall some notation for hypergeometric series (cf. [26]), and basic hypergeometric series (cf. [11]).

We define the *shifted factorial* for all integers k by the following quotient of Gamma functions (cf. [3, Sec. 1.1]),

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Further, the (ordinary) *hypergeometric ${}_rF_s$ series* is defined as

$${}_rF_s\left[\begin{matrix}a_1, a_2, \dots, a_r\\ b_1, b_2, \dots, b_s\end{matrix}; z\right] := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k} \frac{z^k}{k!}, \tag{2.1}$$

and the *bilateral hypergeometric ${}_rH_s$ series* as

$${}_rH_s\left[\begin{matrix}a_1, a_2, \dots, a_r\\ b_1, b_2, \dots, b_s\end{matrix}; z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k} z^k. \tag{2.2}$$

See [26, p. 45 and p. 181] for the criteria of when these series terminate, or, if not, when they converge.

Let q be a complex number such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{and} \qquad (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

For brevity, we employ the usual notation

$$(a_1,\dots,a_m;q)_k \equiv (a_1;q)_k \dots (a_m;q)_k$$

where k is an integer or infinity. Further, we utilize the notations

$$ {}_r\phi_s\left[\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_s\end{matrix};q,z\right]:=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_k}{(q,b_1,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}z^k, \tag{2.3}$$

and

$$ {}_r\psi_s\left[\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_s\end{matrix};q,z\right]:=\sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_k}{(b_1,b_2,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{s-r}z^k, \tag{2.4}$$

for *basic hypergeometric* ${}_r\phi_s$ *series*, and *bilateral basic hypergeometric* ${}_r\psi_s$ *series*, respectively. See [11, p. 25 and p. 125] for the criteria of when these series terminate, or, if not, when they converge.

We want to point out that many theorems for ${}_rF_s$ or ${}_rH_s$ series can be obtained by considering certain "$q\to 1$ limiting cases" of corresponding theorems for ${}_r\phi_s$ or ${}_r\psi_s$ series, respectively. For instance, we describe such a $q\to 1$ limiting case after stating the q-binomial theorem in (3.1). A similar $q\to 1$ limiting case leads from the q-Gauß summation (3.3) to the ordinary Gauß summation (4.7). The situation is different for the ${}_1\psi_1$ series, though. We have a summation for the general ${}_1\psi_1$, but not for the ${}_1H_1$. On the other hand, the general ${}_2H_2$ with unit argument is summable but the general ${}_2\psi_2$ is not. Many theorems for very-well-poised ${}_{r+1}\phi_r$ series can be specialized to theorems for very-well-poised ${}_rF_{r-1}$ series. For the notion of *(very-)well-poised*, see [11, Sec. 2.1]. For detailed treatises on hypergeometric and basic hypergeometric series, we refer to Slater [26], and Gasper and Rahman [11].

In our computations in the following sections, we make heavily use of some elementary identities involving (q-)shifted factorials which are listed in Slater [26, Appendix I], and Gasper and Rahman [11, Appendix I].

3. M. Jackson's proof of Ramanujan's ${}_1\psi_1$ summation

The q-binomial theorem,

$$ {}_1\phi_0\left[\begin{matrix}a\\ -\end{matrix};q,z\right]=\frac{(az;q)_\infty}{(z;q)_\infty}, \tag{3.1}$$

where the series either terminates, or $|z|<1$, for convergence, was first discovered by Cauchy [8] (cf. [11, Sec. 1.3]). It reduces to the ordinary binomial theorem as $a\mapsto q^a$ and $q\to 1^-$.

A bilateral extension of the q-binomial theorem (3.1), the ${}_1\psi_1$ summation, was found by the legendary Indian mathematician Ramanujan [15] (cf. [11, Eq. (5.2.1)]). It reads as follows:

$$ {}_1\psi_1\left[\begin{matrix}a\\ b\end{matrix};q,z\right]=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \tag{3.2}$$

where the series either terminates, or $|b/a|<|z|<1$, for convergence. Clearly, (3.2) reduces to (3.1) when $b=q$.

Unfortunately, Ramanujan did not provide a proof for his summation formula. Hahn [14, $\kappa=0$ in Eq. (4.7)] independently established (3.2) by considering a first order homogeneous q-difference equation. Hahn thus published the first proof of the ${}_1\psi_1$ summation. Not much later, M. Jackson [19, Sec. 4] gave the first elementary proof of (3.2). Her proof derives the ${}_1\psi_1$ summation from the q-Gauß summation, by manipulation of series. It turns out that Jackson's method is effective for proving also other bilateral summation formulae. Since Jackson's short proof of Ramanujan's ${}_1\psi_1$ summation seems to be not so well known, we review her proof in the following.

Before we continue, we want to point out that there are also many other nice proofs of the ${}_1\psi_1$ summation in the literature. A simple and elegant proof of the ${}_1\psi_1$ summation formula was given by Ismail [17] who showed that the ${}_1\psi_1$ summation is an immediate consequence of the q-binomial theorem and analytic continuation.

M. Jackson's elementary proof of (3.2) makes use of a suitable specialization of Heine's [16] q-Gauß summation (cf. [11, Eq. (II.8)]),

$$ {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,\frac{c}{ab}\right]=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}, \tag{3.3}$$

where the series either terminates, or $|c/ab|<1$, for convergence.

In (3.3), we perform the substitutions $a\mapsto aq^n$, $b\mapsto q/b$, and $c\mapsto q^{1+n}$, and obtain

$$ {}_2\phi_1\left[\begin{matrix}aq^n,q/b\\q^{1+n}\end{matrix};q,\frac{b}{a}\right]=\frac{(q/a,bq^n;q)_\infty}{(q^{1+n},b/a;q)_\infty}, \tag{3.4}$$

provided $|b/a|<1$.

Using some elementary identities for q-shifted factorials (see, e. g., Gasper and Rahman [11, Appendix I]) we can rewrite equation (3.4) as

$$\frac{(q,b/a;q)_\infty}{(q/a,b;q)_\infty}\sum_{k=0}^\infty\frac{(q/b;q)_k(a;q)_{n+k}}{(q;q)_k(q;q)_{n+k}}\left(\frac{b}{a}\right)^k=\frac{(a;q)_n}{(b;q)_n}. \tag{3.5}$$

In this identity, we multiply both sides by z^n and sum over all integers n.

On the right side we obtain

$$ {}_1\psi_1\left[\begin{matrix}a\\b\end{matrix};q,z\right].$$

On the left side we obtain

$$\frac{(q,b/a;q)_\infty}{(q/a,b;q)_\infty}\sum_{n=-\infty}^\infty z^n\sum_{k=0}^\infty\frac{(q/b;q)_k(a;q)_{n+k}}{(q;q)_k(q;q)_{n+k}}\left(\frac{b}{a}\right)^k. \tag{3.6}$$

Next, we interchange summations in (3.6) and shift the inner index $n\mapsto n-k$. (Observe that the sum over n is terminated by the term $(q;q)_{n+k}^{-1}$ from below.) We obtain

$$\frac{(q,b/a;q)_\infty}{(q/a,b;q)_\infty}\sum_{k=0}^\infty\frac{(q/b;q)_k}{(q;q)_k}\left(\frac{b}{az}\right)^k\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}z^n.$$

Now, two applications of the q-binomial theorem (3.1) gives us the right side of (3.2), as desired.

Now, we have to admit that M. Jackson did not give her proof in the above precise order. In fact, her proof in [19, Sec. 4] goes backwards. (This is also how the author originally rediscovered Jackson's proof.) She started with the ${}_1\psi_1$ summation (3.2) and equated coefficients of z^n on both sides. The resulting identity is true by the q-Gauß summation.

A reason why M. Jackson's method of proof has so far not been used to prove other bilateral summations could be that the applicability of her derivation was viewed as too limited. Equating coefficients of a power of a Laurent series variable in a bilateral basic hypergeometric series identity is easy if, as in (3.2), there is an argument z which is independent of the other parameters. But this seems to be more particluar to the ${}_1\psi_1$ summation, as not in all bilateral series there is such an independent argument. The starting point for making M. Jackson's proof to a "method" is to read the proof backwards, as displayed above. The essence here is that a unilateral series identity, (3.3), is specialized such that there is the factor $(q;q)_{n+k}^{-1}$ in the series, see (3.5), so that summing over all n again gives a (summable) unilateral series.

In the next two sections, we use Jackson's method to prove two other important bilateral hypergeometric and basic hypergeometric summation theorems. In particular, in Section 4, we give a simple proof of Dougall's ${}_2H_2$ summation, whereas in Section 5, we give a simple proof of Bailey's very-well-poised ${}_6\psi_6$ summation.

4. Dougall's ${}_2H_2$ summation

In Section 3, we multiplied both sides of the identity (3.5) by a suitable factor depending on n and summed over all integers n. On one side, we interchanged sums and found that the inner sum was summable by the q-binomial theorem. Now, what if we start with a different factor following a similar procedure such that we can evaluate the inner sum by, say, the q-Gauß summation? If the analysis works out we may end up with an evaluation for a ${}_2\psi_2$ series. Let us see what happens:

In identity (3.5), let us first replace b by c. Then we multiply both sides by

$$\frac{(b;q)_n}{(d;q)_n}\left(\frac{d}{ab}\right)^n$$

and sum over all integers n.

On the right side we obtain

$${}_2\psi_2\left[\begin{matrix}a,b\\c,d\end{matrix};q,\frac{d}{ab}\right]. \tag{4.1}$$

On the left side we obtain

$$\frac{(q,c/a;q)_\infty}{(q/a,c;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(b;q)_n}{(d;q)_n}\left(\frac{d}{ab}\right)^n\sum_{k=0}^{\infty}\frac{(q/c;q)_k(a;q)_{n+k}}{(q;q)_k(q;q)_{n+k}}\left(\frac{c}{a}\right)^k. \tag{4.2}$$

Next, we interchange summations in (4.2) and shift the inner index $n\mapsto n-k$. We obtain, again using some elementary identities for q-shifted factorials,

$$\frac{(q,c/a;q)_\infty}{(q/a,c;q)_\infty}\sum_{k=0}^{\infty}\frac{(q/c;q)_k(b;q)_{-k}}{(q;q)_k(d;q)_{-k}}\left(\frac{bc}{d}\right)^k\sum_{n=0}^{\infty}\frac{(a,bq^{-k};q)_n}{(q,dq^{-k};q)_n}\left(\frac{d}{ab}\right)^n.$$

Now the inner sum, provided $|d/ab|<1$, can be evaluated by (3.3) and we obtain

$$\frac{(q,c/a;q)_\infty}{(q/a,c;q)_\infty}\sum_{k=0}^{\infty}\frac{(q/c;q)_k(b;q)_{-k}}{(q;q)_k(d;q)_{-k}}\left(\frac{bc}{d}\right)^k\frac{(dq^{-k}/a,d/b;q)_\infty}{(dq^{-k},d/ab;q)_\infty},$$

which can be simplified to

$$\frac{(q,c/a,d/a,d/b;q)_\infty}{(q/a,c,d,d/ab;q)_\infty}\sum_{k=0}^{\infty}\frac{(q/c,aq/d;q)_k}{(q,q/b;q)_k}\left(\frac{c}{a}\right)^k. \tag{4.3}$$

Hence, equating (4.1) and (4.3), we have derived the transformation

$${}_2\psi_2\left[\begin{matrix}a,b\\c,d\end{matrix};q,\frac{d}{ab}\right]=\frac{(q,c/a,d/a,d/b;q)_\infty}{(q/a,c,d,d/ab;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}q/c,aq/d\\q/b\end{matrix};q,\frac{c}{a}\right], \tag{4.4}$$

where the series terminate, or $\max(|d/ab|, |c|, |c/a|) < 1$, for convergence. Unfortunately, the ${}_2\phi_1$ on the right side of (4.4) simplifies only in special cases. If $d = aq$, then the ${}_2\phi_1$ sum reduces just to the first term, 1, and we have the summation

$$
{}_2\psi_2\left[\begin{matrix} a, b \\ aq, c \end{matrix}; q, \frac{q}{b}\right] = \frac{(q, q, aq/b, c/a; q)_\infty}{(aq, q/a, q/b, c; q)_\infty}, \tag{4.5}
$$

where the series terminates, or $\max(|q/b|, |c|) < 1$, for convergence.

We want to add that the transformation in (4.4) is a special case of Bailey's [7, Eq. (2.3)] ${}_2\psi_2$ transformation,

$$
{}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z\right] = \frac{(az, d/a, c/b, dq/abz; q)_\infty}{(z, d, q/b, cd/abz; q)_\infty}\, {}_2\psi_2\left[\begin{matrix} a, abz/d \\ az, c \end{matrix}; q, \frac{d}{a}\right], \tag{4.6}
$$

where the series terminate, or $\max(|z|, |cd/abz|, |d/a|, |c/b|) < 1$, for convergence. Namely, if we perform in (4.6) the simultaneous substitutions $a \mapsto b$, $b \mapsto a$, and $z \mapsto d/ab$, and reverse the order of summation in the truncated series on the right side, we obtain (4.4).

In Section 3, we found, following M. Jackson, a sum for a general ${}_1\psi_1$ series. So far in this section, we applied her method to obtain a transformation for a particular ${}_2\psi_2$ into a (multiple of a) ${}_2\phi_1$ series. As a matter of fact, there is no closed form (as a product of linear factors) for the summation of a general ${}_2\psi_2$ series. The situation is different in the $q \to 1$ case, though.

In the following, we review the classical ${}_2F_1$ and ${}_2H_2$ summations and then prove the latter by our elementary method.

In his doctoral dissertation [12], Gauß showed that

$$
{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{4.7}
$$

where the series either terminates, or $\Re(c-a-b) > 0$, for convergence.

Dougall [10, Sec. 13] extended this result to

$$
{}_2H_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; 1\right] = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \tag{4.8}
$$

where the series either terminates, or $\Re(c+d-a-b-1) > 0$, for convergence. Clearly, the $d \to 1$ case of (4.8) is (4.7).

We are ready to derive (4.8) from (4.7): In (4.7), we perform the simultaneous substitutions $a \mapsto a+n$, $b \mapsto 1-c$, and $c \mapsto 1+n$, and obtain

$$
{}_2F_1\left[\begin{matrix} a+n, 1-c \\ 1+n \end{matrix}; 1\right] = \frac{\Gamma(1+n)\Gamma(c-a)}{\Gamma(1-a)\Gamma(c+n)}, \tag{4.9}
$$

provided $\Re(c-a) > 0$.

Using some elementary identities for shifted factorials (see, e. g., Slater [26, Appendix I]) we can rewrite equation (4.9) as

$$
\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(1-c)_k (a)_{n+k}}{(1)_k (1)_{n+k}} = \frac{(a)_n}{(c)_n}. \tag{4.10}
$$

Alternatively, we could have used (3.5) with the substitutions $a \mapsto q^a$ and $b \mapsto q^c$, and then let $q \to 1^-$, to arrive directly at (4.10).

In (4.10), we multiply both sides by $(b)_n/(d)_n$ and sum over all integers n.

On the right side we obtain

$${}_2H_2\left[\begin{matrix}a,b\\c,d\end{matrix};1\right].$$

On the left side we obtain

$$\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}\sum_{n=-\infty}^{\infty}\frac{(b)_n}{(d)_n}\sum_{k=0}^{\infty}\frac{(1-c)_k(a)_{n+k}}{(1)_k(1)_{n+k}}. \tag{4.11}$$

Next, we interchange summations in (4.11) and shift the inner index $n\mapsto n-k$. (Observe that the sum over n is terminated by the term $(1)_{n+k}^{-1}$ from below.) We obtain

$$\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}\sum_{k=0}^{\infty}\frac{(1-c)_k(b)_{-k}}{(1)_k(d)_{-k}}\sum_{n=0}^{\infty}\frac{(a)_n(b-k)_n}{(1)_n(d-k)_n}.$$

Now, the inner sum, provided $\Re(d-a-b)>0$, can be evaluated by (4.7) and we obtain

$$\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}\sum_{k=0}^{\infty}\frac{(1-c)_k(b)_{-k}}{(1)_k(d)_{-k}}\frac{\Gamma(d-k)\Gamma(d-a-b)}{\Gamma(d-a-k)\Gamma(d-b)},$$

which can be simplified to

$$\frac{\Gamma(1-a)\Gamma(c)\Gamma(d)\Gamma(d-a-b)}{\Gamma(c-a)\Gamma(d-a)\Gamma(d-b)}\sum_{k=0}^{\infty}\frac{(1-c)_k(1+a-d)_k}{(1)_k(1-b)_k}.$$

To the last inner sum, provided $\Re(c+d-a-b-1)>0$, we can again apply (4.7) and eventually obtain the right side of (4.8), as desired.

We note here that to apply Gauß' ${}_2F_1$ summation theorem three times, we needed certain conditions of the parameters, for convergence. These were $\Re(c-a)>0$, $\Re(d-a-b)>0$, and $\Re(c+d-a-b-1)>0$. But in the end the first two of these conditions may be removed by analytic continuation. In particular, both sides of identity (4.8) are analytic in a for $\Re(a)<\Re(c+d-b-1)$ (and excluding some poles). In the course of our derivation, we have shown the identity for $\Re(a)<\min(\Re(c),\Re(d-b),\Re(c+d-b-1))$ (excluding some poles). By analytic continuation, we extend the identity, when defined, to be valid for $\Re(a)<\Re(c+d-b-1)$, the region of convergence of the series.

5. Bailey's very-well-poised ${}_6\psi_6$ summation

One of the most powerful identities for bilateral basic hypergeometric series is Bailey's very-well-poised ${}_6\psi_6$ summation:

$$\begin{aligned}{}_6\psi_6&\left[\begin{matrix}q\sqrt{a},-q\sqrt{a},b,c,d,e\\\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e\end{matrix};q,\frac{a^2q}{bcde}\right]\\&=\frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,a^2q/bcde;q)_\infty},\end{aligned} \tag{5.1}$$

provided the series either terminates, or $|a^2q/bcde|<1$, for convergence.

To prove Bailey's ${}_6\psi_6$ summation, we start with a suitable specialization of Rogers' ${}_6\phi_5$ summation:

$${}_6\phi_5\left[\begin{matrix}a,\,q\sqrt{a},-q\sqrt{a},b,c,d\\\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d\end{matrix};q,\frac{aq}{bcd}\right]=\frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}, \tag{5.2}$$

provided the series either terminates, or $|aq/bcd| < 1$, for convergence. Note that (5.2) is just the special case $e \mapsto a$ of (5.1).

In (5.2), we perform the simultaneous substitutions $a \mapsto c/a$, $b \mapsto b/a$, $c \mapsto cq^n$ and $d \mapsto cq^{-n}/a$, and obtain

$$
{}_6\phi_5\left[\begin{matrix} c/a,\, q\sqrt{c/a}, -q\sqrt{c/a}, b/a, cq^n, cq^{-n}/a \\ \sqrt{c/a}, -\sqrt{c/a}, cq/b, q^{1-n}/a, q^{1+n} \end{matrix}; q, \frac{aq}{bc}\right]
= \frac{(cq/a, q^{1-n}/b, aq^{1+n}/b, q/c; q)_\infty}{(cq/b, q^{1-n}/a, q^{1+n}, aq/bc; q)_\infty}, \qquad (5.3)
$$

where $|aq/bc| < 1$.

Using some elementary identities for q-shifted factorials (see, e. g., Gasper and Rahman [11, Appendix I]) we can rewrite equation (5.3) as

$$
\frac{(cq/b, q/a, q, aq/bc; q)_\infty}{(cq/a, q/b, aq/b, q/c; q)_\infty} \sum_{k=0}^{\infty} \frac{(1-cq^{2k}/a)}{(1-c/a)} \frac{(c/a, b/a; q)_k (c;q)_{n+k} (a;q)_{n-k}}{(q, cq/b; q)_k (q;q)_{n+k} (aq/c;q)_{n-k}} \left(\frac{a}{b}\right)^k
= \frac{(b, c; q)_n}{(aq/b, aq/c; q)_n} \left(\frac{a}{b}\right)^n. \qquad (5.4)
$$

In this identity, we multiply both sides by

$$
\frac{(1-aq^{2n})}{(1-a)} \frac{(d, e; q)_n}{(aq/d, aq/e; q)_n} \left(\frac{aq}{cde}\right)^n
$$

and sum over all integers n.

On the right side we obtain

$$
{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{a^2q}{bcde}\right].
$$

On the left side we obtain

$$
\frac{(cq/b, q/a, q, aq/bc; q)_\infty}{(cq/a, q/b, aq/b, q/c; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})}{(1-a)} \frac{(d, e; q)_n}{(aq/d, aq/e; q)_n} \left(\frac{aq}{cde}\right)^n
\times \sum_{k=0}^{\infty} \frac{(1-cq^{2k}/a)}{(1-c/a)} \frac{(c/a, b/a; q)_k (c;q)_{n+k} (a;q)_{n-k}}{(q, cq/b; q)_k (q;q)_{n+k} (aq/c;q)_{n-k}} \left(\frac{a}{b}\right)^k. \qquad (5.5)
$$

Next, we interchange summations in (5.5) and shift the inner index $n \mapsto n-k$. (Observe that the sum over n is terminated by the term $(q;q)_{n+k}^{-1}$ from below.) We obtain, again using some elementary identities for q-shifted factorials,

$$
\frac{(cq/b, q/a, q, aq/bc; q)_\infty}{(cq/a, q/b, aq/b, q/c; q)_\infty} \sum_{k=0}^{\infty} \frac{(1-cq^{2k}/a)}{(1-c/a)} \frac{(c/a, b/a; q)_k}{(q, cq/b; q)_k}
\times \frac{(1-aq^{-2k})}{(1-a)} \frac{(a;q)_{-2k} (d, e; q)_{-k}}{(aq/c;q)_{-2k} (aq/d, aq/e; q)_{-k}} \left(\frac{cde}{bq}\right)^k
\times \sum_{n=0}^{\infty} \frac{(1-aq^{-2k+2n})}{(1-aq^{-2k})} \frac{(aq^{-2k}, c, dq^{-k}, eq^{-k}; q)_n}{(q, aq^{1-2k}/c, aq^{1-k}/d, aq^{1-k}/e; q)_n} \left(\frac{aq}{cde}\right)^n.
$$

Now the inner sum, provided $|aq/cde| < 1$, can be evaluated by (5.2) and we obtain

$$\frac{(cq/b, q/a, q, aq/bc; q)_\infty}{(cq/a, q/b, aq/b, q/c; q)_\infty} \sum_{k=0}^{\infty} \frac{(1-cq^{2k}/a)}{(1-c/a)} \frac{(c/a, b/a; q)_k (aq; q)_{-2k}}{(q, cq/b; q)_k (aq/c; q)_{-2k}} \times \frac{(d, e; q)_{-k}}{(aq/d, aq/e; q)_{-k}} \left(\frac{cde}{bq}\right)^k \frac{(aq^{1-2k}, aq^{1-k}/cd, aq^{1-k}/ce, aq/de; q)_\infty}{(aq^{1-2k}/c, aq^{1-k}/d, aq^{1-k}/e, aq/cde; q)_\infty},$$

which can be simplified to

$$\frac{(cq/b, q/a, q, aq/bc, aq, aq/cd, aq/ce, aq/de; q)_\infty}{(cq/a, q/b, aq/b, q/c, aq/c, aq/d, aq/e, aq/cde; q)_\infty} \times \sum_{k=0}^{\infty} \frac{(1-cq^{2k}/a)}{(1-c/a)} \frac{(c/a, b/a, cd/a, ce/a; q)_k}{(q, cq/b, q/d, q/e; q)_k} \left(\frac{a^2q}{bcde}\right)^k.$$

To the last sum, provided $|a^2q/bcde| < 1$, we can again apply (5.2) and after some simplifications we finally obtain the right side of (5.1), as desired.

Our derivation of the $_6\psi_6$ summation (5.1) is simple once the nonterminating $_6\phi_5$ summation (5.2) is given. But the latter summation follows by an elementary computation from F. H. Jackson's [18] terminating $_8\phi_7$ summation (cf. [11, Eq. (2.6.2)])

$$_8\phi_7\left[\begin{matrix} a,\ q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix}; q, q\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n} \tag{5.6}$$

as $n \to \infty$. Jackson's terminating $_8\phi_7$ summation itself can be proved by various ways. An algorithmic approach uses the q-Zeilberger algorithm, see Koornwinder [20]. For an inductive proof, see Slater [26, Sec. 3.3.1]. For another elementary classical proof, see Gasper and Rahman [11, Sec. 2.6].

Concluding this section, we would like to add another thought, kindly initiated by an anonymous referee. It is worth comparing our proof with Askey and Ismail's [5] elegant (and now classical) proof of Bailey's $_6\psi_6$ summation. Their proof uses a method in this context often referred to as "Ismail's argument" since Ismail [17] was apparently the first to apply Liouville's standard analytic continuation argument in the context of bilateral basic hypergeometric series. Askey and Ismail use Rogers' $_6\phi_5$ summation once to evaluate the $_6\psi_6$ series at an infinite sequence and then apply analytic continuation. Here, we evaluate the $_6\psi_6$ series on a domain, and, for the full theorem, we also need analytic continuation. In fact, we need, in addition to $|a^2q/bcde| < 1$ two other inequalities on a, b, c, d, e, namely $|aq/bc| < 1$ and $|aq/cde| < 1$, in order to apply the $_6\phi_5$ summation theorem. In the end, these additional conditions can be removed. In particular, both sides of identity (5.1) are analytic in $1/c$ around the origin. So far, we have shown the identity for $|1/c| < \min(|b/aq|, |de/aq|, |bde/a^2q|)$. By analytic continuation, we extend the identity to be valid for $|1/c| < |bde/a^2q|$, the radius of convergence of the series.

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