NONCOMMUTATIVE HYPERGEOMETRIC AND BASIC HYPERGEOMETRIC EQUATIONS

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ABSTRACT. Recently, J. A. Tirao [Proc. Nat. Acad. Sci. 100 (14) (2003), 8138–8141] considered a matrix-valued analogue of the ${}_2F_1$ Gauß hypergeometric function and showed that it is the unique solution of a matrix-valued hypergeometric equation analytic at $z = 0$ with value I, the identity matrix, at $z = 0$. We give an independent proof of Tirao's result, extended to the more general setting of hypergeometric functions over an abstract unital Banach algebra. We provide a similar (but more complicated-looking) result for a second type of noncommutative ${}_2F_1$ Gauß hypergeometric function. We further give q-analogues for both types of noncommutative hypergeometric equations.

1. INTRODUCTION

Hypergeometric series with noncommutative parameters and argument, in the special case involving square matrices, have been the subject of recent study, see e.g. [3, 6, 7, 13, 14, 15, 24, 25, 29]. (For the classical theory of (basic) hypergeometric series, cf. [2, 11, 26].) In particular, Tirao [29] considered a specific type of matrix-valued hypergeometric function ${}_2F_1$, and showed, among other results, that it satisfies a matrix-valued differential equation of the second order (a "matrix-valued hypergeometric equation"), and conversely that any solution of the latter is a matrix-valued hypergeometric function of the considered type. This result was reformulated by one of the present authors [25] in the more general setting of hypergeometric functions with parameters and argument over an unital Banach algebra R. Specifically, in [24, 25] two related types of noncommutative hypergeometric and Q-hypergeometric series were studied, "type I" and "type II", from the view-point of explicit summation theorems they satisfy. In the terminology of [24, 25], Tirao's extension of the Gauß hypergeometric function belongs to type I. As a matter of fact, the explicit form of the noncommutative hypergeometric equation satisfied by the type II Gauß

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hypergeometric function has so far not been determined. (A priori, it is not clear that the type II hypergeometric equation would be of second order or even have a reasonable compact form.) Nor has any of the corresponding noncommutative basic (or Q-)hypergeometric equations been determined. In this paper we give an independent derivation of Tirao's result for the type I Gauß hypergeometric function and succeed in providing an analogous (however, more complicated-looking) result for the type II case. We further give Q-analogues of the above results, hereby establishing the explicit forms of the type I and type II noncommutative *basic* hypergeometric equations. (In the basic type II case we just state the result which is not very elegant and omit the proof.)

To eliminate possible misconception, we would like to stress that the series considered in this paper, as they involve noncommuting parameters and argument, are much more general than the series of (sole) matrix argument as considered e.g. by Gross and Richards [12]. In particular, since in the latter series all parameters (being scalars) commute, the specific issue of noncommutativity does not arise, and the hypergeometric equation is just the usual one.

We point out that certain first order ordinary differential operators applied to the Gauß hypergeometric function have been successfully used in order to study representations of quadratic R-matrix algebras appearing in quantum inverse scattering method, see [20], while the study of q-hypergeometric functions (and difference equations for them) have been recently used to study representation theory of Lie algebras, quantum affine algebras, quantum groups, Yangians and quantum Yang–Baxter equations, see [10, 16, 27, 28, 30, 31]. We refer to [4, 21] for surveys on R-matrices and Yangians, and to [8, 9] for comprehensive references on dynamical Yang–Baxter equations and quantum theory. Our results may cast new light and give a better insight on the investigation of certain twisted versions of Yangians related to the representation theory of infinite dimensional Lie algebras, see [22], and apt to spring further inspection upon quantum conformal field theory using infinite dimensional noncommutative geometry, see [17, 32, 33].

We refer to [5, 23, 34], and [1, 18, 19], for comprehensive references on Banach algebras, and on differential equations in Banach spaces, respectively.

In the following section, we collect some definitions and notations, taken almost verbatim from [24, 25]. These are needed in Sections 3 and 4 for the study of the type I and type II noncommutative (basic) hypergeometric equations.

2. Preliminaries

Let R be a unital Banach algebra, i.e., an associative ring (over some field K) with a multiplicative identity element, together with some norm $\|\cdot\|$ such that R is norm-complete. Throughout this paper, the elements of R will be denoted by capital letters $A, B, C, \ldots$. In general these elements need not commute with each other; however, we may sometimes specify certain commutation relations explicitly. We denote the identity by I and the zero element by O. Whenever a multiplicative

inverse element exists for any $A \in R$, we denote it by A^{-1}. (Since R is a unital ring, we have $AA^{-1} = A^{-1}A = I$.) On the other hand, as we shall implicitly assume that all the expressions which appear are well defined, whenever we write A^{-1} we assume its existence. For instance, in (2.1a) and (2.1b) we assume that $C_i + jI$ is invertible for all $1 \le i \le r$, $0 \le j < k$.

An important special case is when R is the ring of $n \times n$ square matrices (our notation is certainly suggestive with respect to this interpretation), or, more generally, one may view R as a space of some abstract operators.

For any nonnegative integers m and l with $m \ge l-1$ we define the noncommutative product as follows:

$$\prod_{j=l}^{m} A_j = \begin{cases} I & m = l-1 \\ A_l A_{l+1} \cdots A_m & m \ge l. \end{cases}$$

In [24, 25] a more general definition was given, which however we will not need here.

For nonnegative integers k and r we define the generalized *noncommutative shifted factorial of type I* by

$$\left\lceil \begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix}; Z \right\rceil_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (C_i + (k-j)I)^{-1} (A_i + (k-j)I) \right) Z \right], \quad (2.1a)$$

and the *noncommutative shifted factorial of type II* by

$$\left\lfloor \begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix}; Z \right\rfloor_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (C_i + (j-1)I)^{-1} (A_i + (j-1)I) \right) Z \right]. \quad (2.1b)$$

Note the unusual usage of brackets ("floors" and "ceilings" are intermixed) on the left-hand sides of (2.1a) and (2.1b) which is intended to suggest that the products involve noncommuting factors in a prescribed order. In both cases, the product, read from left to right, starts with a denominator factor. The brackets in the form "$\lceil - \rfloor$" are intended to denote that the factors are *falling*, while in "$\lfloor - \rceil$" that they are *rising*.

We define the *noncommutative hypergeometric series of type I* by

$${}_{r+1}F_r \left\lceil \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r \end{matrix}; Z \right\rceil := \sum_{k \ge 0} \left\lceil \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r, I \end{matrix}; Z \right\rceil_k,$$

and the *noncommutative hypergeometric series of type II* by

$${}_{r+1}F_r \left\lfloor \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r \end{matrix}; Z \right\rfloor := \sum_{k \ge 0} \left\lfloor \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r, I \end{matrix}; Z \right\rfloor_k.$$

In each case, the series terminates if one of the upper parameters A_i is of the form $-nI$. If the series is nonterminating, then the series converges in R if $\|Z\| < 1$. If $\|Z\| = 1$ the series may converge in R for some particular choice of upper and lower parameters. Exact conditions depend on the Banach algebra R.

Throughout this paper, Q will be a parameter which commutes with any of the other parameters appearing in the series. (For instance, a central element such as $Q = qI$, a scalar multiple of the unit element in R, for $qI \in R$, trivially satisfies this requirement.)

For nonnegative integers k and r we define the generalized *noncommutative Q-shifted factorial of type I* by

$$\left[\begin{matrix} A_1, A_2, \dots, A_r \\ C_1, C_2, \dots, C_r \end{matrix}; Q, Z\right]_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (I - C_i Q^{k-j})^{-1} (I - A_i Q^{k-j}) \right) Z \right],$$

and the *noncommutative Q-shifted factorial of type II* by

$$\left\lfloor\begin{matrix} A_1, A_2, \dots, A_r \\ C_1, C_2, \dots, C_r \end{matrix}; Q, Z\right\rfloor_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (I - C_i Q^{j-1})^{-1} (I - A_i Q^{j-1}) \right) Z \right].$$

We define the *noncommutative basic hypergeometric series of type I* by

$${}_{r+1}\phi_r \left[\begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r \end{matrix}; Q, Z\right] := \sum_{k \geq 0} \left[\begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r, Q \end{matrix}; Q, Z\right]_k,$$

and the *noncommutative basic hypergeometric series of type II* by

$${}_{r+1}\phi_r \left\lfloor\begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r \end{matrix}; Q, Z\right\rfloor := \sum_{k \geq 0} \left\lfloor\begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r, Q \end{matrix}; Q, Z\right\rfloor_k.$$

We also refer to the respective series as *(noncommutative) Q-hypergeometric series*. In each case, the series terminates if one of the upper parameters A_i is of the form Q^{-n}. If the series does not terminate, then it converges if $\|Z\| < 1$.

Finally we recall the following well known definition.

A Banach $*$-algebra is a Banach algebra R over a field K equipped with an involutive antiautomorphism, i.e. a map $* : R \longrightarrow R$ which satisfies the following properties for every $X, Y \in R$:

$$\begin{aligned} &(X^*)^* = X, \quad \text{viz. the map } * \text{ is an involution,} \\ &(X+Y)^* = X^* + Y^*, \\ &(XY)^* = Y^* X^*, \\ &\|X^*\| = \|X\|, \end{aligned}$$

and such that the restriction $* : K \longrightarrow K$ is an involutive automorphism, since K is commutative.

For instance, the ring of complex $n \times n$ square matrices is a Banach $*$-algebra, where the map $*$ is the adjoint operator, viz. conjugate transposition.

3. Type I and type II noncommutative hypergeometric equations

Tirao [29] proved the following result:

Proposition 3.1. *For a positive integer n, let $R = M_{n\times n}(\mathbb{C})$ be the ring of complex $n \times n$ square matrices. Let $A, B, C, F_0 \in R$ be such that the spectrum of C contains no negative integers, and let $z \in \mathbb{C}$. Then $F(z) = {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; zI\right] F_0$ is the unique solution analytic at $z = 0$ of the matrix-valued hypergeometric equation*

$$z(1-z)F''(z) + (C - z(1 + A + B))F'(z) - ABF(z) = 0,$$

where $F(0) = F_0$.

As was indicated without proof in [25, Remark 2.1] this readily extends to the following:

Theorem 3.1. *Let R be a unital Banach algebra with norm $\|\cdot\|$, let $A, B, C, F_0 \in R$ such that $C + jI$ is invertible for all nonnegative integers j. Further let Z be central (i.e., $Z \in \{X \in R : XY = YX,\ \forall Y \in R\}$) with $\|Z\| < 1$. Then*

$$F(Z) = {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right] F_0 \tag{3.1}$$

is the unique solution analytic at $Z = O$ of the noncommutative hypergeometric equation

$$Z(I - Z)F''(Z) + (C - Z(I + A + B))F'(Z) - ABF(Z) = O, \tag{3.2}$$

where $F(O) = F_0$.

We provide an operator proof of Theorem 3.1. On the contrary, Tirao's proof of the above Proposition given in [29] is essentially different. Starting with the matrix-valued hypergeometric equation it involves the computation of the coefficients F_k in the analytic series $F(z) = \sum_{k\geq 0} F_k z^k$ by a generic Ansatz.

Proof. First of all, the (right multiple of the) type I noncommutative hypergeometric series

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right] F_0\\ &= \left[\sum_{k\geq 0}\left(\prod_{j=1}^{k}(C + (k-j)I)^{-1}(A + (k-j)I)(B + (k-j)I)\right)\frac{Z^k}{k!}\right] F_0\end{aligned}$$

is clearly analytic at $Z = O$ and ${}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; O\right] F_0 = F_0$.

Next we show that ${}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right] F_0$ is a solution of the differential equation (3.2). We define the linear operator

$$\mathcal{D}_T := T + Z\frac{\mathrm{d}}{\mathrm{d}\,Z},$$

where $T \in R$, acting (from the left) on functions of Z over R.

If $F(Z)$ is analytic at $Z = O$ we can write $F(Z) = \sum_{k\geq 0} F_k Z^k$, where $F_k \in R$ for any nonnegative integer k. It is immediate that

$$\mathcal{D}_T\, F(Z) = \sum_{k\geq 0} (T + kI)\, F_k Z^k.$$

Hence

$$\mathcal{D}_A \left(\mathcal{D}_B\ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \right) = \sum_{k\geq 0} (A + kI)(B + kI) \left[\begin{matrix} A, B \\ C, I \end{matrix}; Z\right]_k,$$

and

$$\begin{aligned}
\mathcal{D}_{C-I}\ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] &= \sum_{k\geq 0} (C + (k-1)I) \left[\begin{matrix} A, B \\ C, I \end{matrix}; Z\right]_k \\
&= C - I + \sum_{k\geq 1} (A + (k-1)I)(B + (k-1)I) \\
&\quad \times \left(\prod_{j=1}^{k-1} (C + (k-1-j)I)^{-1} (A + (k-1-j)I)(B + (k-1-j)I) \right) \frac{Z^k}{k!} \\
&= C - I + \sum_{k\geq 0} (A + kI)(B + kI) \\
&\quad \times \left(\prod_{j=1}^{k} (C + (k-j)I)^{-1} (A + (k-j)I)(B + (k-j)I) \right) \frac{Z^{k+1}}{(k+1)!}.
\end{aligned}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}\, Z} \left(\mathcal{D}_{C-I}\ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \right) = \mathcal{D}_A \left(\mathcal{D}_B\ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \right). \tag{3.3}$$

Since the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\, Z} \left(\mathcal{D}_{C-I}\, F(Z) \right) = \mathcal{D}_A \left(\mathcal{D}_B\, F(Z) \right), \tag{3.4}$$

or, more explicitly,

$$\frac{\mathrm{d}}{\mathrm{d}\, Z} \left(C - I + Z \frac{\mathrm{d}}{\mathrm{d}\, Z} \right) F(Z) = \left(A + Z \frac{\mathrm{d}}{\mathrm{d}\, Z} \right) \left(B + Z \frac{\mathrm{d}}{\mathrm{d}\, Z} \right) F(Z),$$

is equivalent to (3.2), it follows from (3.3) (and multiplication of a constant from the right) that ${}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] F_0$ satisfies the differential equation (3.2).

The uniqueness of the solution (3.1) of (3.2) with $F(O) = F_0$ readily follows from the theorem of existence and uniqueness of solutions of differential equations in Banach spaces (hence in Banach algebras), cf. e.g. [19]. All we need to show is that

if there were two solutions $F_1(Z)$ and $F_2(Z)$ then $F_1'(O) = F_2'(O)$. (As we are considering a second order differential equation, two initial conditions, fixing $F(O)$ and $F'(O)$, are required to make the solution unique.)

Assume that $F_1(Z)$ and $F_2(Z)$ are solutions of (3.2) with $F_1(O) = F_2(O) = F_0$. Then we have

$$\begin{aligned} &Z(I-Z)F_1''(Z) + (C - Z(A+B+I))F_1'(Z) - ABF_1(Z) \\ &= Z(I-Z)F_2''(Z) + (C - Z(A+B+I))F_2'(Z) - ABF_2(Z). \end{aligned}$$

Evaluating this equation in $Z = O$ we get $C\,F_1'(O) = C\,F_2'(O)$ and since C is invertible the claim follows. □

Now we are ready to state and prove the following new result concerning type II noncommutative hypergeometric series. It appears to lie in the nature of the type II series that the result is not as simple and elegant as in the corresponding type I case. In particular, the following theorem as stated requires the condition $C(C-A-B)+AB$ being invertible, which has no counterpart in the type I case.

Theorem 3.2. *Let R be a unital Banach $*$-algebra with norm $\|\cdot\|$, let $A, B, C, F_0 \in R$ such that $C(C-A-B)+AB$ and $C+jI$ are invertible for all nonnegative integers j. Further let Z be central (i.e., $Z \in \{X \in R : XY = YX,\ \forall Y \in R\}$) with $\|Z\| < 1$. Then*

$$F(Z) = F_0\ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \tag{3.5}$$

is the unique solution analytic at $Z = O$ of the noncommutative hypergeometric equation

$$\begin{aligned} &Z(I-Z)F''(Z) + ZF'(Z)(C - I - A - B) \\ &+ \big((I-Z)F'(Z) - F(Z)C^{-1}AB\big)\big(C(C-A-B)+AB\big)^{-1}C\big(C(C-A-B)+AB\big) \\ &\qquad = O, \end{aligned} \tag{3.6}$$

where $F(O) = F_0$.

Remark 3.1. If, instead of the condition of $C(C-A-B)+AB$ being invertible, C would commute with A and B, then we would have a much simpler hypergeometric equation, as the type II hypergeometric function would essentially be a starred type I hypergeometric function. More precisely, if C commutes with A and B, one has (as one readily verifies)

$${}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] = {}_2F_1\left[\begin{matrix} B^*, A^* \\ C^* \end{matrix}; Z^*\right]^*,$$

and the corresponding hypergeometric equation in place of (3.6) is just

$$F''(Z)Z(I-Z) + F'(Z)(C - Z(I+A+B)) - F(Z)AB = O.$$

As this would not yield anything really new, we prefer not to impose the strong condition of C commuting with A and B, but nevertheless, in order to make progress

at a particular point in the following proof (namely, after arriving at (3.10)), impose the (slightly awkward-looking) condition of $C(C-A-B)+AB$ being invertible.

Proof of Theorem 3.2. First of all, the (left multiple of the) type II noncommutative hypergeometric series

$$\begin{aligned}&F_0\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]\\&=F_0\sum_{k\geq 0}\left(\prod_{j=1}^{k}(C+(j-1)I)^{-1}(A+(j-1)I)(B+(j-1)I)\right)\frac{Z^k}{k!}\end{aligned}$$

is clearly analytic at $Z=O$ and $F_0\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};O\right]=F_0$.

Next we show that $F_0\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};O\right]$ is a solution of the differential equation (3.6). We have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\,Z}\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]\\&=\sum_{k\geq 1}\left(\prod_{j=1}^{k}(C+(j-1)I)^{-1}(A+(j-1)I)(B+(j-1)I)\right)\frac{Z^{k-1}}{(k-1)!}\\&=\sum_{k\geq 0}\left(\prod_{j=1}^{k+1}(C+(j-1)I)^{-1}(A+(j-1)I)(B+(j-1)I)\right)\frac{Z^{k}}{k!}\\&=\sum_{k\geq 0}\left[\begin{matrix}A,B\\C,I\end{matrix};Z\right]_k(C+kI)^{-1}(A+kI)(B+kI).\end{aligned}\tag{3.7}$$

We define the linear operator $\widetilde{\mathcal{D}}_T$ by

$$\widetilde{\mathcal{D}}_T:=T+\frac{\widetilde{\mathrm{d}}}{\mathrm{d}\,Z}Z,$$

where $T\in R$, acting from the *right* on functions over R. Here $\frac{\widetilde{\mathrm{d}}}{\mathrm{d}\,Z}$ is the differential operator applied from the right side. With other words

$$F(Z)\,\frac{\widetilde{\mathrm{d}}}{\mathrm{d}\,Z}=\frac{\mathrm{d}}{\mathrm{d}\,Z}\,F(Z),$$

and

$$F(Z)\,\widetilde{\mathcal{D}}_T=F(Z)\,T+Z\frac{\mathrm{d}}{\mathrm{d}\,Z}\,F(Z),$$

where $F(Z)$ is any function of Z (Z being central) over R.

In particular, we have

$$F(Z)\,\widetilde{\mathcal{D}}_T=(\mathcal{D}_T^*\,(F(Z)^*))^*$$

where $\mathcal{D}^*_T := T^* + Z^* \frac{\mathrm{d}}{\mathrm{d}\, Z^*}$.

If $F(Z)$ is analytic at $Z = O$ we can write $F(Z) = \sum_{k\geq 0} F_k Z^k$, where $F_k \in R$ for any nonnegative integer k. It is immediate that

$$F(Z)\, \widetilde{\mathcal{D}}_T = \sum_{k\geq 0} F_k Z^k\, (T + kI), \tag{3.8a}$$

and

$$F(Z)\, \widetilde{\mathcal{D}}_U^{-1} = \sum_{k\geq 0} F_k Z^k\, (U + kI)^{-1}, \tag{3.8b}$$

provided $U + kI$ is invertible in R for all nonnegative integers k. (As U is invertible, so is $\widetilde{\mathcal{D}}_U^{-1} = (U + \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z} Z)^{-1} = U^{-1}(I + \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z} Z U^{-1})^{-1}$ where the last expression can be expanded as a geometric series and becomes meaningful. Independently, it is easy to see from (3.8a)/(3.8b) that the action of $\widetilde{\mathcal{D}}_U^{-1}$ on a power series is inverse to that of $\widetilde{\mathcal{D}}_U$.)

Hence

$$\begin{aligned}\left(\left({}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right] \widetilde{\mathcal{D}}_C^{-1}\right) \widetilde{\mathcal{D}}_A\right) \widetilde{\mathcal{D}}_B &= \sum_{k\geq 0} \left[\begin{matrix}A, B\\ C, I\end{matrix}; Z\right]_k (C + kI)^{-1}(A + kI)(B + kI) \\ &= {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right] \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z}\,,\end{aligned}$$

by (3.7).

It follows that

$$G(Z) = {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right] \widetilde{\mathcal{D}}_C^{-1}$$

is a solution of the differential equation

$$\Big(G(Z)\, \widetilde{\mathcal{D}}_A\Big)\, \widetilde{\mathcal{D}}_B = \Big(G(Z)\, \widetilde{\mathcal{D}}_C\Big)\, \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z}.$$

This is simply a "reversed" version of (3.4) with A and B interchanged and $C + I$ in place of C. It thus follows from Theorem 3.1 that $G(Z)$ satisfies the reversed type I noncommutative hypergeometric equation:

$$Z(I - Z)G''(Z) + G'(Z)(C + I - Z(I + A + B)) - G(Z)\, AB = O. \tag{3.9}$$

We now need to rewrite (3.9) in terms of $F(Z) = {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; Z\right]$. We have

$$F(Z) = G(Z)\, \widetilde{\mathcal{D}}_C = G(Z)C + ZG'(Z),$$

and

$$F'(Z) = G'(Z)(C + I) + ZG''(Z),$$

which, in conjunction with (3.9), gives

$$(I-Z)F'(Z)+F(Z)(C-A-B)-F(Z)\,\widetilde{\mathcal{D}}_C^{-1}\big(C(C-A-B)+AB\big)=O. \quad (3.10)$$

Next, we multiply both sides of (3.10) from the right with

$$\big(C(C-A-B)+AB\big)^{-1}\widetilde{\mathcal{D}}_C\big(C(C-A-B)+AB\big)$$

(which is $\big(C(C-A-B)+AB\big)^{-1}C\big(C(C-A-B)+AB\big)+\frac{\widetilde{\mathrm{d}}}{\mathrm{d}\,Z}Z$). After a series of computations, including the simplifiction

$$I-(C-A-B)\big(C(C-A-B)+AB\big)^{-1}C=C^{-1}AB\big(C(C-A-B)+AB\big)^{-1}C,$$

we eventually arrive at (3.6). Further, by multiplying a constant from the left, it follows that $F_0\,{}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]$ satisfies the differential equation (3.6).

Using the same argument as in the proof of Theorem 3.1, one readily establishes the uniqueness of the solution (3.5) of (3.6) with $F(O)=F_0$. □

4. Type I and type II noncommutative basic hypergeometric equations

For a parameter $Q\in R$ which commutes with all other parameters (that appear in the respective expressions) we define the Q-difference operator $\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}$ by

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}F(Z)=(I-Q)^{-1}Z^{-1}(F(Z)-F(QZ)).$$

Its action on monomials is

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}Z^k=(I-Q)^{-1}(I-Q^k)Z^{k-1},$$

while in combination with the multiplication operator Z one has

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}\left(ZF\left(Z\right)\right)=F\left(Z\right)+QZ\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}F\left(Z\right). \quad (4.1)$$

Clearly, as $Q\to I$, the Q-difference operator $\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}$ approaches the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}\,Z}$.

We have the following Q-analogue of Theorem 3.1:

Theorem 4.1. *Let R be a unital Banach algebra with norm $\|\cdot\|$, let $A,B,C,F_0,Q\in R$ such that Q commutes with A,B,C,F_0 and such that $I-CQ^j$ is invertible for all nonnegative integers j. Further let Z be central (i.e., $Z\in\{X\in R: XY=YX,\ \forall Y\in R\}$) with $\|Z\|<1$. Then*

$$F(Z)={}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]F_0 \quad (4.2)$$

is the unique solution analytic at $Z = O$ of the noncommutative basic hypergeometric equation

$$\begin{aligned} &Z(C - ABQZ)\frac{\mathrm{d}_Q^2}{\mathrm{d}_Q Z^2}F(Z) \\ +(I-Q)^{-1}\big[(I-C)+(I-A)(I-B)Z-(I-ABQ)Z\big]&\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}F(Z) \\ &-(I-Q)^{-2}(I-A)(I-B)F(Z)=O, \end{aligned} \tag{4.3}$$

where $F(O) = F_0$.

For commuting parameters Theorem 4.1 reduces to [11, Ex. 1.13]. We prove Theorem 4.1 in a similar way to our proof of Theorem 3.1.

Proof. First of all, the (right multiple of the) type I noncommutative basic hypergeometric series

$$\begin{aligned} &{}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]F_0 \\ &=\left[\sum_{k\geq 0}\left(\prod_{j=1}^{k}(I-CQ^{k-j})^{-1}(I-AQ^{k-j})(I-BQ^{k-j})(I-Q^{1+k-j})^{-1}\right)Z^k\right]F_0 \end{aligned}$$

is clearly analytic at $Z = O$ and ${}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,O\right]F_0 = F_0$.

Next we show that ${}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]F_0$ is a solution of the difference equation (4.3). We define the linear operator

$$\mathcal{D}_{Q,T} := (I-Q)^{-1}(I-T) + TZ\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z},$$

where $T \in R$, acting (from the left) on functions of Z over R.

If $F(Z)$ is analytic at $Z = O$ we can write $F(Z) = \sum_{k\geq 0} F_k Z^k$, where $F_k \in R$ for any nonnegative integer k. Since

$$\mathcal{D}_{Q,T}\, Z^k = (I-Q)^{-1}(I-TQ^k)Z^k,$$

it is immediate that

$$\mathcal{D}_{Q,T}\, F(Z) = (I-Q)^{-1}\sum_{k\geq 0}(I-TQ^k)\, F_k Z^k.$$

Hence

$$\mathcal{D}_{Q,A}\left(\mathcal{D}_{Q,B}\,{}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]\right) = (I-Q)^{-2}\sum_{k\geq 0}(I-AQ^k)(I-BQ^k)\left[\begin{matrix}A,B\\C,Q\end{matrix};Q,Z\right]_k,$$

and

$$\begin{aligned}
\mathcal{D}_{Q,CQ^{-1}}\,{}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right] &= (I-Q)^{-1}\sum_{k\geq 0}(I-CQ^{k-1})\left[\begin{matrix}A,B\\C,Q\end{matrix};Q,Z\right]_k\\
&= (I-Q)^{-1}(I-CQ^{-1})+(I-Q)^{-1}\sum_{k\geq 1}(I-AQ^{k-1})(I-BQ^{k-1})(I-Q^k)^{-1}\\
&\quad\times\left(\prod_{j=1}^{k-1}(I-CQ^{k-1-j})^{-1}(I-AQ^{k-1-j})(I-BQ^{k-1-j})(I-Q^{k-j})^{-1}\right)Z^k\\
&= (I-Q)^{-1}(I-CQ^{-1})+(I-Q)^{-1}\sum_{k\geq 0}(I-AQ^{k})(I-BQ^{k})(I-Q^{k+1})^{-1}\\
&\quad\times\left(\prod_{j=1}^{k}(I-CQ^{k-j})^{-1}(I-AQ^{k-j})(I-BQ^{k-j})(I-Q^{k+1-j})^{-1}\right)Z^{k+1}.
\end{aligned}$$

Thus we have

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\left(\mathcal{D}_{Q,CQ^{-1}}\,{}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]\right)=\mathcal{D}_{Q,A}\left(\mathcal{D}_{Q,B}\,{}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]\right). \tag{4.4}$$

Since the Q-differential equation

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\left(\mathcal{D}_{Q,CQ^{-1}}\,F(Z)\right)=\mathcal{D}_{Q,A}\left(\mathcal{D}_{Q,B}\,F(Z)\right), \tag{4.5}$$

or, more explicitly,

$$\begin{aligned}
&\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\left((I-Q)^{-1}(I-CQ^{-1})+CQ^{-1}Z\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\right)F(Z)\\
&=\left((I-Q)^{-1}(I-A)+AZ\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\right)\left((I-Q)^{-1}(I-B)+BZ\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\right)F(Z),
\end{aligned}$$

is equivalent to (4.3) (which can be verified using (4.1) and simple identities such as $(I-Q)^{-1}(I-CQ^{-1})+CQ^{-1}=(I-Q)^{-1}(I-C)$), it follows from (4.4) (and multiplication of a constant from the right) that ${}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]F_0$ satisfies the differential equation (4.3).

The uniqueness of the solution (4.2) of (4.3) with $F(O)=F_0$ readily follows from the theorem of existence and uniqueness of solutions of differential equations in Banach spaces, just as in the proof of Theorem 3.1. □

In a Banach $*$-algebra R, for a parameter $Q\in R$ which commutes with all other parameters (that appear in the respective expressions) we define $\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q Z}$ as the Q-difference operator acting from the *right* on functions over R, i.e.,

$$F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q Z} = \left(\frac{\mathrm{d}^*{}_Q}{\mathrm{d}^*{}_Q Z}\,(F(Z)^*)\right)^*$$

where

$$\frac{\mathrm{d}^*{}_Q}{\mathrm{d}^*{}_Q Z}F(Z) = (I-Q^*)^{-1}(Z^*)^{-1}(F(Z)-F(QZ)).$$

By a similar analysis as in the proof of Theorem 3.2 one can also work out a type II noncommutative basic hypergeometric equation which is again of second order. As one would expect, the result has a significantly more complicated form than in the type I case (compare Theorem 3.2 with Theorem 3.1). In particular, a required condition is that C and $(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)$ have to be invertible, which has no counterpart in the basic type I case. Since the proof (which essentially follows the lines of the proofs of Theorems 3.2 and 4.1) is just tedious but not very illuminating, we state the result without it.

Theorem 4.2. *Let R be a unital Banach $*$-algebra with norm $\|\cdot\|$, let $A, B, C, F_0, Q \in R$ such that Q commutes with A, B, C, F_0 and such that C, $(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)$ and $I - CQ^j$ are invertible for all nonnegative integers j. Further let Z be central (i.e., $Z \in \{X \in R : XY = YX,\ \forall Y \in R\}$) with $\|Z\| < 1$. Then*

$$F(Z) = F_0\ {}_2\phi_1\left(\begin{matrix}A, B\\ C\end{matrix}; Q, Z\right) \tag{4.6}$$

is the unique solution analytic at $Z = O$ of the noncommutative basic hypergeometric equation

$$\begin{aligned}
&F(Z)\frac{\widetilde{\mathrm{d}_Q^2}}{\mathrm{d}_Q Z^2}Z(I-C^{-1}ABQZ)\\
&\qquad\times(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)^{-1}C(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)\\
&+F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q Z}(I-Q)^{-1}\\
&\qquad\times(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)^{-1}(I-C)(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)\\
&+F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q Z}Z(I-Q)^{-1}\Big[C-A-B+(C^{-1}AB+C^{-1}ABQ-I)\\
&\qquad\times(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)^{-1}C(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)\Big]\\
&+F(Z)(I-Q)^{-2}\Big[A+B-C-I-(C^{-1}AB-I)\\
&\qquad\times(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)^{-1}C(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)\Big]=O,
\end{aligned} \tag{4.7}$$

where $F(O) = F_0$.

Remark 4.1. Similarly as in Theorem 3.2 (see the Remark 3.1), if instead of the condition of $(I-C^{-1}A-C^{-1}(I-C^{-1}A)B)$ being invertible, C would commute with A and B, then we would have a much simpler Q-hypergeometric equation (as in type I), stemming from the observation that in this case the type II Q-hypergeometric function is a starred type I Q-hypergeometric function (with starred parameters, but with the upper parameters A and B being interchanged). More precisely, if C commutes with A and B, one has (as one readily verifies)

$$ {}_2\phi_1\left[\begin{matrix}A,B\\C\end{matrix};Q,Z\right]={}_2\phi_1\left[\begin{matrix}B^*,A^*\\C^*\end{matrix};Q^*,Z^*\right]^*, $$

and the corresponding Q-hypergeometric equation in place of (4.7) is

$$\begin{aligned}&F(Z)\frac{\widetilde{\mathrm{d}_Q^2}}{\mathrm{d}_Q Z^2}Z(C-ABQZ)\\ &+F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q Z}(I-Q)^{-1}[(I-C)+(I-A)(I-B)Z-(I-ABQ)Z]\\ &+F(Z)(I-Q)^{-2}(I-A)(I-B)=O.\end{aligned}$$

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