Programme on

"Infinite-dimensional Riemannian geometry with applications to image matching and shape analysis"

January 7 – February 27, 2015

organized by

Martin Bauer (U Vienna), Martins Bruveris (Brunel), Peter W. Michor (U Vienna)

Workshop on

"Infinite-dimensional Riemannian geometry"

January 12 – 16, 2015

- **Monday, January 12, 2015**

9:30–10:30
Local and global well-posedness of the fractional order EPDiff equation on $\mathbb{R}^d$
Boris Kolev
Université d'Aix-Marseille I

Of concern is the study of fractional order Sobolev-type metrics on the group of H^∞-diffeomorphism of $\mathbb{R}^d$ and on its Sobolev completions $\mathcal{D}^q(\mathbb{R}^d)$. It is shown that the H^s-Sobolev metric induces a strong and smooth Riemannian metric on the Banach manifolds $\mathcal{D}^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$. As a consequence a global well-posedness result of the corresponding geodesic equations, both on the Banach manifold $\mathcal{D}^s(\mathbb{R}^d)$ and on the smooth regular Fréchet-Lie group of all H^∞-diffeomorphisms is obtained. In addition a local existence result for the geodesic equation for metrics of order $\frac{1}{2} \leq s < 1 + d/2$ is derived.

This is joint work with Martin Bauer and Joachim Escher.

11:00–12:00
Uniqueness of the Fisher–Rao metric on the space of smooth densities
Peter W. Michor
Universität Wien

On a closed manifold of dimension greater than one, every smooth weak Riemannian metric on the space of smooth positive probability densities, that is invariant under the action of the diffeomorphism group, is a multiple of the Fisher–Rao metric.

This is joint work with Marin Bauer and Martins Bruveris.

14:00–15:00
On the convexity of the KdV Hamiltonian
Thomas Kappeler
Universität Zürich

We show that the KdV Hamiltonian $\mathcal{H}$, defined on the Sobolev space H^1 of real valued functions on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, is strictly concave near 0 in the strongest possible sense: it means that when expressed in action variables $I = (I_n)_{n\geq 1}$, the nonlinear part of $\mathcal{H}$ extends analytically to the sequence space ℓ^2 and its Hessian is negative definite in a neighborhood of 0 in ℓ^2. A key element of the proof is to show that the KdV equation admits Birkhoff coordinates when considered on the Fourier Lebesgue spaces $\mathcal{L}^{s,p}(\mathbb{T}, \mathbb{R})$ for any $-1/2 \leq s \leq 0$ and $2 \leq p < \infty$. The convexity property of Hamiltonians plays a crucial role in the perturbation theory of Hamiltonian systems.

This is joint work with Alberto Maspero, Jan Molnar, and Peter Topalov.

15:00-16:00
Lobachevsky geometry and image recognition
Valentin Lychagin
University of Tromsø

In paper [2] E. Sharon and D. Mumford suggested an approach for analysis of 2-dimensional shapes by the conformal group. Namely, they study plane domains with smooth boundaries up to conformal equivalence. Domains in their approach are assumed to be empty, i.e. they do not curry any geometrical structure except the conformal one.

In this paper we extend their approach to analyze 2-dimensional domains equipped with a some geometrical object. We study in defails cases when domains are considered together with functions, differential 1-forms or foliations on its and two such "decorated" domains are equivalent if there is a conformal mapping of them which equalizes the corresponding geometrical objects.

First of all we remark that the Riemann conformal mapping theorem transforms the problem to study equivalence of functions, differential 1-forms or foliations on the Lobachevsky plane with respect to the group of Möbius transformations.

Secondly, application of Lie–Tresse theorem [1] allows us to find fields of rational differential invariants for the different actions of the group $PSL_2(\mathbb{R})$ of the Möbius transformations. These fields separate regular orbits of actions of the Möbius transformations on the jet-spaces of functions, differential 1-forms and foliations respectively and give complete solutions of the equivalence problem for regular cases.

This is joint work with Nadia Konovenko.

[1] B. Kruglikov, V. Lychagin. Global Lie-Tresse theorem, 2013, 48p., `http://arxiv.org/abs/1111.5480`

[2] E. Sharon, D. Mumford. 2D-Shape Analysis Using Conformal Mapping. International Journal of Computer Vision 70(1), 55–75, 2006.
DOI: 10.1007/s11263-006-6121-z

16:30–17:30
Invariants of diffeomorphism group actions
Valentin Lychagin
University of Tromsø

We'll discuss invariants (local or differential) of diffeomorphism group actions on (pseudo-)Riemann metrics as well as invariants of solutions of Einstein and Einstein–Maxwell equations. In all these cases we'll describe the fields of rational differential invariants and show that they do separate regular orbits. Later these invariants shall be used to construct differential factor-equations in all three cases and solve the equivalence problem.

- **Tuesday, January 13, 2015**

9:00–10:00
The group of diffeomorphisms of the circle and its completions
Yuri Neretin
Universität Wien

The group of diffeomorphisms of the circle is a well-known object in representation theory. Consider a unitary representation of this group and its closure in bounded operators. It produces a strange boundaries. We discuss them and some unsolved problems related to such boundaries.

10:00–11:00

On Lie groups of diffeomorphisms generated by time-dependent vector fields

Armin Rainer
Universität Wien

By applying a construction due to Trouvé, we define groups $\mathcal{G}^{\infty,p}(\mathbb{R}^n)$ and $\mathcal{G}^{[M],p}(\mathbb{R}^n)$ of diffeomorphisms on $\mathbb{R}^n$, generated by the flows of time-dependent $W^{\infty,p}$ and $W^{[M],p}$ vector fields. We denote by $W^{\infty,p}(\mathbb{R}^n)$ the intersection of all Sobolev spaces $W^{k,p}(\mathbb{R}^n)$, $k \in \mathbb{N}$, for $1 \le p \le \infty$. And $W^{[M],p}(\mathbb{R}^n)$ is the subspace consisting of functions in $W^{\infty,p}(\mathbb{R}^n)$ subject to a certain growth condition for the L^p-norms of its partial derivatives in terms of a weight sequence $M = (M_k)$. We show that $\mathcal{G}^{\infty,p}(\mathbb{R}^n)$ and $\mathcal{G}^{[M],p}(\mathbb{R}^n)$ coincide with the identity components of the groups $\mathrm{Diff}^{\infty,p}(\mathbb{R}^n)$ and $\mathrm{Diff}^{[M],p}(\mathbb{R}^n)$ of diffeomorphisms on $\mathbb{R}^n$ that differ from the identity by a $W^{\infty,p}$- or a $W^{[M],p}$-mapping, respectively. As a consequence we conclude that $\mathcal{G}^{\infty,p}(\mathbb{R}^n)$ and $\mathcal{G}^{[M],p}(\mathbb{R}^n)$ are C^∞- and $C^{[M]}$-regular Lie groups, respectively.

Joint work with Andreas Kriegl and Peter Michor.

11:30–12:30

The exponential map on the area-preserving diffeomorphism group for a bounded surface

Stephen Preston
University of Colorado at Boulder

The Riemannian exponential map is the solution operator of the geodesic equation (taking an initial velocity to a final position). On the area-preserving diffeomorphism group, this map describes the solution of the two-dimensional ideal Euler equations of a fluid in Lagrangian coordinates. Singularities of the exponential map correspond to conjugate points on the area-preserving diffeomorphism group, and hence to stable Lagrangian perturbations.

The singularities of an infinite-dimensional map (that is, the failure of its differential to be either injective or surjective) can be much more complicated than those of a finite-dimensional map. However if the map is Fredholm, they are essentially the same, and one can prove global theorems in geometry using Fredholmness and the local structure. It is known that when the fluid domain is a surface without boundary, the exponential map is Fredholm, while if the domain is three-dimensional, the map is not Fredholm.

In this talk I will give an overview of Fredholmness, how one proves it for a given Euler-Arnold type equation, and what its known applications are. Then I will discuss some new work with Gerard Misiolek on the case when the surface has a boundary.

14:30–15:15

Geometry of the group of axisymmetric volumorphisms

Pearce Washabaugh
University of Colorado at Boulder

The sectional curvature of the volume preserving diffeomorphism group of a Riemannian manifold M can give information about the stability of inviscid, incompressible fluid flows on M. We will discuss properties of the curvature of the submanifold of the volumorphism group of the solid flat torus generated by axisymmetric fluid flows with swirl. In particular, it has positive sectional curvature in every section containing the field $X = u(r)\partial_\theta$ iff $\partial_r(ru^2) > 0$. This is in sharp contrast to the situation on the rest of the volume preserving diffeomorphism group, where only Killing fields X have nonnegative sectional curvature in all sections containing them. We will also discuss how this criterion guarantees the existence of conjugate points along the geodesic defined by X.

This is joint work with Stephen Preston.

15:15–16:00

Conjugate points on the group of contactomorphisms

Boramey Chhay
University of Colorado at Boulder

In this talk we investigate the geometry of the contactomorphism group. We find the conjugate points along the geodesic determined by the flow of the Reeb field and determine that they are monoconjugate of finite order. We then prove that the exponential map is Fredholm.

16:30–17:15
L^2-geometry of the symplectomorphism group
James Benn
University of Notre Dame

It was shown by Ebin, Misiolek and Preston that the exponential map of the L^2 metric on the volume-preserving diffeomorphism group of a compact, boundaryless, $2D$ manifold is a non-linear Fredholm map of index zero. Although this is no longer true for $3D$-manifolds, I will show in this talk that the exponential map of the L^2 metric on the symplectic diffeomorphism group of a closed symplectic $2nD$ manifold is a non-linear Fredholm map of index zero, for all $n \geq 1$. Examples of conjugate points are given and it is shown that the multiplicity of every conjugate point along a geodesic of isometries in the symplectic diffeomorphism group is of even multiplicity.

- **Wednesday, January 14, 2015**

9:00–10:00
The initial value problem for fluids with free boundary and surface tension
David G. Ebin
Stony Brook University

We will consider the initial value problem for the motion of incompressible inviscid fluids with free boundary and surface tension on the boundary. We will show how to find solutions. We also show that if the initial boundary has constant mean curvature, then the motion converges to motion of a fluid with fixed boundary as the surface tension constant gets large.

10:00–11:00
Diffeomorphic density matching using Fisher–Rao geodesics
Klas Modin
Chalmers University

We use recently discovered connections between information geometry and topological hydrodynamics to obtain novel, geometrically founded algorithms for diffeomorphic density matching, significantly more efficient than standard LDDMM algorithms. Images are treated as probability densities. The objective is to find a diffeomorphism that warps a source density so that its gross features match a target density. The diffeomorphism should be optimal with respect to an energy functional derived from the Fisher–Rao metric. We obtain the solution by a gradient flow on the space of diffeomorphisms. This flow admits a geometric reduction to a 2-component density gradient flow. Connections to optimal mass transport will be discussed.

This is joint work with Marin Bauer and Sarang Joshi.

11:30–12:30
String connections via the caloron correspondence
Christian Becker
Universität Potsdam

The string group is by definition a 3-connected cover of the spin group. It is well known that it cannot be realized as a finite dimensional Lie group. However there exists an infinite dimensional Fréchet Lie group version.

We describe certain fiber bundles with structure group the string group over finite dimensional manifolds, so called string structures. These lift spin-bundles in the same way as spin structures lift frame bundles. Like for spin structures, there is an obstruction to such lifts, a certain characteristic class of the underlying spin bundle, called string class. We construct connections on string structures by using the so-called caloron correspondence.

- **Thursday, January 15, 2015**

9:00–10:00
Infinite-dimensional sub-Riemannian geometry
Sylvain Arguillère
John Hopkins University

In this talk, I will start by defining weak and strong sub-Riemannian geometry on Banach manifolds, and I will compare these manifolds to the finite dimensional case. In particular, we will see that there is no Pontryagin maximum principle in infinite dimensions, because a third type of geodesic called "elusive geodesics" can appear. They correspond to curves for which the range of the differential of the endpoint mapping is dense and not closed. However, we will see that we still have the existence of a Hamiltonian geodesic flow for normal geodesics, at least in the strong case. I will also give a variant of Chow-Rashevski's theorem, and a variant of Sussmann's orbit theorem.

10:00–11:00
Sub-Riemannian structures on groups of diffeomorphisms
Emmanuel Trélat
Université Paris 6

In this talk, I will explain how to define and study strong right-invariant sub-Riemannian structures on the group of diffeomorphisms of a manifold with bounded geometry. In these structures, we establish some approximate and exact reachability properties (with consequences such as "horizontal" versions of Moser's theorems), and we derive the Hamiltonian geodesic equations for such structures. We recover normal and abnormal geodesics in that infinite-dimensional context, but we have also a new type of geodesic called elusive.

This is a work in collaboration with Sylvain Arguillère.

11:30–12:15
An example of a shape space modelled on Hilbert space
Jakob Møller-Andersen
TU Copenhagen

If M and N are manifolds then the shape space $B(M,N)$ is heuristically the collection of unique images of M in N under maps $M \to N$. It is usually constructed as the quotient of immersions or embeddings (of some regularity) modulo reparametrizations of M; $B(M,N) = \mathrm{Imm}(M,N)/\operatorname{Diff}(M)$. For the case of Sobolev immersions of $M = I, S^1$ into $N = \mathbb{R}^d$, it is a fact that the resulting shape space does not carry a differentiable structure, but is a length space. As any curve in H^2 has a constant speed reparametrization, we can consider the alternative space of constant speed curves. Using the implicit function theorem, we show that it has a Hilbert structure which carries an induced strong Riemannian metric. We derive the geodesic equations, calculate curvature and discuss completeness properties.

15:00–16:00
Reconstruction and pattern recognition via the Petitot model of the primary visual cortex
Jean-Paul Gauthier
Université de Toulon

The Petitot-Citti-Sarti Model of the primary visual cortex leads to natural mechanisms of image completion, via a certain associated Hypoelliptic diffusion.

Here, we discuss a semi-discrete version of this model, that looks also very natural, and that provides very elementary methods (algorithms) for pattern recognition and texture discrimination.

The main fact is that the group that comes is a (noncompact) Moore group, subject to Chu duality (which is a noncompact generalization of Tannaka duality).

16:30–17:30
Probabilities and manifolds
Andrea Mennucci
Scuola Normale Superiore

Probability theory has been widely studied for almost four centuries. Yet little is known regarding probabilities defined on infinite-dimensional manifolds. At the same time, infinite-dimensional manifolds appear often in shape theory. One such example is the Stiefel manifold. In this seminar we will see some negative and some positive results.

- **Friday, January 16, 2015**

9:00–10:00
Matrix-valued kernels for shape deformation
Mario Micheli
University of Washington

In recent years the rapid development of precise acquisition techniques for medical data has prompted applied mathematical work on the quantification of geometric deformation, for the ultimate purpose of performing statistics (e.g. template estimation, classification, regression analysis, and so on) on "shape spaces"; examples of shapes are curves in two or three dimensions, surfaces, images, tensor fields, or sets of feature points. In particular, the action of groups of diffeomorphisms induces Riemannian metrics on shape spaces; such approach is known as Large Deformation Diffeomorphic Metric Mapping (LDDMM). One may choose different metrics (inner products of vector fields) on the tangent space of the diffeomorphisms group, and these will induce different metrics and geometries on the shape spaces. In this talk we shall characterize the class of translation- and rotation-invariant metrics on group of diffeomorphisms, and provide examples of metrics whose geodesics in the group are generated by curl-free or divergence-free vector fields. The former are are suitable for the longitudinal studies of deformations that are purely caused by the growth or loss of matter (e.g. during the growth of a child's brain or the development of neurodegenerative diseases); the latter are instead especially useful in medical applications where deformations are known to preserve volume (e.g., for deformations of the tissues of the heart).

10:00–11:00
PCA on manifolds: application to spaces of landmarks
Sergey Kushnarev
National University of Singapore

Principal Component Analysis (PCA) is a widely used tool to reduce the dimensionality of the data. In many shape applications data lies on a manifold, and PCA cannot be applied in this non-linear space. The traditional approach is to linearize the data via the tangent space representation, but this linearization fails to account for the curvature of the underlying manifold. In this talk I will demonstrate how to take into the account the curvature of the manifold to have a better estimate of the principal components, and better capture the variance of the data. Numerical experiments on the manifold of landmarks will be demonstrated.

11:30–12:30
The gradient and the weighted metrics for the space of Kähler metrics
Kai Zheng
Universität Hannover

The geometric structure of the space of Kahler metric plays the fundamental role in the study of the variational problem of Calabi's extremal Kähler metrics. I will first describe various Riemannian metrics on the space of Kähler metrics. Then I will present their application to the current progress of the Calabi flow.

14:30–15:30
The intrinsic hypoelliptic Laplacian on sub-Riemannian manifolds
Ugo Boscain
École Polytechnique

In this talk I will discuss the question of defining a Laplacian in a sub-Riemannian manifold. I will give a "macroscipic" definition starting from a notion of intrinsic volume, and a "microsopic" definition based on random walks. I will then attack the problem of understanding when the two notions coincide. Surprisingly, different results are obtained in the contact and quasi-contact case.

15:30–16:30
Infinite dimensional manifold structures on spaces of submanifolds with boundary
François Gay-Balmaz
ENS Paris

Many variational optimization problems in image matching require the use of differential calculus on infinite dimensional spaces of submanifolds with boundary. In this talk I will present the construction of infinite dimensional Fréchet manifold structures on various spaces of submanifolds with boundary of a given manifold. I will also consider the associated principal bundle structures on spaces of embeddings as well as the corresponding volume preserving versions by using Moser's Lemma. This extends several earlier works on the subject.

All talks take place at the ESI, Boltzmann Lecture Hall.