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Generating functions for directed animals, convex following their direction

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Abstract

We give a generating function for one source under-diagonal directed animals convex following their direction. This family of animals is a subset of the general directed animals introduced in theoretical physics as a model for the study of the directed percolation. The generating function parameters are the horizontal semi-perimeter, the vertical semi-perimeter, the area and the first column height of a family of polyominoes in bijection with those animals.

1 Introduction

Consider the infinite square lattice $\Pi = Z \times Z$. A *unit step* is a couple of points $(p_1, p_2) \in \Pi^2$ such that $p_1 = (i, j)$ and $p_2 = (i + \epsilon, j + \sigma)$ where the couple (ϵ, σ) can take the subsequent values: $(1, 0,)$ for the *East* step, $(-1, 0)$ for the *West* step, $(0, 1)$ for the *North* step and $(0, -1)$ for the *South* step. A *path* in a subset P of Π between two points p and q is a sequence of points $p = p_1, p_2, ..., p_k = q$ all in P, such that, for all i, $1 \leq i \leq k-1$, (p_i, p_{i+1}) is a unit step.
A *directed animal* is a subset P of Π such that every point in P can be reached from particular points called *sources* (or *roots*), following a path in P using only North and East unit steps. Such a path is called a *directed path*. The direction of the animal is then North-East. Usually, the sources of P are located on a line perpendicular to the principle diagonal for which the equation is $y = x$.
The associated *polyomino* of an animal is obtained by centering every point of the animal in a unit square.
The enumeration of directed animals and polyominoes took its sources in the statistic physics theory and was studied by physicists like H.N.V. Temperley [Te], Dhar [Dh], Enting & Guttmann [EG], Lin [Li], V. Hakim & J.P. Nadal [HN] as well as by combinatorists like D. Gouyou-Beauchamps & X.G. Viennot

[GV], J.G. Penaud [Pe1], M.P. Delest [De], S. Dulucq [DD], Fedou [Fe], M. Bousquet-Mélou [BM1], E. Barcucci & R. Pinzani & R. Sprugnoli [BPS], S. Freretić & D. Svrtan [FS] and J.C. Lalanne [La].
We are interested here by the enumeration of one source directed animals, convex following their direction. We call them *diagonally convex directed animals* (*dcda* for short). We give an example of 26 points *dcda* in figure 1. An *under-diagonal dcda* is a *dcda* without points over the line crossing its source and parallel to the diagonal of equation $y = x$.

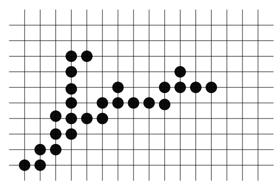

Figure 1: A 26 points *dcda*

2 Definitions

In relation to polyominoes, we normally think in terms of columns rather than diagonals. So, accordingly, we "stand up" our animals transforming diagonals into columns.
Let φ be an application which transforms the under-diagonal *dcda*'s into column convex directed polyominoes. We call $\mathcal{F}$ the image of the under-diagonal *dcda*'s by φ. φ is a simple rotation of the under-diagonal *dcda* diagonals around their base (i.e. intersection with the axis Ox) of $\Pi/4$ in the trigonometric sense (Fig. 2).

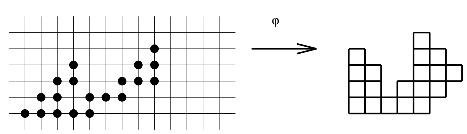

Figure 2

Let us now give a characterisation of the set $\mathcal{F}$:
Let F be a column convex polyomino. $C_1, C_2, ..., C_n$ are the columns of F. Consider that the plan is provided with a reference system (O, x, y) such that F is in the rectangle $0 \leq x \leq n$, $0 \leq y \leq m$ (Fig. 3). Let $[b_i, s_i]$ the projection of C_i over the axis Oy, in a parallel direction to Ox; the *base* (resp. *top*) of the column C_i is b_i (resp. s_i). An element F of $\mathcal{F}$ can have only one column, and in this case, it is reduced to one box. If it has two columns or more, it verifies the next four conditions:
(i)$b_1 = b_2 = 0$ and $s_2 \geq s_1 - 1$
(ii)$\forall i$, such that $2 \leq i \leq n-1$, $b_i \leq b_{i+1} \leq s_i - 1$
(iii)$\forall i$, such that $2 \leq i \leq n-1$ if $s_{i-1} < s_i$ then $s_{i+1} \geq s_i - 1$ and $b_{i+1} \leq s_{i-1} - 1$

(iv) $s_n \leq s_{n-1}$
φ is roughly a bijection between the *dcda*'s and $\mathcal{F}$.

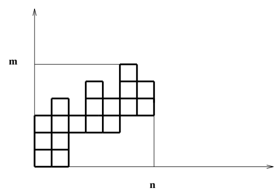

Figure 3

We can consider that a column convex polyomino is a *path* which begins by a North step and finishes by a West step in the point of coordonates $(0,0)$ using North, East, South and West steps without crossing the same point more than once and without containing the factor WSE. This path constitutes the border of the polyomino.
The vertical perimeter (resp. horizontal perimeter) of a polyomino is the number of North and South (resp. East and West) steps in the path representing it. The perimeter of a polyomino is then the sum of the vertical and the horizontal perimeter.
The area of a polyomino (resp. animal) is the number of squares (resp. points) it contains.
We define the *diagonal perimeter* of an under-diagonal *dcda* as the perimeter of its image by φ.

3 Generating function for $\mathcal{F}$

We always cover polyominoes from the left to the right. The first column is the left hand column.
Let $G(x, y, z, t)$ be the generating function of the polyominoes of $\mathcal{F}$ involving the parameters: horizontal semi-perimeter (by x), vertical semi-perimeter (by y), area (by z) and the height of the first column (by t).
To simplify, we put: $G(1) = G(x, y, z, 1)$ and $G(tz) = G(x, y, z, tz)$.
The next lemma gives us an equation satisfied by $G(x, y, z, t)$:

Definition 3.1 *Let $\mathcal{M}$ be the set of polyominoes being not in $\mathcal{F}$ and if we duplex their first column they become elements of $\mathcal{F}$.*
$M(x, y, z, t)$ is the generating function of the elements of $\mathcal{M}$.

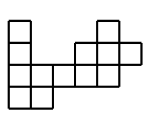

Figure 4: An element of $\mathcal{M}$

Lemma 3.2 *The generating function $G(t)$ of the family $\mathcal{F}$ and the generating function $M(t)$ of the family $\mathcal{M}$ satisfy the next system of equations:*

$$\begin{aligned} G(t) &= xyzt + \frac{tzx}{(1-yz^2t)(1-tz)}G(1) - \frac{x}{(1-yz^2t)(1-tz)}G(tz) \\ &\quad + x(1+zty)(G(tz)+M(tz)) \qquad (1) \\ M(t) &= \frac{xy^2z^2t^2}{(1-zyt)^2}(G(tz)+M(tz)) + \frac{zty}{1-zty}G(t) \qquad (2) \end{aligned}$$

proof: The method we use is the same used by M. Bousquet-Mélou in [BM2]. It consists in taking off the first column of the polyominoes we want to enumerate and in giving the generating function of the rest. In our case, taking off the first column of the polyominoes of $\mathcal{F}$, such that the height of their first column is lower or equal to the height of their second column, leads us to another family of polyominoes $\mathcal{L}$. Obviously, this new family of polyominoes contains $\mathcal{F}$.
Note that $\mathcal{L} = \mathcal{F} \cup \mathcal{M}$ and $\mathcal{F} \cap \mathcal{M} = \emptyset$.

□

To solve the system of equation $\{(1),(2)\}$, we introduce the following notations:

$$a(t) = \frac{tzx}{(1-yz^2t)(1-tz)}, \qquad b(t) = \frac{x}{(1-z^2yt)(1-tz)},$$

$$c(t) = x(1+yzt), \qquad d(t) = xyzt,$$

$$f(t) = \frac{xy^2z^2t^2}{(1-zyt)^2}, \qquad e(t) = \frac{1}{1-zty}$$

$L(t) = G(t) + M(t)$.
The equality (2) gives:

$$L(t) = e(t)G(t) + f(t)L(tz) \qquad (3)$$

When we substitute t by tz in (3), we obtain:

$$L(tz) = e(tz)G(tz) + f(tz)L(tz^2) \qquad (4)$$

By doing the same substitution in (1), we obtain the following equation:

$$G(tz) = a(tz)G(1) - b(tz)G(tz) + c(tz)L(tz) + d(tz) \qquad (5)$$

The equalities (1) and (5) give expressions for $L(tz)$ and $L(tz^2)$. Lets transfer them into (4). We can then write the following theorem:

Theorem 3.3 *The generating function G(x,y,z,t) of the polyominoes of $\mathcal{F}$ according to the horizontal semi-perimeter (by x), vertical semi-perimeter (by y), area (by z) and the height of the first column (by t) satisfies the following equation:*

$$G(t) = A(t)G(1) + B(t)G(tz) + C(t)G(tz^2) + D(t) \qquad (6)$$

where:

$$A(t) = a(t) - \frac{c(t)f(tz)a(tz)}{c(tz)}, \qquad B(t) = -b(t) + c(t)e(tz) + \frac{c(t)f(tz)}{c(tz)}$$

$$C(t) = \frac{f(tz)a(tz)c(t)}{tz^2c(tz)}, \qquad D(t) = d(t) - \frac{d(tz)f(tz)c(t)}{c(tz)}$$

and so: $$G(t) = \frac{E(t)F(1) + F(t)(1-E(1))}{1-E(1)}$$

where:

$$E(t) = \sum_{i\geq 0} U_i A(tz^i), \text{ et} \qquad F(t) = \sum_{i\geq 0} U_i D(tz^i)$$

where: U_n *is a sequence which satisfies the following equation:*

$$U_0 = 1, \quad U_1 = B(t) \quad \text{and} \quad U_n = C(tz^{n-2})U_{n-2} + B(tz^{n-1})U_{n-1}$$

Corollary 3.4 *The area generating function* $G(1,1,z,1)$ *of* $\mathcal{F}$ *is:*

$$G(1,1,z,1) = \frac{F}{1-E}$$

where: $$E = \sum_{i\geq 0} z^{i+1} \frac{(1-z^{2i+4})(1-z^{i+3})(1-z^{i+2}) - (1-z^{2i+2})z^{2i+5}}{(1-z^{2i+4})(1-z^{i+3})(1-z^{i+2})^2(1-z^{i+1})} U_i$$

and $$F = \sum_{i\geq 0} z^{i+1} \frac{(1-z^{2i+4})(1-z^{i+2}) - (1+z^{i+1})z^{2i+5}}{(1-z^{2i+4})(1-z^{i+2})} U_i$$

where (U_n) *is a sequence satisfying the following recurrence:*

$$U_0 = 1, \qquad U_1 = \frac{-z^2}{(1-z^4)(1-z)}$$

and $$U_n = \frac{(1+z^{n-1})z^{2n}}{(1-z^{2n})(1-z^{n+1})(1-z^n)^2} U_{n-2} + \frac{(z^2-1)z^{2n}}{(1-z^{2n+2})(1-z^{n+1})(1-z^n)} U_{n-1}$$

4 Semi-perimeter generating function of $\mathcal{F}$

We note that $G(t) = \sum_{r\geq 1} g_r t^r$ and $M(t) = \sum_{r\geq 2} m_r t^r$.

By developing the formulas (1) and (2), we obtain the following formulas for g_r and m_r:

for $r \geq 1$,

$$g_r = xz^r((g_r + m_r) + y(g_{r-1} + m_{r-1}) + \frac{1-(yz)^r}{1-yz}G(1) - \sum_{k=1}^{r}\sum_{h=1}^{k}(yz)^{r-k}g_h) \tag{7}$$

for $r \geq 2$,

$$m_r = xz^r \sum_{k=1}^{r-2}(r-k-1)y^{r-k}(g_k + m_k) + \sum_{k=1}^{r-1} z^{r-k}y^{r-k}g_k \tag{8}$$

Consider now the particular case where *z:=1 and y:=x* :
In this case, we note that$G(1) = G(x,x,1,1)$.
Using (1) and (2), we notice that G(t) is rational. Hence,

$$G(t) = \frac{N(t)}{D(t)} \tag{9}$$

where:

$$N(t) = t(b_1 - b_2t + b_3t^2 - b_4t^3 + b_5t^4) \tag{10}$$

$$D(t) = 1 - a_1t + a_2t^2 - a_3t^3 + a_4t^4 \tag{11}$$

The coefficients are:

$$\begin{aligned} &a_1 = (1+x)^2,\ a_2 = x(3+2x+x^2-x^3),\ a_3 = x^2(3+x),\ a_4 = x^3,\\ &b_1 = x^2 + xG(1),\ b_2 = x^2(1+3x) + 2x^2G(1),\\ &b_3 = x^3(3+3x-x^2) + x^3(1-x)G(1),\ b_4 = x^4(3-x),\\ &b_5 = x^5(1-x), \end{aligned} \tag{12}$$

Let $$G_1(t) = \sum_{r\geq 2} g_r t^{r-2} = \frac{G(t) - g_1t}{t^2}$$

we have: $$G_1(t) = \frac{N_1(t)}{D(t)} \tag{13}$$

Where:

$$\begin{aligned} N_1(t) = &-x^4(x^2+G(1))t^3 + x^3(x^3+x^2+x+3G(1))t^2\\ &+x^2(x^4-2x^3+x^2+(x^3-2x^2-x-3)G(1))t\\ &+x(x^3-x^2+(x^2+1)G(1)) \end{aligned} \tag{14}$$

We notice, then, that the series $G_1(t)$ is rational according to t such that the upper term has a lower degree than the lower term.
So, using [Vi2], we have to solve the homogeneous linear recurrence with constant coefficents of degree 4:

$$\text{for } r \geq 2 \qquad g_{r+4} - a_1 g_{r+3} + a_2 g_{r+2} - a_3 g_{r+1} + a_4 g_r = 0 \qquad (15)$$

If we write $D(t) = \prod_{i=1}^{4}(1-\lambda_i t)$, the g_r's where $r \geq 2$ are written as follows:

$$\text{for } r \geq 2 \qquad g_r = \sum_{i=1}^{4} A_i \lambda_i^{r-2}$$

[$\lambda_1, \lambda_2, \lambda_3$ et λ_4 **are given in the appendix**]

When $x \to 0$, we have:

$\lambda_1 = O(1) \qquad \lambda_2 = O(x)$
$\lambda_3 = O(x) \qquad \lambda_4 = O(x)$

But $g_r = O(x^{r+1})$, because the height of the first column is r. So, $A_1 = 0$

Hence, $\qquad$ for $r \geq 2 \qquad g_r = A_2\lambda_2^{r-2} + A_3\lambda_3^{r-2} + A_4\lambda_4^{r-2}$, $\quad$ (16)

By summing (16) for $r \geq 2$ and by knowing that in this case, (7) gives us: $g_1 = x^2 + xG(1)$, we obtain:

$$G(x,x,1,1) = (= G(1))\frac{1}{1-x}\left(x^2 + \sum_{i=2}^{4}\frac{A_i}{1-\lambda_i}\right) \qquad (17)$$

and so:

$$\begin{aligned} G(t) &= g_1 + t^2\sum_{r\geq 2} g_r t^{r-2} \\ &= t\left(x^2 + \frac{x}{1-x}(x^2 + \sum_{i=2}^{4}\frac{A_i}{1-\lambda_i})\right) + t^2\sum_{i=2}^{4}\frac{A_i}{1-\lambda_i t} \qquad (18) \end{aligned}$$

Using (13) and (16), we can say that λ_1^{-1} is a root of $N_1(t)$. So, the polynomials $D(t)$ and $N_1(t)$ have a same root. By calculating the resultant of the two polynomials by eliminating the variable t, we obtain the following algebraic equation :

$$\begin{aligned} &G(1)^4 + 2x(1-x+x^2)G(1)^3 + x(-1+3x-7x^2+5x^3-2x^4+x^5)G(1)^2 \\ &+x^3(2-5x+6x^2-3x^3+2x^4)G(1) + x^5(-1+2x-x^2+x^3) = 0 \qquad (21) \end{aligned}$$

By writing $N_1(\lambda_1^{-1}) = 0$, we obtain the following theorem:

Theorem 4.1 *The diagonal semi-perimeter generating function, of the under-diagonal diagonally convex directed animals, is:*

$$G(x,x,1,1) = \sum_{n\geq 2} a_n x^n = \frac{x^3(x^2-(x^2+x+1)\lambda_1 - (x-1)^2\lambda_1^2 + (1-x)\lambda_1^3)}{-x^3+3x^2\lambda_1 + x(x^3-2x^2-x-3)\lambda_1^2 + (x^2+1)\lambda_1^3}$$

where a_n is the number of the under-diagonal diagonally convex directed animals of diagonal semi-perimeter n. λ_1 is given in the appendix.

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Appendix:

$$\lambda_1 = p_5 + \frac{1}{12}\sqrt{\frac{6(p_6-p_7)}{\sqrt{p_4}}}$$
$$\lambda_2 = p_5 - \frac{1}{12}\sqrt{\frac{6(p_6-p_7)}{\sqrt{p_4}}}$$
$$\lambda_3 = p_5 + \frac{1}{12}\sqrt{\frac{6(p_6+p_7)}{\sqrt{p_4}}}$$
$$\lambda_4 = p_5 - \frac{1}{12}\sqrt{\frac{6(p_6+p_7)}{\sqrt{p_4}}}$$

where:

$$p_1 = x^4(3x^2(-4+32x-88x^2+172x^3-312x^4+251x^5+8x^6+98x^7 +48x^8+55x^9-20x^{10}))^{(1/2)}$$
$$p_2 = \frac{1}{3}x^5 - \frac{28}{27}x^6 + \frac{23}{18}x^7 - \frac{2}{9}x^8 + \frac{5}{54}x^9 + \frac{1}{9}x^{11} - \frac{1}{27}x^{12} - \frac{1}{18}p_1$$
$$p_3 = \frac{1}{3}x^5 - \frac{28}{27}x^6 + \frac{23}{18}x^7 - \frac{2}{9}x^8 + \frac{5}{54}x^9 + \frac{1}{9}x^{11} - \frac{1}{27}x^{12} + \frac{1}{18}p_1$$
$$p_4 = 11x^4 + 4x^3 + 2x^2 - 12x + 3 + 12\sqrt[3]{p_3} + 12\sqrt[3]{p_2}$$
$$p_5 = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4} + \frac{1}{12}\sqrt{3p_4}$$
$$p_6 = (11x^4 + 4x^3 + 2x^2 - 12x + 3 - 6\sqrt[3]{p_3} - 6\sqrt[3]{p_2})\sqrt{p_4}$$
$$p_7 = \sqrt{3}(12x^3 - 30x^5 - 15x^6 - 9x^4 - 21x^2 + 18x - 3)$$