ON ELEMENTARY METHODS IN POSITIVITY THEORY

J. Gillis

The Department of Applied Mathematics
The Weizmann Institute of Science
Rehovot, Israel

B. Reznick

The Department of Mathematics
University of Illinois
Urbana, IL 61801

and

D. Zeilberger

The Department of Theoretical Mathematics
The Weizmann Institute of Science
Rehovot, Israel

Abstract: We raise conjectures concerning the positivity of the power series coefficients of multi-variate rational functions. We also give a short proof of a result of Askey and Gasper that $(1-x-y-z+4xyz)^{-\beta}$ has positive power series coefficients for $\beta \geq (\sqrt{17}-3)/2$. We show how Ismail and Tamharkar's proof that

$$(1-(1-\lambda)x - \lambda y - \lambda xz - (1-\lambda)yz + xyz)^{-\alpha} \quad (0 \leq \lambda \leq 1)$$

has positive power series coefficients for $\alpha=1$ implies Koorwinder's result that it does so for $\alpha \geq 1$.

1. We are interested in the following

Problem 1: Give necessary and sufficient conditions on multi-variate polynomials $P(x_1,\ldots,x_n)$ and $Q(x_1,\ldots x_n)$ such that $P/Q = \sum a_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ has non-negative power-series coefficients, i.e. $a_{\alpha_1,\ldots\alpha_n} \geq 0$ for all $(\alpha_1,\ldots,\alpha_n) \in N^n$.

Szego [6] proved that $1/[(1-r)(1-s) + (1-r)(1-t) + (1-s)(1-t)]$ has positive power series coefficients. Askey and Gasper [2] established a similar result for $[1 - r - s - t + 4rst]^{-1}$ and also for some other rational functions. However, it seems that no attack on the general problem has yet been made. The most general result to date is that of Askey and Gasper [2] :

Theorem 2 : $[1 + (1-\rho)\rho^{-1}(r+s+t) + (\rho-2)\rho^{-1}(rs+rt+st) + (3-\rho)\rho^{-1}rst]^{-1}$ expanded as a series of powers of r,s,t, has non-negative coefficients for $\rho \geq 2$.

We refer the reader to Askey's monograph [1] (Ch. 6) for a fascinating account of the history of this problem up to 1975. In 1978 Koorwinder [5] proved that for $0 \leq \lambda \leq 1$ and $\alpha \geq 0$, $[1-(1-\lambda)r - \lambda s - \lambda rt - (1-\lambda)st + rst]^{-\alpha-1}$ has positive coefficients. Ismail and Tamhankar [4] gave an elementary proof of this result (for $\alpha = 0,1,\ldots$) using MacMahon's Master theorem. This result (for $\lambda = 1/2$) was proved independently by Gillis and Kleeman [3] who also gave an elementary proof of a result equivalent to Koorwinder's theorem [5] that $[1-x-y-z-u + 4xyz + 4xyu + 4xzu + 4yzu - 16xyzu]^{-1}$ has positive power series coefficients.

In the general case, write

$$P = \sum p_\beta x^\beta = \sum_{|\beta| \leq N} p_{\beta_1 \ldots \beta_n} x_1^{\beta_1} \ldots x_n^{\beta_n}$$

and

$$Q = \sum_{|\beta| \leq N} q_\beta x^\beta = 1 + \sum_{\substack{\beta \neq 0 \\ |\beta| \leq N}} q_\beta x^\beta = 1 + Q' \text{ , (say), (here}$$

$|\beta| = \beta_1 + \ldots + \beta_n)$.

Now, $P/Q = P(1 + Q')^{-1} = P \sum_{n=0}^{\infty} (-1)^n Q'^n$.

Using the multinomial theorem and collecting terms one sees that the a_α's in $P/Q = \sum a_\alpha x^\alpha$ are polynomials in the p_β's and q_β's ; $a_\alpha = a_\alpha(p_\beta, q_\beta)$. The conditions on the p_β's and q_β's are therefore that they satisfy the infinite set of polynomial inequalities

(1) $$a_\alpha (p_\beta, q_\beta) \geq 0 \text{ , } \alpha \in N^n .$$

We conjecture that the required p_β , q_β are in fact solutions of a <u>finite</u> set of polynomial inequalities.

<u>Conjecture 3</u> : For every integer M, there are polynomials $\{A_i\}_{i=1}^{K}$ such that for P and Q of degree M, P/Q has non-negative power series coefficients if and only if

$$A_i(p_\beta, q_\beta) \geq 0, \quad i = 1,\dots K.$$

If the above conjecture turns out to be true then it may be that the A_i's can be taken among the a_α's thus leading to

<u>Conjecture 4</u> : For every pair of positive integers M, n there exists an integer L such that if P and Q are polynomials in n variables of degree M, and $P/Q = \sum_{\alpha \in N^n} a_\alpha x^\alpha$, then $a_\alpha \geq 0$ for $|\alpha| \leq L$ implies $a_\alpha \geq 0$ for every α.

These conjectures are still unproved. However for the special case of functions of the type $1/[1-x-y-z + Axy + Bxz + Cyz + Dxyz]^{-1}$ we have established the following result, based on an observation of Askey [1] and a theorem of Koorwinder [5].

<u>Proposition 5</u> : Let $0 \leq \mu_1, \mu_2, \mu_1 + \mu_2 \leq 1$, $\mu_3 \leq 1$, then

$$F(\mu_1,\mu_2,\mu_3 ; x,y,z) = 1/[1-(1-\mu_1)x - (1-\mu_2)y - (1-\mu_3)z + (1-\mu_1-\mu_2)xy + (1-\mu_1-\mu_3)xz + (1-\mu_2-\mu_3)yz + (\mu_1+\mu_2+\mu_3-1)xyz]$$

has positive power series coefficients.

<u>Proof</u> : Koorwinder [5] (see also Ismail and Tamhankar [4]) proved that

$$(2) \quad \int_0^\infty e^{-u} L_n(\lambda u)\, L_m((1-\lambda)u)\, L_k(u)\,du \geq 0,$$

for $0 \leq \lambda \leq 1$. It is known that for any c,

(3) $L_n(cu) = \sum \binom{n}{k} (1-c)^{n-k} c^k L_k(u)$.

It follows from (2) and (3) that for $0 \leq c_1, c_2, c_3 \leq 1$,

$$\int_0^\infty e^{-u} L_n(c_1 \lambda u)\, L_m(c_2(1-\lambda)u)\, L_k(c_3 u)\, du \geq 0$$

and hence

$$\int_0^\infty e^{-u} L_n(\mu_1 u)\, L_m(\mu_2 u)\, L_k(\mu_3 u)\, du \geq 0$$

for $\mu_1 + \mu_2 \leq 1$, $0 \leq \mu_3 \leq 1$, $0 \leq \mu_1, \mu_2$. But the generating function of these numbers (Askey [1], p. 47) is precisely $F(\mu_1,\mu_2,\mu_3,x,y,z)$. This establishes proposition 5.

The transformation $x \leftarrow (1-\mu_1)x$, $y \leftarrow (1-\mu_2)y$, $z \leftarrow (1-\mu_3)z$ yields

Corollary 6 : $(1-x-y-z + Axy + Bxz + Cyz + Dxyz)^{-1}$ has non-negative power series coefficients if there exist $0 \leq \mu_1, \mu_2$, such that $\mu_1 + \mu_2 \leq 1$ and $0 \leq \mu_3 \leq 1$ such that

$$A = (1-\mu_1-\mu_2)/(1-\mu_1)(1-\mu_2) , \quad B = (1-\mu_1-\mu_3)/(1-\mu_2)(1-\mu_3) ,$$

$$C = (1-\mu_2-\mu_3)/(1-\mu_2)(1-\mu_3) , \quad D \leq (\mu_1+\mu_2+\mu_3-1)/(1-\mu_1)(1-\mu_2)(1-\mu_3) .$$

By eliminating μ_1, μ_2, μ_3 it is possible to obtain polynomial inequalities in A,B,C,D which would ensure the positivity of the coefficients of $(1-x-y-z + Axy + Bxz + Cyz +Dxyz)^{-1}$, as in conjecture 3 .

2. Some operations preserving the positivity of the coefficients .

If p and q have positive coefficients then so does, of course, pq . Slightly less trivial is

Proposition 7 : Let a,b,c,d be polynomials in $x_1,\dots,x_n$. If $(a-bx_{n+1})^{-1}$

and $(c-dx_{n+1})^{-1}$ have positive coefficients, then so does $[a(\underline{x})c(\underline{y}) - b(\underline{x})d(\underline{y})]^{-1}$.

<u>Proof :</u> $(a-bx_{n+1})^{-1} = \sum(b^r/a^{r+1})\, x^r_{n+1}$ thus b^r/a^{r+1} has positive coefficients for every r. Similarly d^r/c^{r+1} has positive coefficients. The same will therefore be true for $b^r d^r/a^{r+1}c^{r+1}$ and $\sum(bd)^r/(ac)^{r+1} = [a(\underline{x})c(\underline{y}) - b(\underline{x})d(\underline{y})]^{-1}$.

<u>Example 8 :</u> Ismail and Tamhankar [4] and Gillis and Kleeman [3] gave an elementary proof of Koorwinder's result that $[1-x-y-wx-wy + 4xyw]^{-1} = [(1-x-y)-w(x+y-4xy)]^{-1}$ has positive coefficients. Hence so also have

$[(1-x-y)(1-z-u) - (4xy-x-y)(4zu-z-u)]^{-1} =$

$[1-x-y-z-ut + 4(xyz + zyu + yzu) - 16xyzu]^{-1}$ but this is the generating function for $\int_0^\infty e^{-2u} L_m L_n L_k L_r$, and so this quadruple integral is positive (see Koorwinder [5] and Gillis and Kleeman [3]).

<u>Observation 9 :</u> If $p(x_1,\ldots,x_n)$ has positive coefficients and so do $q_1,\ldots,q_n$ then $p(q_1,\ldots,q_n)$ also has positive coefficients.

<u>Example 10 :</u> We know that $(1-x-y-zx-zy + 4xyz)^{-1}$ has positive coefficients (example 8); setting $z \leftarrow z/1-z$ we see that $(1-z)/(1-x-y-z + 4xyz)$ also has positive coefficients. This result is stronger than Askey and Gasper's positivity result since $1/(1-x-y-z + 4xyz) = (1-z)^{-1} \cdot (1-z)/(1-x-y-z + 4xyz)$. Indeed, writing

$$(1-x-y-z + 4xyz)^{-1} = \sum a_{m,n,k}\, x^m y^n z^k \text{ , then}$$

$$(1-z)(1-x-y-z + 4xyz)^{-1} = \sum(a_{m,n,k} - a_{m,n,k-1})\, x^m y^n z^k$$

and we get that $a_{m,n,k} \geq a_{m,n,k-1}$, and by symmetry we get the monotonicity result: -

$$a_{m,n,k} \geq a_{m',n',k'} \quad \text{if} \quad m \geq m', \ n \geq n', \quad k \geq k' .$$

Since $L_n^{(-1)} = L_n - L_{n-1}$ the above result is also equivalent to

$$\int_0^\infty e^{-2x} L_n^{(-1)}(x)\, L_m(x)\, L_k(x)\, dx \geq 0 .$$

Proposition 10: Suppose that $a(x_1,\ldots,x_{n-1})$ and $b(x_1,\ldots,x_{n-1})$ are polynomials. If (i) $(a - bx_n)^{-1}$ has positive coefficients and (ii) $a^{-\beta}$ has positive coefficients for all $\beta > 0$, then so does $(a-bx_n)^{-\alpha}$ for all $\alpha \geq 1$.

Proof : By hypothesis $(a-bx_n)^{-1} = \sum (b^r/a^{r+1}) x_n^r$ has positive coefficients, implying that so also does b^r/a^{r+1} for every r. Since $\binom{-\alpha}{r}(-1)^r$ is positive and $a^{1-\alpha}$ has positive coefficients, we see that

$$(a-bx_n)^{-\alpha} = a^{1-\alpha} \sum_{r=0}^{\infty} \binom{-\alpha}{r} (-1)^r\, b^r/a^{r+1} \quad \text{has positive}$$

coefficients.

Example 11 : Ismail and Tamhankar [4] give a proof of the positivity of the coefficients of $[1-(1-\lambda)x - \lambda y - \lambda xz - (1-\lambda)yz + xyz]^{-1}$. This, together with proposition 10, yields an elementary proof of Koorwinder's [5] result that $(1-(1-\lambda)x - \lambda y - \lambda xz - (1-\lambda)yz + xyz)^{-\alpha}$ has positive coefficients for all real $\alpha \geq 1$.

Proposition 12: If $[a(x,y) - b(x,y)z]^{-\alpha}$ and $[c(x,y) - d(x,y)z]^{-\alpha}$ have positive coefficients, so also does $[a(x,y)c(z,u) - b(x,y)d(z,u)]^{-\alpha}$.

Proof: Similar to that of Proposition 7.

Example 13 : By Example 11 with $\lambda = 1/2$,

$$(1 - \tfrac{1}{2}x - \tfrac{1}{2}y - \tfrac{1}{2}xz - \tfrac{1}{2}yz + xyz)^{-\alpha} = (1 - \tfrac{1}{2}x \quad \tfrac{1}{2}y - (\tfrac{1}{2}x + \tfrac{1}{2}y - xy)z]^{-\alpha}$$

has positive coefficients. Hence

$$[(1 - \tfrac{1}{2}x - \tfrac{1}{2}y)(1 - \tfrac{1}{2}x' - \tfrac{1}{2}y') - (\tfrac{1}{2}x + \tfrac{1}{2}y - xy)(\tfrac{1}{2}x' + \tfrac{1}{2}y' - x'y')]^{-\alpha}$$

$$= [1 - \tfrac{1}{2}x - \tfrac{1}{2}y - \tfrac{1}{2}x' - \tfrac{1}{2}y' + \tfrac{1}{2}xyx' + \tfrac{1}{2}xyy' + \tfrac{1}{2}xx'y' + \tfrac{1}{2}yx'y' - xyx'y']^{-\alpha}$$

has positive coefficients for $\alpha \geq 1$. This yields an elementary proof of Koorwinder's [5] result that

$$\int_0^\infty e^{-2x} L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x)\, L_k^{(\alpha)}(x)\, L_r^{(\alpha)}(x)\, dx \geq 0 \quad (\alpha \geq 0) .$$

3. A short proof of a result of Askey and Gasper

It follows from Example 9 and Proposition 10 that $(1-x-y-z + 4xyz)^{-\beta}$ has positive coefficients for $\beta \geq 1$. Askey and Gasper [2] extended this result to $\beta \geq (\sqrt{17} - 3)/2$. This can be obtained quite simply by an extension of a method used in [3] .

Suppose that $\beta > (\sqrt{17}-3)/2$. Write

$R = 1-x-y-z + 4xyz$, it is readily seen that

$$\frac{\partial}{\partial x} R^{-\beta} = (1+2z)\left[x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \beta\right]R^{-\beta} + 2\left(y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\right)R^{-\beta} .$$

Substitute $R^{-\beta} = \sum D_{a+1,b,c}\, x^{a+1}y^bz^c$ above, compare coefficients of $x^ay^bz^c$, and set $a \leftarrow a-1$ to get

$$aD_{a,b,c} = (a + b - c + \beta - 1)D_{a-1,b,c} + 2(a - b + c - 2 + \beta)D_{a-1,b,c-1} .$$

Now, by symmetry, it is enough to prove positivity for $a \geq b \geq c$. The coefficients of the above recurrence are positive if $a \geq b \geq c > 1$ and the result will follow by induction if $D_{a,a,1} \geq 0$ for all a.

Now

$$D_{a,a,1} = \frac{\beta(\beta+1)\cdots(\beta+2a-2)}{(a-1)!^2}\left[\frac{(\beta+2a-1)(\beta+2a)}{a^2} - 4\right]$$

But $(\beta+2a-1)(\beta+2a) - 4a^2 = \beta^2-\beta+2a(2\beta-1)$ increases with a since $\beta \cong 0.56 > 0.5$, while $D_{1,1,1} = \beta(\beta^2 + 3\beta - 2) > 0$, and the result follows.

4. Does $(1-(x_1+\dots+x_n) + n!\, x_1 \cdots x_n)^{-1}$ have positive power series coefficients?

We have already mentioned Askey and Gasper's result that $[1-(x+y+z) + 4xyz)^{-1}$ has positive power series coefficients. We are interested in A_n, the largest A for which $(1-(x_1+\dots+x_n) + Ax_1\cdots x_n)^{-1}$ has non-negative coefficients. Since the coefficients of $x_1\cdots x_n$ in the above expansion is $n! - A_n$, we must certainly have $A_n \leq n!$. We conjecture that for $n \geq 4$, $A_n = n!$. It may be seen that $A_n \geq (n-1)!$, i.e. that $[1-(x_1+\dots+ x_n) + (n-1)!\, x_1\cdots x_n)^{-1}$ has positive coefficients. The reason is that the coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is the above expansion has combinatorial significance, namely, it is the number of words with α_1 1's,..., α_n n's such that no substring of n letters which ends with the letter "n" can be a permutation (e.g. with n=4, the six words 1234, 1324, 2134, 2314, 3124, 3214 are not allowed as subwords) (see Zeilberger [7] for details).

Let us state

Proposition 14 : Let $(1-(x_1+\dots+x_n) + n!\, x_1\cdots x_n)^{-1} = \sum A_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. If $A_{r,\dots,r} \geq 0$ for all r then $A_{\alpha_1,\dots\alpha_n} \geq 0$ for all $(\alpha_1,\dots,\alpha_n) \in N^n$.

The proof is rather long and we omit it here. Note that

$$A^{(n)}_{r,\ldots,r} = \sum_{j=0}^{r} (-1)^j \frac{(rn-(n-1)j)!\ (n!)^j}{(r-j)!^n\ j!}$$

and it would therefore suffice to show that this binomial sum is positive. This has been verified by computer for $n=4$ and $1 \leqslant r \leqslant 220$. In this range $a^{(4)}_{r\ldots r}$ increases monotonically and appears to have faster than exponential growth. This would appear to support our conjecture.

Acknowledgement : Many thanks are due to Gilad Bandel for his competent programming.

References

1. R. Askey, "Orthogonal Polynomials and Special Functions", CBMS, vol. 21, SIAM, Philadelphia, 1975.

2. R. Askey and G. Gasper, Convolution structures of Laguerre polynomials, J. Analyse Math. 31 (1977), pp. 48-68.

3. J. Gillis and J. Kleeman, A combinatorial proof of a positivity result, Math. Proc. Camb. Phil. Soc. 86 (1979), pp. 13-19.

4. M.E.H. Ismail and M.V. Tamhankar, A combinatorial approach to some positivity problems, SIAM J. Math. nal., 10 (1979), pp. 478-485.

5. T. Koorwinder, Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition theorem, J. London Math. Soc. (2), 18 (1978), pp. 101-114.

6. G. Szegö, Über gewisse Potenzreihen mit lauter positiven Koeffizienten, Math. Z., 37 (1933), pp. 674-688 .

7. D. Zeilberger, Enumeration of words by their number of mistakes, Discrete Mathematics, 34 (1981), pp. 89-91.