Counting Stable Sets In Trees

by

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Abstract

Cook [1] has shown that the problem of generating all the stable sets of a given graph is NP complete while several authors have shown that it is possible to generate all the maximal stable sets of G in polynomial-time. In [3] this is done in O(n = number of vertices, m = number of edges, μ = number of maximal stable sets).

In this paper we shall

1) prove that the problem of counting all stable sets in a tree is no harder than listing all maximal stable sets in a tree
2) calculate the upper bound on the number of maximal stable sets in a tree of n vertices.

§ I. Introduction:

Theorem 1: If P is a path of n vertices then the number of stable sets (always including ∅) is the (n+2)nd Fibonacci number F(n+2) where F(1) = F(2) = 1 .

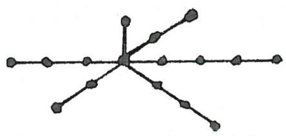
Proof: By induction. If the first vertex is in the stable set the rest is chosen from P_{n-2} and there are F(n) of them.

If the first vertex is not in the set the rest are chosen from (P_{n-1}) in F(n+1) ways. In total F(n)+F(n+1) = F(n+2) sets.□

Although it has no baring on our topic we mention that the same argument proves.

Theorem 2: The number of stable sets in an n-cycle is F(n-1) + F(n+1).□

A tree with only one vertex of degree > 2 is called a star, e.g.



Definition: In a tree a maximal path of vertices of degree 2 ending in a leaf (a vertex of degree 1) is called a leg.□

Example: The star above has legs of length 1,2,4,3,2 and 3 reading clockwise from the top.

The following is trivial:

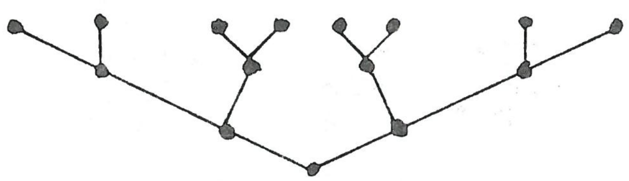
Theorem 3: A star with legs of length $n_1, n_2, \ldots$ has exactly

$$\prod_i F(n_i+1) + \prod_i F(n_i+2)$$

stable sets.

§ II. Side trip:

A tree in which the root has degree 2, all other vertices have degree 1 or 3 and each leaf is height (n-1) from the root is called the Binary Tree of rank n , B_n . B_4 is shown below:



Let b_n be the number of stable sets in Bn.. Clearly

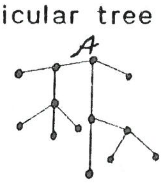
$$b_n = b_{n-1}^2 + b_{n-2}^4$$

$$b_0 = 1 \ , \ b_1 = 2 \ , \ b_2 = 5 \ , \ b_3 = 41 \ , \ b_4 = 2306 \ \ldots . \ ■$$

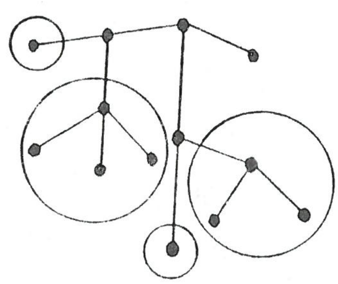
This sequence is not in Sloane [2] .□

§ III. The Principle of deletion to count stable sets in trees:

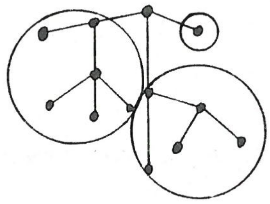
Let us study the particular tree drawn below:

T = 

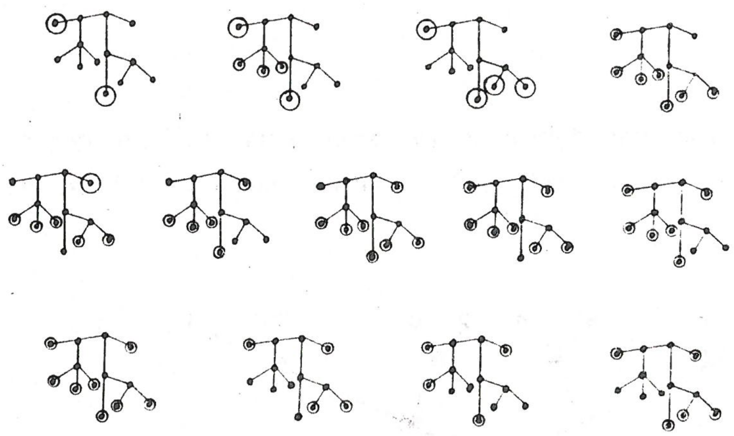
If A is in the stable set the rest of the vertices come from the circled subrees below



If A is not the rest come from these circles subtrees



Continuing this process we finally write T =

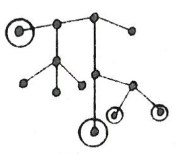


A tree with x circled vertices counts for 2^x stable subsets therefore the total number of stable sets in the original tree is exactly

$$4+32+16+128+64+32+128+128+64+256+16+8+32 = 908$$

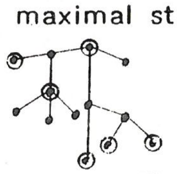
Each of these 13 trees with circles corresponds to a maximal stable set of T where the leafs used are the ones circled and the other vertices are internal. For example

1



corresponds to the maximal stable set.

2

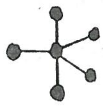


This gives us the following algorithm.

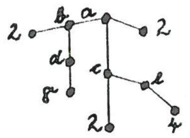
§ IV. The algorithm:

We begin with a definition.

Definition: If a vertex of degree (n+1) is connected to n leaves it is called an n-fan.

e.g  is a 4-fan.

Step 1: For every n=fan, replace all the leaves with one leaf labeled 2^n . Label all other leaves 2 . Label all internal vertices with letters.



Step 2: List all maximal Stable sets of this simplified tree

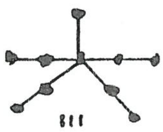
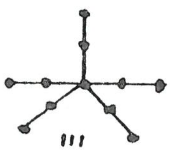
a2d2e
a2d24
a282e
a2824
b8c42
b8242
b82e2
2dc42
2d2e2
2d242
28c42
282e2
28242

Step 3: Set all letters =1 , take the product of the labels of every maximal stable set, add these numbers and obtain the total number of stable sets of the original tree. □

Theorem 4: The complexity of counting all stable sets in a tree is no greater than the complexity of listing all maximal stable sets of a simpler tree. □

§ V. Which tree has the most stable sets?

Wilf [4] has found the trees of n vertices with the most maximal stable sets. They are

 n-even  n-odd

It is not a priori true that these trees must have the most stable sets. The maximal sets might have many common subsets, as we shall see.

Lemma 1: For every tree there are only two stable sets such that both it and its complement are both stable.

Proof: Every such stable set would give a 2-coloring of the tree. But a tree has essentially one 2-coloring. Therefore the special sets are the red set and (its complement) the blue set. □

Lemma 2: A tree of n vertices can have at most $2^{n-1}+1$ stable sets.

Proof: Except for the red set and the blue set the complement of a stable set is nonstable. Therefore of the 2^n-2 nonspecial subsets of vertices at most half are stable.□

Theorem 5: The tree with only one non-leaf has the most stable sets.

□

References:

[1] COOK, S.A.: The complexity of theorem proving procedures, Proc. 3rd Annual ACM Symposium of Theory of Computing (1971) pp. 151-8

[2] SLOANE, N.A.: Handbook of integer sequences

[3] TSUKIYAMA, S./IDE, M./ARIYOSHI, M./SHIRAKAWA, I.: A new algorithm for generating all maximal independent sets, SIAM J. Comput. vol.6, no. 3 (1977) pp. 505-517

[4] WILF, H.S.: The number of maximal independent sets in a tree, to appear