Regular polytopes and equivariant tessellations
from a combinatorial point of view

Research announcement

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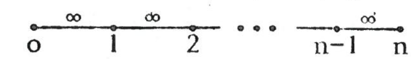
Let $\Sigma = \Sigma(n) = \langle \sigma_o, \sigma_1, \ldots, \sigma_n \mid \sigma_i^2 = (\sigma_i \sigma_k)^2 = 1$ for $i,k = 0,\ldots,n$; $|i-k| \geq 2 \rangle$ denote the Coxeter group associated to the diagram  .
To any equivariant tessellation (M^n, T, Γ) consisting of an n-dimensional manifold M^n, a tessellation T of M^n and a group Γ of homeomorphisms of M^n respecting the tessellation T we associate a Σ-set $\mathcal{D} = \mathcal{D}(M^n, T, \Gamma)$ and n functions $r_1, \ldots, r_n : \mathcal{D} \to \mathbb{N}$ which in case $\pi_o(M^n) = \pi_1(M^n) = 1$ characterize (M^n, T, Γ) completely up to isomorphism.

Several consequences and examples are being discussed.

§ 1 Tessellations

Let T be a partially ordered set *). For any such T we define the derived semisimplicial complex (or the barycentric subdivision) $\mathring{T} =: \{B \subseteq T \mid B$ finite and linearly ordered$\}$ and the topological realization

$$|T| =: |\mathring{T}| =: \left\{ \sum_{t \in T} x_t\, t \in \bigoplus_{t \in T} \mathbb{R}t \;\middle|\; x_t \geq 0,\ \sum_{t \in T} x_t = 1,\ \{t \mid x_t > 0\} \in \mathring{T} \right\} \subseteq \bigoplus_{t \in T} \mathbb{R}t$$

with $\bigoplus_{t \in T} \mathbb{R}t$ denoting the real vectorspace, freely generated by T and topologized by the "direct limit topology" (i.e. in such a way that a subset $\mathcal{O} \subseteq \bigoplus_{t \in T} \mathbb{R}t$ is open if and only if the intersections with all finite dimensional subspaces, topologized as usual, are open).

Any homomorphism $\varphi : T_1 \to T_2$ between two partially ordered sets defines a pl-map $|\varphi| : |T_1| \to |T_2|$ which is injective if and only if φ is injective.

Let X be a pl-space or a "polyhedron" in the sense of [6]. We define a <u>tessellation</u> of X to consist of a partially ordered set T together with a pl-homeomorphism $X \cong |T|$.

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*) In the definition of a partially ordered set we include the axiom " $s \leq t$ and $t \leq s \Rightarrow s = t$ ".

Examples: $T = \{(a_1,\dots,a_n) \mid a_i \in \{0,1,2\}\}$ with "$(a_1,\dots,a_n) \leq (b_1,\dots,b_n)$" if and only if $(a_i - b_i)(b_i - 1) = 0$ for all $i = 1,\dots,n$ gives the standard tessellation of the n-dimensional cube I^n; $T = \mathbb{Z}^n$ with "$(a_1,\dots,a_n) \leq (b_1,\dots,b_n)$" if and only if $|a_i - b_i| \leq 1$ and $(a_i - b_i)(b_i - 1) \equiv 0(2)$ gives the cubic tessellation of $\mathbb{R}^n$; $T = T_0 \dot{\cup} T_1 \dot{\cup} T_2$ with $T_0 = \langle\alpha_1,\alpha_2\rangle \setminus (A_5 \times \{\pm 1\})$, $T_1 = \langle\alpha_0,\alpha_2\rangle \setminus (A_5 \times \{\pm 1\})$, $T_2 = \langle\alpha_0,\alpha_1\rangle \setminus (A_5 \times \{\pm 1\})$; $\alpha_0 = (13)(45) \times (-1)$, $\alpha_1 = (14)(32) \times (-1)$, $\alpha_2 = (14)(35) \times (-1) \in A_5 \times \{\pm 1\}$ a standard choice of generating involutions, identifying $A_5 \times \{\pm 1\}$ with the Coxeter group $\underset{0}{\bullet}\overset{5}{\text{—}}\underset{1}{\bullet}\text{—}\underset{2}{\bullet}$ and with $\langle \alpha_j \mid j \neq i \rangle \beta \leq \langle \alpha_j \mid j \neq k \rangle \gamma$ if and only if $i \leq k$ and $\langle \alpha_j \mid j \neq i \rangle \beta \cap \langle \alpha_j \mid j \neq k \rangle \gamma \neq \emptyset$ gives the dodecahedral decomposition of the 2-sphere.

Standard constructions: For T_1 and T_2 two partially ordered sets we have $|T_1| \times |T_2| \cong |T_1 \times T_2|$ with $T_1 \times T_2$ the partially ordered set consisting of the cartesian product of T_1 and T_2 with "$(t_1,t_2) \leq (s_1,s_2)$" if and only if $t_1 \leq s_1$ and $t_2 \leq s_2$ $(t_i,s_i \in T_i;\ i = 1,2)$. We have $|T_1| * |T_2| = |T_1 * T_2|$ for the join of $|T_1|$ and $|T_2|$ with $T_1 * T_2$ denoting the partially ordered set $T_1 \dot{\cup} T_2 = T_1 \times \{1\} \cup T_2 \times \{2\}$ with $(t,i) \leq (s,j)$ if and only if $i = j$ and $t \leq s$ (in T_i) or $i = 1$ and $j = 2$. For T a partially ordered set we have $\mathring{\hat{T}} = \mathring{T}$ and thus $|\hat{T}| = |T|$ with $\hat{T}$ the dual partially ordered set, consisting of the same elements as T but with $s \mathrel{\hat{\leq}} t$ if and only if $t \leq s$. Thus for a tessellation $|T| \cong X$ of a polyhedron X we have the dual tessellation $|\hat{T}| \cong X$ and for tessellations $|T_1| \cong X_1$ and $|T_2| \cong X_2$ of two polyhedra X_1 and X_2 we have the tessellations $|T_1 \times T_2| \cong X_1 \times X_2$ and $|T_1 * T_2| \cong X_1 * X_2$ of the product $X_1 \times X_2$ and the join $X_1 * X_2$ of X_1 and X_2.

Cellular and smooth tessellations: For any element or tile $t \in T$ of a partially ordered set T we define the boundary

$\partial_* t =: \{s \in T \mid s < t\}$, the co-boundary $\partial^* t = \{s \in T \mid t < s\}$,

the closure $e_* t = \{s \in T \mid s \leq t\}$ and the co-closure $e^* t = \{s \in T \mid t \leq s\}$ of t. T is defined to be cellular (co-cellular) if all $|\partial_* t|$ $(|\partial^* t|)$ are (pl-) spheres in which case the topological realizations $|e_* t|$ $(|e^* t|)$ give rise to a cell decomposition of $|T|$ in the sense of [6],chapter 2. It follows from [6],p.24,exercise 2.24,(5), that $|T|$ is a pl-manifold if and only if T is cellular and co-cellular, in which case T will be called <u>smooth</u>.

Since there are obvious obstructions for deciding whether or not a partially ordered set is smooth (i.e. the unproved Poincaré conjecture in dimension 3, it seems reasonable to consider certain weaker, purely combinatorial conditions on T.

<u>Dimension and finiteness:</u> For a partially ordered set we define $\dim T = \max \{ \# B \mid B \in \mathring{T} \} - 1$ and we define $\dim t = \dim e_* t$ and $\operatorname{codim} t = \dim e^* t$ for any $t \in T$. T is <u>finite dimensional</u> if $\dim T < \infty$ and it is <u>locally finite dimensional</u> if $\dim t < \infty$ for all $t \in T$. T is finite if $\# T < \infty$ and T is <u>locally finite</u> if $\# e_*(t) < \infty$ for all $t \in T$.

<u>Flags:</u> A maximal linearly ordered subset of T is called a <u>flag</u> F, the set of all maximal linearly ordered subsets of T is called the flag-space $\mathcal{F} = \mathcal{F}(T)$ of T. If T is locally finite dimensional and $F \in \mathcal{F}(T)$ we denote the i-th element in the linearly ordered set F by $F(i)$, starting with $i=0$, i.e. if $F = \{t_0, t_1, \ldots, t_i, \ldots\}$ and $t_0 < t_1 < \cdots < t_i < \cdots$, then $F(i) = t_i$.
We define two flags $F, F' \in \mathcal{F}(T)$ to be <u>wall-neighbours</u> and denote this by $F \vee F'$, if they differ by one element only, i.e. if there exist $t, t' \in T$ with $t \neq t'$ and $F = (F \cap F') \dot{\cup} \{t\}$, $F' = (F \cap F') \dot{\cup} \{t'\}$, in which case we have necessarily
$\{s \in F \mid s < t\} = \{s' \in F' \mid s' < t'\}$ and
$\{s \in F \mid s > t\} = \{s' \in F' \mid s' > t'\}$. If T is locally finite dimensional, we define F and F' to be <u>k-wall-neighbours</u> if $t = F(k)$ or – equivalently – $t' = F'(k)$ and we denote this by $F \overset{k}{\vee} F'$.

<u>Pure and locally pure tessellations:</u> We define a tessellation T to be <u>pure</u> if it is finite-dimensional and if all flags in $\mathcal{F}(T)$ have the same cardinality. T is defined to be <u>locally pure</u> if $e_*(t)$ is pure for all $t \in T$. Note that T is pure if and only if T and $\hat{T}$ are locally pure and $\dim t + \operatorname{codim} t = \dim T$ holds for all $t \in T$. Note also that for a locally pure tessellation T, a flag $F \in \mathcal{F}(T)$ and an element $t \in F$ we have $\dim t = i$ if and only if $F(i) = t$.

Flag-connected and locally flag-connected tessellations:

A tessellation T is defined to be flag-connected if for any two flags $F, F' \in \mathcal{F}(T)$ there exists a string of flags $F = F_o, F_1, \ldots, F_\ell = F'$ with $F_o \vee F_1$, $F_1 \vee F_2, \ldots, F_{\ell-1} \vee F_\ell$. Note that a flag-connected tessellation is pure if and only if it is finite dimensional. T is defined to be locally flag-connected if $e_* t$ is flag-connected for all $t \in T$. Note that T is flag-connected if T and $\hat{T}$ are locally flag-connected and T is "connected", i.e. if for $t, t' \in T$ there exists a string of elements $t = t_o, t_1, \ldots, t_\ell = t' \in T$ with $t_o \leq t_1$, $t_1 \geq t_2, \ldots, t_{\ell-1} \geq t_\ell$, but that flag-connectedness does not imply local flag-connectedness.

T is defined to be strongly locally flag-connected, if T and $\hat{T}$ are locally flag-connected and if moreover for any $t, t' \in T$ with $t \leq t'$ the partially ordered subset $e^*(t) \cap e_*(t') = \{s \in T \mid t \leq s \leq t'\}$ is flag-connected. This is easily seen to be equivalent to the following condition: If B is a non-empty, linearly ordered subset of T and if $F, F' \in \mathcal{F}(T)$ are two flags containing B, then ther exists a string of flags $F_o = F$, $F_1, \ldots, F_\ell = F'$ with $F_o \vee F_1$, $F_1 \vee F_2, \ldots, F_{\ell-1} \vee F_\ell$ and $B \subseteq \bigcap_{\lambda=0}^{\ell} F_\lambda$.

Pseudo-smooth tessellations: A tessellation T is defined to be pseudo-smooth if it is pure and if for any $F \in \mathcal{F}(T)$ and $k \in \{0,1,\ldots,\dim T\}$ there exists precisely one k-wall-neighbour $F' \in \mathcal{F}(T)$ of F. We denote this F' by $\sigma_k(F)$. It is easy to see that this way, an action of the Coxeter-group $\Sigma = \Sigma(\dim T)$ defined above on the flag-space $\mathcal{F}(T)$ of a pseudo-smooth tessellation T is being defined, i.e. that $\sigma_k^2(F) = F$ and $\sigma_k\sigma_i(F) = \sigma_i\sigma_k(F)$ for $|i-k| \geq 2$; $i,k=0,1,\ldots,n$ hold.
We shall study the Σ-set $\mathcal{F}(T)$ in the next section.

Another way to describe pseudo-smoothness is by interpreting the derived complex $\dot{T}$ as a partially ordered with respect to inclusion and to look at its dual $\hat{\dot{T}}$: it is easily seen that the 1-skeleton $\hat{\dot{T}}^1 = \{B \in \dot{T} \mid \# B \geq \dim T\}$ of $\hat{\dot{T}}$ is cellular if and only if T is pseudo-smooth. Moreover, one can prove that the 2-skeleton $\hat{\dot{T}}^2 = \{B \in \dot{T} \mid \# B \geq \dim T-1\}$ is cellular if T is pseudo-smooth, strongly locally connected and T and $\hat{T}$ are locally finite.

One can also show that a tessellation T of an n-dimensional manifold M^n

is necessarily pseudo-smooth of dimension n, strongly locally connected and – together with $\hat{T}$ – locally finite. Moreover, M^n is compact if and only if T is finite.

For two tessellations T_1 and T_2 we have $\dim(T_1 \times T_2) = \dim T_1 + \dim T_2$ and $\dim(T_1 * T_2) = \dim T_1 + \dim T_2 + 1$.
$T_1 \times T_2$ is cellular, co-cellular or smooth if T_1 and T_2 are cellular, co-cellular or smooth, respectively, whereas – as a consequence of [6], chapter 2, p. 24, exercise 2.24,(5), $T_1 * T_2$ is cellular, co-cellular or smooth if and only if $|T_1|$ is a sphere and T_2 is cellular, T_1 is co-cellular and $|T_2|$ is a sphere or $|T_1|$ and $|T_2|$ are spheres, respectively.

We have $T_1 * T_2 = \hat{T}_2 * \hat{T}_1$, $F(T_1 * T_2) = F(T_1) \times F(T_2)$ and $(T_1 \dot{*} T_2) = \dot{T}_1 \times \dot{T}_2$. $T_1 * T_2$ is pure if and only if T_1 and T_2 are pure. $T_1 * T_2$ is flag-connected if and only if T_1 and T_2 are flag-connected and it is locally flag-connected if and only if T_1 and T_2 are locally flag-connected and T_1 is flag-connected. $T_1 * T_2$ is strongly locally flag-connected if T_1 and T_2 are strongly locally flag-connected and connected. $T_1 * T_2$ is pseudo-smooth if and only if T_1 and T_2 are pseudo-smooth in which case we have for $F = (F_1, F_2) \in F(T_1 * T_2) = F(T_1) \times F(T_2)$:

$$\sigma_k(F_1,F_2) = \begin{cases} (\sigma_k F_1, F_2) & \text{for } k \leq \dim T_1 \\ (F_1 , \sigma_{k-\dim T_1 - 1} F_2) & \text{for } k > \dim T_1 . \end{cases}$$

This shows in particular that pseudo-smoothness is a much weaker notion than smoothness, since – as we have stated above – $T_1 * T_2$ is smooth if and only if $|T_1|$ and $|T_2|$ are spheres. $T_1 \times T_2$ is (locally) pure if and only if T_1 and T_2 are (locally) pure. In the pure case we have

$$F(T_1 \times T_2) = F(T_1) \times F(T_2) \times \Phi(n_1 + n_2; n_1, n_2) \quad \text{with } n_i = \dim T_i$$

and $\Phi(n_1 + n_2; n_1, n_2)$ denoting the set of pairs (φ_1, φ_2) of monotonic maps $\varphi_1 : \{0,1,\ldots,n_1 + n_2\} \to \{0,\ldots,n_1\}$ and $\varphi_2 : \{0,\ldots,n_1 + n_2\} \to \{0,\ldots,n_2\}$ with $\varphi_1(k) + \varphi_2(k) = k$ for all $k = 0,1,\ldots,n_1 + n_2$ *) – once we identify an element
$(F_1, F_2; (\varphi_1, \varphi_2)) \in F(T_1) \times F(T_2) \times \Phi(n_1 + n_2; n_1, n_2)$ with the flag
$F \in F(T_1 \times T_2)$ defined by

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*) This set is easily seen to correspond to the set of subsets N_1 of cardinality n, of $\{1,2,\ldots,n_1 + n_2\}$ via $N_1 \to (\varphi_{N_1}, \varphi_{\bar{N}_1})$ with $\varphi_M(k) =: \#(M \cap \{0,\ldots,k\})$.

$$F(k) = (F_1(\varphi_1(k)),\ \ F_2(\varphi_2(k))).$$

If T_1 and T_2 are pure, $T_1 \times T_2$ is ((strongly) locally) flag-connected if and only if T_1 and T_2 are ((strongly) locally) flag-connected.

$T_1 \times T_2$ is pseudo-smooth if and only if T_1 and T_2 are pseudo-smooth, in which case we have – extending φ_1 and φ_2 artificially by $\varphi_i(-1) = -1$ and $\varphi_i(n_i + 1) = n_i + 1$ –

$$\sigma_k(F_1,F_2;\ (\varphi_1,\ \varphi_2)) = \begin{cases} (\sigma_{\varphi_1(k)} F_1,F_2;(\varphi_1,\varphi_2)) & \text{if } \ \varphi_1(k+1) = \varphi_1(k-1) + 2 \\ (F_1,\ \sigma_{\varphi_2(k)}F_2;(\varphi_1,\varphi_2)) & \text{if } \ \varphi_2(k+1) = \varphi_2(k-1) + 2 \\ (F_1,F_2;\ (\bar\varphi_1,\ \bar\varphi_2)) & \text{otherwise with} \\ \qquad \bar\varphi_i(j) = \begin{cases} \varphi_i(j) & \text{for } \ j \neq k \\ \varphi_i(k-1) + \varphi_i(k+1) - \varphi_i(k) & \text{for } \ j = k \end{cases} \end{cases}$$

One has always $\widehat{T_1 \times T_2} = \hat{T}_1 \times \hat{T}_2$.

Let us finally consider the derived complex $\mathring{T}$ of a tessellation T. $\mathring{T}$ is a partially ordered set with respect to inclusion. Being a semi-simplicial complex, it is always cellular and thus locally finite, locally pure and locally flag-connected. We have $\dim T = \dim \mathring{T}$. T is pure if and only if $\mathring{T}$ is pure. If $\dim T < \infty$, T is flag-connected if and only if $\mathring{T}$ is flag-connected and T is strongly locally flag-connected if and only if $\hat{\mathring{T}}$ is locally flag-connected in which case $\mathring{T}$ is strongly locally flag-connected. T is pseudo-smooth if and only if $\mathring{T}$ is pseudo-smooth. For T being pure of dimension n the flag-space $F(\mathring{T})$ can be identified with the cartesian product $F(T) \times S_{\{0,1,\ldots,n\}}$ of $F(T)$ and the full symmetric group $S_{\{0,\ldots,n\}}$, consisting of all permutations of the set $\{0,1,\ldots,n\}$, by identifying an element $(F,\pi) \in F(T) \times S_{\{0,\ldots,n\}}$ with the flag

$$(\{F(\pi(0))\},\ \{F(\pi(0)),\ F(\pi(1))\}\ ,\ldots,\ \{F(\pi(0)),\ldots,F(\pi(n))\}\ ) \in F(\mathring{T}).$$

If T is pseudo-smooth, this identification is a Σ-isomorphism once we define

$$\sigma_k(F,\pi) = \begin{cases} (F,\ \pi \circ (k,k+1)) & \text{for } k < n \\ (\sigma_{\pi(n)} F,\ \pi) & \text{for } k = n. \end{cases}$$

§ 2 Pseudo-smooth tessellations and Σ-sets.

In this section we want to study the relations between pseudo-smooth tessellations of dimension n and Σ-sets, Σ being defined as above.

For any $I \subseteq \{0,1,\ldots,n\}$ let $\Sigma^I = \{\sigma_k \in \Sigma \mid k \notin I\}$ and $\Sigma_I = \{\sigma_i \in \Sigma \mid i \in I\}$. For $I = \{i\}$ write Σ^i instead of $\Sigma^{\{i\}}$.

If T is pseudo-smooth, then $F(T)$ satisfies

(T0) $\sigma_k F \neq F$ for all $k = 0,1,\ldots,n$ and all $F \in F(T)$,

(T1) $\bigcap_{i=0}^{n} \Sigma^i F = \{F\}$ for all $F \in F(T)$,

(T2) $\bigcap_{i \neq k} \Sigma^i F = \{F, \sigma_k F\}$ for all $k = 0,\ldots,n$ and all $F \in F(T)$.

T is flag-connected if and only if Σ acts transitively on $F(T)$.
T and $\hat{T}$ are locally flag-connected if and only if for any $t \in T$ the subgroup $\Sigma^{\dim t}$ acts transitively on the set $F_t(T) =: \{F \in F(T) \mid t \in F\}$.

T is strongly locally flag-connected if and only if for any linearly ordered subset $B \subseteq T$ the subgroup $\Sigma^{\{\dim t \mid t \in B\}}$ acts transitively on $F_B(T) = \{F \in F(T) \mid B \subseteq T\} = \bigcap_{t \in T} F_t(B)$. Thus, if T and $\hat{T}$ are locally flag-connected, T is strongly locally flag-connected if and only if

(T3) $\bigcap_{i \in I} \Sigma^i F = \Sigma^I F$ for all $F \in F(T)$ and all $I \subseteq \{0,1,\ldots,n\}$

holds.

T is finite if and only if $F(T)$ is finite and, if T and $\hat{T}$ are locally flag-connected, T is locally finite if and only if $\Sigma^i F$ is finite for all $F \in F(T)$ and all $i = 0,1,\ldots,n$ or – equivalently – for all $F \in F(T)$ and $i = 0, n$.

Vice-versa - we can associate to any Σ-set $\mathcal{F}$ a pure, partially ordered set of dimension n defined by

$$T(\mathcal{F}) = \{(i, \Sigma^i F) \mid i = 0,1,\ldots,n \;;\; F \in \mathcal{F}\}$$

with "$(i, \Sigma^i F) \leq (k, \Sigma^k F')$" if and only if $i \leq k$ and $\Sigma^i F \cap \Sigma^k F' \neq \emptyset$.

If $\mathcal{F} = \mathcal{F}(T)$ for T a pseudo-smooth tessellation we have a natural, well-defined and surjective homomorphism of partially ordered sets $T(\mathcal{F}(T)) \twoheadrightarrow T : (i, \Sigma^i F) \mapsto F(i)$, which is an isomorphism if and only if T and $\hat{T}$ are locally flag-connected. Again, vice-versa, for any Σ-set $\mathcal{F}$ we have a natural, surjective map $\mathcal{F} \twoheadrightarrow \mathcal{F}(T(\mathcal{F})) : F \mapsto ((0, \Sigma^0 F), (1, \Sigma^1 F), \ldots, (n, \Sigma^n F))$, which is injective if and only if $\mathcal{F}$ satisfies (T1). In this case, $T(\mathcal{F})$ is pseudo-smooth if and only if $\mathcal{F}$ satisfies in addition (T0) and (T2), in which case $\mathcal{F} \hookrightarrow\!\!\!\twoheadrightarrow \mathcal{F}(T(\mathcal{F}))$ is an isomorphism of Σ-sets. Thus we have

Theorem 1: There is a 1-1 correspondance between pseudo-smooth tessellations T of dimension n, for which T and $\hat{T}$ are locally flag-connected, - such tessellations will be called Σ-tessellations - and Σ-sets $\mathcal{F}$ which satisfy (T0), (T1) and (T2).

As a consequence, one can derive

Theorem 2 (see [3]): For any Σ-tessellation T we have a canonical isomorphism

$$\mathrm{Aut}(T) \cong \mathrm{Aut}_\Sigma(\mathcal{F}(T)).$$

In particular, if T is flag-connected, this gives for any $F \in \mathcal{F}(T)$ the isomorphisms $\mathrm{Aut}(T) \cong \mathrm{Aut}_\Sigma(\mathcal{F}(T)) \cong \mathrm{Aut}_\Sigma(\Sigma / \Sigma_F) \cong N_\Sigma(\Sigma_F) / \Sigma_F$ with $\Sigma_F = \{\tau \in \Sigma \mid \tau F = F\}$ the stabilizer group of F and $N_\Sigma(\Sigma_F) = \{\tau \in \Sigma \mid \tau\Sigma_F = \Sigma_F \tau\}$ the normalizer of Σ_F in Σ.
Another application of the relation between tessellations and Σ-sets is

Theorem 3: Let T be a smooth tessellation of dimension n, let $F \in \mathcal{F}(T)$ be a flag and define $x_F = \sum_{t \in F} \frac{1}{n+1} t \in |T|$. Then

$$\pi_1(|T|, x_F) \cong \Sigma_F \;/\; \langle \tau^{-1} \langle \sigma_{k-1}, \sigma_k \rangle_{\tau F} \tau \mid k = 1,2,\ldots,n;\ \tau \in \Sigma \rangle$$

(with $\langle \sigma_{k-1}, \sigma_k \rangle_{\tau F} = \langle \sigma \in \langle \sigma_{k-1}, \sigma_k \rangle \mid \sigma\tau F = \tau F \rangle$ the stabilizer group of τF in $\langle \sigma_{k-1}, \sigma_k \rangle$).

Since $\pi_1(|T|, x_F) = \pi_1(|\mathring{T}|, x_F) = \pi_1(|\hat{\mathring{T}}|, F) = \pi_1(|\hat{\mathring{T}}^2|, F)$, this follows easily from

Theorem 3': Let T be pseudo-smooth of dimension n and strongly locally connected and let $F \in F(T)$ be a flag and thus a vertex in $\hat{\mathring{T}}$. Let $\hat{\mathring{T}}^2$ denote the 2-skeleton of $\hat{\mathring{T}}$, i.e.

$$\hat{\mathring{T}}^2 = \{B \in \mathring{T} \mid \# B \geq n-1\} \text{ with } "B \leq B'"$$

if and only if $B' \subseteq B$. Then

$$\pi_1(|\hat{\mathring{T}}^2|, F) \cong \Sigma_F / \langle \tau^{-1} \langle \sigma_{k-1}, \sigma_k \rangle_{\tau F} \tau \mid k = 1,2,\ldots,n;\ \tau \in \Sigma \rangle .$$

To rephrase this result observe that

$$\tau^{-1} \langle \sigma_{k-1}, \sigma_k \rangle_{\tau F} \tau = \tau^{-1}(\langle \sigma_{k-1}, \sigma_k \rangle \cap \Sigma_{\tau F})\, \tau$$

$$= \tau^{-1} \langle \sigma_{k-1}, \sigma_k \rangle \tau \cap \Sigma_F .$$

So, for any subgroup $\Delta \leq \Sigma$ we define $\tilde{\Delta} = \langle \tau^{-1} \langle \sigma_{k-1}, \sigma_k \rangle \tau \cap \Delta \mid k=1,\ldots,n;\ \tau \in \Sigma \rangle$ and observe that $\tilde{\Delta} \leq \Delta$, $\tilde{\tilde{\Delta}} = \tilde{\Delta}$ and $\pi_1(|\hat{\mathring{T}}^2|, F) \cong \Sigma_F / \tilde{\Sigma}_F$ if T is pseudo-smooth and strongly locally connected. We define $\Delta \leq \Sigma$ to be <u>dihedrally generated</u> if $\Delta = \tilde{\Delta}$ and thus we have as a corollary: if T is pseudo-smooth, flag-connected and strongly locally connected, then Σ_F is dihedrally generated for all $F \in F(T)$ if and only if $|\hat{\mathring{T}}^2|$ is simply connected, which in case T is smooth, is equivalent to $|T|$ being simply connected.

Furthermore we have for any pseudo-smooth tessellation T and any $F \in F(T)$ the relation $\langle \sigma_i, \sigma_k \rangle_F \subsetneq \langle (\sigma_i \sigma_k) \rangle$ for all $i,k = 0,\ldots,n$, i.e. we have $\langle \sigma_i, \sigma_k \rangle_F = 1$ for $|i-k| \geq 2$ and $\langle \sigma_{k-1}, \sigma_k \rangle_F = \langle (\sigma_{k-1}\sigma_k)^{r_k(F)} \rangle$ for $r_k(F) = (\langle (\sigma_{k-1}\sigma_k) \rangle : \langle \sigma_{k-1}, \sigma_k \rangle_F) \in \{2,3,\ldots;\infty\}$ (with the convention $\sigma^\infty = 1$). In other words, if we define a subgroup $\Delta \leq \Sigma$ to be polygonal if $\langle \sigma_i, \sigma_k \rangle \cap \tau \Delta \tau^{-1} \subsetneq \langle (\sigma_i \sigma_k) \rangle$ for all $i,k = 0,\ldots,n$, then Σ_F is polygonal for any flag F of a pseudo-smooth tessellation T. It seems reasonable to conjecture that for any dihedrally generated and polygonal subgroup $\Delta \leq \Sigma$ the Σ-set Σ/Δ satisfies (T0) and (T3) (and thus (T1) and (T2) !) so that $T = T(\Sigma/\Delta)$ is a pseudo-smooth, strongly locally connected and flag-connected tessellation with a simply connected $|\hat{T}^2|$.

For any polygonal subgroup $\Delta \leq \Sigma$ we define

$$r_k = r_k^{\Delta} : \Sigma \longrightarrow \mathbb{N} \cup \{\infty\} : \tau \mapsto r_k(\tau) = (\langle \sigma_{k-1}\sigma_k \rangle : \langle \sigma_{k-1}\sigma_k \rangle \cap \tau\Delta\tau^{-1})$$

for $k = 1,2,\ldots,n$. We have obviously

(P0) $r_k(\tau) \geq 2$ for all $\tau \in \Sigma$; $k = 1,2,\ldots,n$.

(P1) $r_k(\sigma_i \tau) = r_k(\tau)$ for all $\tau \in \Sigma$; $k,i \in \{1,\ldots,n\}$ and $i \neq k-2, k+1$,

and we have

(P2) (i) $r_k(\sigma_{k-2}\tau) = r_k(\tau)$ if $r_{k-1}(\sigma\tau) = 2$ for all $\sigma \in \langle \sigma_{k-1}\sigma_k \rangle$

(ii) $r_k(\sigma_{k+1}\tau) = r_k(\tau)$ if $r_{k+1}(\sigma\tau) = 2$ for all $\sigma \in \langle \sigma_{k-1}\sigma_k \rangle$

for all $\tau \in \Sigma$ and all $k = 1,2,\ldots,n$.

(P3) $r_k(\tau \cdot \rho^{-1}(\sigma_{i-1}\sigma_i)^{r_i(\rho)}\rho) = r_k(\tau)$ for all $\tau,\rho \in \Sigma$; $i,k \in \{1,\ldots,n\}$.

Another reasonable conjecture is that for any set of functions $r_k : \Sigma \longrightarrow \mathbb{N} \cup \{\infty\}$ satisfying the compatibility conditions (P0), (P1), (P2) and (P3), the subgroup

$$\Delta = \Delta(r_1,r_2,\ldots,r_n) =: \left\langle \tau^{-1}(\sigma_{k-1}\sigma_k)^{r_k(\tau)}\tau \mid k = 1,\ldots,n;\ \tau \in \Sigma \right\rangle$$

is polygonal (it is obviously dihedrally generated) and satisfies $r_k(\tau) = r_k^{\Delta}(\tau)$ for all $\tau \in \Sigma$; $k = 1,2,\ldots,n$.

If both conjectures were true - and they can probably be proved generalizing the methods of Bourbaki/Tits, [1] - we would have a nice 1-1 correspondance between

(a) pseudo-smooth, strongly locally connected and flag-connected tessellations T with a simply connected $|\hat{T}^2|$,

(b) transitive Σ-sets F, satisfying (T0) and (T3), with $\Sigma_F = \tilde{\Sigma}_F$ dihedrally generated for all $F \in F$,

(c) conjugacy classes of dihedrally generated, polygonal subgroups $\Delta \leq \Sigma$.

(d) equivalence classes of families of functions $r_1, r_2, \ldots, r_n : \Sigma \to \mathbb{N} \cup \{\infty\}$, satisfying (P0), (P1), (P2) and (P3), with the equivalence defined by "$(r_1, r_2, \ldots, r_n) \sim (r_1', r_2', \ldots r_n')$" if and only if there exists some $\tau \in \Sigma$ with $r_k(\sigma) = r'(\sigma\tau)$ for all $\sigma \in \Sigma$; $k = 1, 2, \ldots, n$.

So far we have a 1-1 correspondance between (a) and (b) and we have to any object in (b) a unique object in (c) and to any object in (c) a unique object in (d).

It seems worthwhile to observe finally in this context that for a dihedrally generated, polygonal subgroup $\Delta \leq \Sigma$ we have

$$N_\Sigma(\Delta) = \{\sigma \in \Sigma \mid r_k^\Delta(\tau\sigma) = r_k^\Delta(\tau) \text{ for all } \tau \in \Sigma \text{ and all } k = 1, 2, \ldots, n\}.$$

§ 3 Equivariant tessellations

An <u>equivariant tessellation</u> (T, Γ) consists of a tessellation T and a group Γ of automorphisms of T, acting on T from the right.

If T is pseudo-smooth of dimension n and T and $\hat{T}$ are locally connected, - i.e. if T is a Σ-tessellation - this corresponds - by Theorem 2 - to a Σ-set $\mathcal{F}$, satisfying (T0), (T1) and (T2), together with a group Γ of Σ-automorphisms of $\mathcal{F}$, acting from the right on $\mathcal{F}$ - i.e. to an "equivariant Σ-set" $(\mathcal{F}, \Gamma)$. Thus we can form the Σ-set $\mathcal{D} = \mathcal{D}(T, \Gamma) = \mathcal{F}/\Gamma$ of Γ-orbits of flags of T, which we call the <u>Delaney - symbol</u> of (T, Γ). From $\mathcal{D}$ we get a canonical tessellation of the orbit space $|T|/\Gamma$ via

<u>Theorem 4:</u> For any Σ-set $\mathcal{F}$ define the "derived Σ-set" $\mathring{\mathcal{F}}$ by

$$\mathring{\mathcal{F}} = \mathcal{F} \times S_{\{0,\ldots,n\}}$$

with

$$\sigma_k(F, \pi) = \begin{cases} (F, \pi \cdot (k, k+1)) & \text{for } k < n \\ (\sigma_{\pi(n)} F, \pi) & \text{for } k = n\,. \end{cases}$$

Then for any equivariant tessellation (T, Γ) we have a canonical homeomorphism

$$|T| / \Gamma \cong |T(\mathring{\mathcal{D}}(T, \Gamma))| \,.$$

In particular, $\mathcal{D}(T, \Gamma)$ is finite if and only if $|T| / \Gamma$ is compact.

Now we observe that for any equivariant Σ-set $(\mathcal{F}, \Gamma)$ we have functions $r_k : \mathcal{F}/\Gamma \longrightarrow \mathbb{N} \cup \{\infty\}$ defined by $r_k(F\Gamma) = (\langle \sigma_{k-1}\sigma_k \rangle : \langle \sigma_{k-1}\sigma_k \rangle_F)$, since the r.h.s. of this equation does not depend on the chosen representative F of the Γ-orbit $F\Gamma$. These functions have properties similar to those listed as (P0), (P1) and (P2).

The following theorem follows immediately from the foregoing results:

<u>Theorem 5:</u> To any equivariant tessellation (M^n, T, Γ) of a manifold M^n or - more generally - to any equivariant Σ-tessellation (T, Γ) we can associate the Delaney-symbol $\mathcal{D} = \mathcal{D}(T, \Gamma)$ and a family of functions $r_1, \ldots, r_n : \mathcal{D} \longrightarrow \mathbb{N}\ (\cup \{\infty\})$ - the <u>ramification parameters</u> of the equivariant tessellation (T, Γ) - having the properties

(P0') $r_k(f) \geq 2$ for all $f \in \mathcal{D}$ and $k \in \{1, \ldots, n\}$

(P1') $r_k(\sigma_i f) = r_k(f)$ for all $f \in \mathcal{D}$ and $i, k \in \{1, \ldots, n\}$ with $i \neq k-2, k+1$.

(P2') (i) $r_k(\sigma_{k+1} f) = r_k(f)$ if $r_{k+1}(\sigma f) = 2$ for all $\sigma \in \langle \sigma_{k-1}\sigma_k \rangle$

(ii) $r_k(\sigma_{k-2} f) = r_k(f)$ if $r_{k-1}(\sigma f) = 2$ for all $\sigma \in \langle \sigma_{k-1}\sigma_k \rangle$.

If M^n is connected and simply connected or - more generally - if T is connected and strongly locally flag-connected and $|\hat{T}^2|$ is simply connected, then (M^n, T, Γ) (or just (T, Γ)) is uniquely determined by its Delaney-symbol and its ramification parameters, - i.e. if (M'^n, T', Γ') (or just (T', Γ')) is another equivariant tessellation and M'^n is also connected and simply connected (or T' is also a flag-connected Σ-tessellation with a simply connected $|\hat{T}'^2|$), then we have an isomorphism $(M^n, T, \Gamma) \cong (M'^n, T', \Gamma')$ (or $(T, \Gamma) \cong (T', \Gamma')$) if and only if we have a Σ-isomorphism $\mathcal{D}(T, \Gamma) \overset{\alpha}{\xrightarrow{\sim}} \mathcal{D}(T', \Gamma')$, such that $r_k(f) = r'_k(\alpha f)$ for all $f \in \mathcal{D}(T, \Gamma)$ - r'_k denoting the ramification parameters of (T', Γ').

The following results are of interest in this context:

(1) Γ acts transitively on the i-dimensional tiles if and only if Σ^i acts transitively on $\mathcal{D}(T, \Gamma)$, - more precisely, we have a

natural bijection between $\Sigma^i \setminus \mathcal{D}(T, \Gamma)$ and T_i / Γ with $T_i = \{t \in T \mid \dim t = i\}$: $\Sigma^i \setminus \mathcal{D}(T, \Gamma) = \Sigma^i \setminus \mathcal{F}(T) / \Gamma = T_i / \Gamma$.

(2) Γ acts fixed point free on the i-dimensional tiles T_i if and only if $\Sigma^i_F = \Sigma^i_{\Gamma F}$ for all $F \in \mathcal{F}(T)$.

Finally we state

<u>Theorem 6:</u> Let (M^n, T, Γ) be an equivariant tessellation of the connected and simply connected manifold M^n. Assume Γ to act sharply transitive (i.e. transitive and fixed point free) on the vertices or zero-dimensional tiles of T. Then Γ can be presented as follows:

Choose some $F \in \mathcal{F}(T)$. For any flag $A = \alpha F \in \Sigma^o F$ $(\alpha \in \Sigma^o)$ in the Σ^o-orbit of F there exist a unique flag $\bar{A} = \bar{\alpha} F \in \Sigma^o F$ $(\bar{\alpha} \in \Sigma^o)$ and a unique element $\gamma_A \in \Gamma$ with $\sigma_o A = \bar{A}\,\gamma_A$.

We have $\bar{\bar{A}} = A$, $\gamma_{\bar{A}} = \gamma_A^{-1}$ and $\overline{\sigma A} = \sigma \bar{A}$ as well as $\gamma_{\sigma A} = \gamma_A$ for $\sigma \in \Sigma^{o,1} =: \Sigma^{\{o,1\}} = \langle \sigma_k \mid k \geq 2 \rangle$, so γ_A depends only on the $\Sigma^{o,1}$-orbit $a = \Sigma^{o,1} A$ of A – so we write γ_a instead of γ_A for $a = \Sigma^{o,1} A$ – and the involution $A \mapsto \bar{A}$ of $\Sigma^o F$ defines an involution

$$a = \Sigma^{o,1} A \mapsto \bar{a} = \Sigma^{o,1} \bar{A} = \overline{\Sigma^{o,1} A}$$

on the orbit space $\Sigma^{o,1} \setminus \Sigma^o F$.

For any $A \in \Sigma^o F$ define $A_1 = A$, $A_{k+1} = \sigma_1 \bar{A}_k$ and $a_k(A) = \Sigma^{o,1} A_k$.

Then the homomorphism of the free group $\mathbb{F} = \mathbb{F}(\Sigma^{o,1} \setminus \Sigma^o F)$, generated by the $\Sigma^{o,1}$-orbits $a = \Sigma^{o,1} A$ of flags A in $\Sigma^o F$, into Γ, defined by $a \mapsto \gamma_a$ is surjective and its kernel is generated as a normal subgroup $K = K_F$ of $\mathbb{F}$ by the elements

(1) $\quad a\,\bar{a} \qquad (a \in \Sigma^{o,1} \setminus \Sigma^o F)$,

and

(2) $\quad a_{r_1(A)}(A) \circ a_{r_1(A)-1}(A) \circ \cdots \circ a_2(A) \circ a_1(A) \qquad (A \in \Sigma^o F)$.

If we do not assume M^n to be simply connected, we have instead an exact sequence

$$1 \longrightarrow \pi_1(M^n) \longrightarrow \mathbb{F} / K \longrightarrow \Gamma \longrightarrow 1.$$

As a corollary we get: For any equivariant tessellation (M^n, T, Γ) of a connected manifold M^n for which Γ acts sharply transitive on the vertices of T, and for any $F \in \mathcal{F}(T)$ we have an exact diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \tilde{\Sigma}_F & \longrightarrow & \Sigma_F & & \longrightarrow & & \mathbb{F} / K_F & \longrightarrow \Gamma \longrightarrow & 1 \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & \pi_1(M^n) & & & & \\
 & & & & & \nearrow & & \searrow & & & \\
 & & & & 1 & & & & 1 & &
\end{array}$$

Theorem 6 can be proved more or less purely topologically or by using topology only to prove that for any equivariant tessellation (M^n, T, Γ), for which Γ acts sharply transitively on the vertices of T, and for any flag $F \in (T)$ the subgroup Σ_F, defined above, is generated as a normal subgroup of $\Sigma_{F\Gamma}$ by $\Sigma^o \cap \Sigma_{F\Gamma} = \Sigma^o \cap \Sigma_F$ and the elements $\tau^{-1}(\sigma_1\sigma_o)^{r_1(\tau F)}\tau$ $(\tau \in \Sigma^o)$ and then applying the following, basically probably well-known lemma, which states the group theoretical background of Theorem 6 :

<u>Lemma:</u> Let G be a group, let U, V, W be subgroups of G and assume $UV = VU = G$ and $U \cap V \subseteq W \subseteq V$.

(a) The map $V / W \longrightarrow U \backslash G / W : vW \longrightarrow UvW$ is a bijection.

(b) If $W \trianglelefteq V$, $\Gamma = V / W$ and $G = \langle U, g_i \mid i \in I \rangle$

then we can define a system of generators
$\{\gamma_{i,uW} \mid i \in I,\ uW \in UW / W = \{xW \mid x \in U\}\}$ of Γ by observing, that for any $i \in I$ and any $uW \in UW / W$ there exists a unique coset $h_i(uW) \in UW / W$ and a unique element $\gamma = \gamma_{i,uW}$ with $g_i uW = h_i(uW)\gamma$.
(Here we use that $\Gamma = V / W$ acts naturally on G / W from the right. It also acts naturally and sharply transitively on $U \backslash G / W$.)

(c) If W is generated as a normal subgroup of V by $U \cap V$ and certain elements $y_j \in W$ $(j \in J)$, we can define a complete system of relations for these generators in the following way:

(1) For each sequence $K = ((g_{i_k}, u_k), (g_{i_{k-1}}, u_{k-1}), \dots (g_{i_1}, u_1))$ define $h^1(K) = u_1 W \in UW / W$, $h^{\kappa+1}(K) = u_{\kappa+1} \cdot h_{i_\kappa}(h^\kappa(K))$ and $\gamma_\kappa(K) = \gamma_{i_\kappa, h^\kappa(K)} \in \{\gamma_{i,uW} \mid i \in I, uW \in UW / W\}$.

(2) Express each y_j as a product

$$g_{i_{j,k_j}} \cdot u_{j,k_j} \cdot g_{i_{j,k_j-1}} \cdot u_{j,k_j-1} \cdot \dots \cdot g_{i_{j,1}} \cdot u_{j,1}$$

thereby associating to each y_j a certain (of course not uniquely determined) sequence K_j of the form considered (1).

(3) Then the relations

$$\gamma_{k_j}(K_j) \cdot \gamma_{k_j-1}(K_j) \cdot \dots \cdot \gamma_1(K_j) = 1 \quad (j \in J)$$

are a complete system of relations for Γ with respect to the generators $\{\gamma_{i,uW} \mid i \in I, uW \in U / W\}$.

§ 4 Some applications

(a) In the two-dimensional case one verifies easily that a pseudo-smooth tessellation T is smooth if and only if T and $\hat{T}$ are locally connected and locally finite.

If (S^2, T, Γ) is an equivariant tessellation of the 2-sphere with Γ acting by isometries with respect to the elliptic metric on S^2, one has a finite Delaney-symbol $\mathcal{D} = \mathcal{D}(S^2, T, \Gamma) = \mathcal{F}(T) / \Gamma$ and – using $\chi(S^2) = 2$ – one can prove that $K = K(\mathcal{D}; r_1, r_2) =: \sum_{f = F\Gamma \in \mathcal{D}} (\frac{1}{r_1(f)} + \frac{1}{r_2(f)} - \frac{1}{2})$ is positive and that $|\Gamma| = 4 \cdot K^{-1}$.

If $(\mathbb{E}^2, T, \Gamma)$ is an equivariant tessellation of the euclidean plane with Γ acting by euclidean isometries, one has – as always – $|\mathcal{D}| < \infty$ if and only if $\mathbb{E}^2 / \Gamma$ is compact (i.e. Γ is crystallographic), in which case $K(\mathcal{D}(\mathbb{E}^2, T, \Gamma); r_1, r_2) = 0$ holds.

If $(\mathbb{H}^2, T, \Gamma)$ is an equivariant tessellation of the hyperbolic plane with Γ acting by hyperbolic isometries and with $\mathbb{H}^2 / \Gamma$ compact one has $K(\mathcal{D}; r_1, r_2) < 0$. This result can probably be extended to groups Γ with $vol(\mathbb{H}^2 / \Gamma) < \infty$ using an appropriate definition of $K(\mathcal{D}; r_1, r_2)$.

These results can be used to classify metrically equivariant tessellations (S^2, T, Γ) and $(\mathbb{E}^2, T, \Gamma)$ for which there are not too many Γ-orbits of vertices (or edges or faces) in T.

They give also rise to the conjecture, which has been proved in very many special cases already, that any equivariant tessellation (M^2, T, Γ) with M^2 / Γ compact (so that $K = K(\mathcal{D}(M^2, T, \Gamma); r_1, r_2)$ is defined) is isomorphic to a metrically equivariant tessellation (M'^2, T', Γ') with

$$M'^2 \cong \begin{cases} S^2 & \text{if } K > 0 \\ \mathbb{E}^2 & \text{if } K = 0 \\ \mathbb{H}^2 & \text{if } K < 0 \end{cases} .$$

They can also be used to reduce the classification problem of regular polyhedra in the sense of Branko Grünbaum (cf. [4]) to the (wider) problem of classifying all discrete subgroups of the full isometry group of the euclidean 3-space $\mathbb{E}^3$, which are generated by 3 involutions.

(b) In the <u>platonic</u> case, which is defined by the requirement that Γ acts transitively on the flag-space, so that the Delaney-symbol $\mathcal{D}$ becomes the trivial one-point-set, one can use the well-known classification of Coxeter-groups (see [1] or [2]) to give a complete description of all possible platonic pseudo-smooth tessellations T for which T and $\hat{T}$ are locally flag-connected and $|\hat{T}^2|$ is simply connected.

Since $\# \mathcal{D} = 1$, they are completely characterized by the sequence of numbers $\{r_1, r_2, \dots, r_n\}$ which of course is just the "Schläfli-symbol" of the platonic tessellation (T, Γ).

(c) If $(b_{ij})_{i,j \in \{0,1,\dots,n\}}$ is the Coxeter matrix of a Coxeter group (so $b_{ij} = b_{ji} \geq 2$ (for $i \neq j$), $b_{ij} \in \mathbb{N}$, $b_{ii} = 1$) we can associate to (b_{ij}) the $\Sigma = \Sigma(n)$-set $S_{\{0,1,\dots,n\}}$ on which Σ acts via the homomorphism

$$\beta : \Sigma \twoheadrightarrow S_{\{0,1,\dots,n\}} : \sigma_i \mapsto \begin{cases} (i,i+1) & i < n \\ \mathrm{Id} & i = n \end{cases},$$

the ramification parameters

$$r_k : S_{\{0,1,\dots,n\}} \to \mathbb{N} : f \mapsto \begin{cases} 3 & \text{if } k < n \\ 2b_{ij} & \text{if } n = k \text{ and } f = \begin{pmatrix} 0 \dots & n-1 & n \\ \dots\dots & i & j \end{pmatrix} \end{cases}$$

and the subgroup

$$\Delta = \left\langle \tau^{-1} (\sigma_{k-1}\, \sigma_k)^{r_k(\beta\tau)} \tau \mid \tau \in \Sigma;\ k = 1,2,\dots,n \right\rangle .$$

It can be shown that the tessellation $T = T(\Sigma / \Delta)$ corresponds to the simplicial complex associated to any Coxeter group by Tits (see [1]) and that the group $\Gamma = \mathrm{Ke}(\beta) / \Delta$ is isomorphic to the Coxeter group associated to (b_{ij}), the generating involutions being the cosets

$$\sigma_i\, \sigma_{i+1} \dots \sigma_{n-1}\, \sigma_n\, \sigma_{n-1} \dots \sigma_{i+1}\, \sigma_i\, \Delta .$$

It follows that for any two Coxeter matrices $(b_{ij})_{i,j \in \{0,\dots,n\}}$ and $(c_{ij})_{i,j \in \{0,\dots,n\}}$ of spherical type (i.e. for any two positive definite Coxeter matrices (b_{ij}) and (c_{ij})) we get a smooth tessellation T' of dimension $n+1$ if we define $S_{\{0,\dots,n\}}$ to be a $\Sigma = \Sigma(n+1)$-set via the homomorphism

$$\beta' = \Sigma \twoheadrightarrow S_{\{0,\dots,n\}} : \sigma_i \mapsto \begin{cases} (i-1,i) & 0 < i < n+1 \\ \mathrm{Id} & i = 0 \text{ or } i = n+1, \end{cases}$$

the ramification parameters

$$r_k : S_{\{0,\dots,n\}} \to \mathbb{N} : f \mapsto \begin{cases} 3 & \text{for } k \neq 1,\ n+1 \\ 2b_{ij} & \text{for } k = n+1,\ f = \begin{pmatrix} 0 \dots & n & n+1 \\ \dots\dots & i & j \end{pmatrix} \\ 2c_{ij} & \text{for } k = 1,\ f = \begin{pmatrix} 0\,1 \dots & n+1 \\ i\,j \dots\dots & \end{pmatrix} \end{cases},$$

$$\Delta' = \left\langle \tau^{-1} (\sigma_{k-1}\, \sigma_k)^{r_k(\beta')} \tau \mid \tau \in \Sigma;\ k = 1,\dots,n+1 \right\rangle$$

and put

$$T' = T(\Sigma / \Delta') .$$

It would be nice to know whether or not the associated topological space $|T'|$ can always be identified with the (n+1)-dimensional euclidean space $\mathbb{E}^{n+1}$ in such a way that the group of automorphisms $\Gamma' = Ke(\beta') / \Delta'$ acts isometrically and to determine the explicit structure of Γ'.

More generally it seems tempting to ask the following question: Let G be a Lie-group and let $U \leq G$ be a closed subgroup with $\pi_1(G/U) = 1$. Give necessary and perhaps even sufficient conditions for a $\Sigma = \Sigma(\dim G/U)$-set $\mathcal{D}$ and ramification parameters $r_k : \mathcal{D} \longrightarrow \mathbb{N}$ in order to ensure that the associated equivariant tessellation $(|T|, T, \Gamma)$ with $T = T(\Sigma / \Delta)$, $\Gamma = \Sigma_f / \Delta$, f some element in $\mathcal{D}$ and

$$\Delta = \left\langle \tau^{-1} (\sigma_{k-1} \sigma_k)^{r_k(\tau f)} \tau \;\middle|\; \tau \in \Sigma,\ k = 1,2,\ldots,\dim G/U \right\rangle$$

is isomorphic $(G/U, T, \Gamma')$ with Γ' a discrete subgroup of G acting in the natural way on G/U.

Applications of this theory towards planar patterns have appeared meanwhile in [7].

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